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Nonlinear weak generalized interpolation (**)

Introduction

Very often, the same operator is investigated on several different function spaces. Thus, it is valuable to have theorems which give relationships between properties of the same operator considered in different function spaces. The well known Marcel Riesz interpolation theorem [**19**] which was published in 1926 is a nontrivial example of such a theorem.

Since 1926, much work has been done in interpolation theory. A. P. Calderon [**4**], in 1964, used Banach space valued functions on the strip S in the complex plane to construct his complex interpolation spaces, $(X_0, X_1)_s$.

Lions-Peetre [**13**], in 1964, used Banach space valued « weighted » functions with real domain to construct their « mean » spaces, $S_X = S(P_0, E_0, X_0, P_1, E_1, X_1)$.

Later, M. Schechter [**21**]$_3$, in 1967, used Banach space valued functions defined in the complex strip together with a two-dimensional distribution T with compact support to construct interpolation spaces, $(X_0, X_1)_T$, which generalize the Calderon interpolation spaces above.

More recently, V. Williams [**23**]$_1$, in 1971 defined a generalized interpolation space, $X_{(T,C)}$, which generalized each of the above-mentioned interpolation

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spaces. Also, a generalized interpolation theorem is proved in [**23**]$_1$ which generalizes the Calderon, Lions-Peetre, and Schechter interpolation theorems.

Interpolation theory is very useful; as consequences, we get the classical theorems of Riesz [**19**] and Marcinkiewicz [**14**].

Also, complex interpolation theory has applications in differential equations (see [**21**]$_{1,2}$).

Up to now, most of the work in interpolation space theory has been concerned with obtaining bounded linear operators between interpolation spaces. It is natural to ask if there is also a compact interpolation theory.

Compact interpolation theory has been considered by several authors. In 1960, M. A. Krasnoselskii [**11**] studied compact interpolation between L^p spaces. In 1963, A. Persson [**17**] studied compact interpolation by using a compactness theorem of Lions-Peetre [**13**] together with a special H condition.

In 1964, Lions-Peetre [**13**] stated two compactness theorems where certain classes of Banach spaces are considered.

More recently, in 1974 R. K. Juberg [**10**] studied compact interpolation for a special operator between L^p spaces in which he presented in a paper at the January, 1974, Mathematics Meeting in San Francisco.

Compact interpolation theory can be applied directly to the study of compactness of certain integral operators.

In addition to the applications, the reader will note that a lot of elegant mathematics is used in compact interpoation theory.

The object of this paper is to consider nonlinear interpolation, where all the operators involved are not necessarily linear.

We also consider the case where all spaces are not necessarily Banach. Boundedness and compactness are considered for nonlinear operators; fixed point propositions are considered for nonlinear operators; finally, we consider nonexpansive operators in the nonlinear case.

This nonlinear paper is significant because Theorem 1 generalizes the generalized interpolation theorem for linear operators [**23**]$_1$.

This paper also motivates the following definition.

Definition. Let X be a normed linear space. Let T be a bounded linear operator from a Banach space C into X, then the Banach space $T(C) = \{T(f): f \in C\}$ under the norm

$$\|x\| = \inf_{f \in C} \{\|f\|_C : T(f) = x\},$$

is said to be a *weak generalized interpolation space* in X (relative to T and C), which is denoted, ${}_wX_{(T,C)}$.

The word « weak » is used because the interpolation space is no longer in a Banach space necessarily.

Clearly, generalized interpolation spaces are weak generalized interpolation spaces.

As a consequence of V. Williams' work [**23**]$_1$, Theorem 1 generalizes the interpolation theorems of Calderon [**4**], Lions-Peetre [**13**], and Schechter [**21**]$_3$. It follows from the work of the author [**9**] where a new diagram proof of Riesz's classical interpolation theorem is given that Theorem 1 also generalizes the classical Riesz interpolation Theorem, the Hausdorff-Young theorem, and Young's inequality.

Theorem 2 generalizes a generalized compactness proposition in [**9**].

Propositions 3 and 4 give sufficient conditions for an operator on a weak generalized interpolation space to have a nonzero fixed point and for the operator to be nonexpansive respectively.

There are corollaries for the theorems and propositions for the Calderon, Lions-Peetre, and Schechter interpolation spaces which give sufficient conditions for operators on all three of the aforementioned interpolation spaces to have nonzero fixed points.

We begin with the following definitions and notation.

If X and Y are normed linear spaces, then $L\colon X \to Y$ is *bounded* (not necessarily linear) if there is a constant $M \geqslant 0$ such that $\|Lx\|_Y \leqslant M\|x\|_X$ for every $x \in X$. Clearly, L takes bounded sets to bounded sets.

Definition. A *compatible triplet* $\{X_0, X_1, \chi_0\}$ consists of two Banach spaces X_0 and X_1 which are continuously embedded in a Hausdorff topological vector space χ_0.

From [**4**], we define the Calderon interpolation space

(1) $X_0 + X_1$ with norm defined by

$$\|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1}\} \qquad [x = x_0 + x_1,\ x_i \in X_i,\ i = 0, 1]$$

is a Banach space.

(2) $X_0 \cap X_1$ with norm defined by

$$\|x\|_{X_0 \cap X_1} = \max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\},$$

is a Banach space.

Let S be the open strip between 0 and 1 in the complex plane, that is, the set of complex numbers with real part greater than zero and less than 1. Let $H(X_0, X_1)$ be the set of all functions f from the closure of S to $X_0 + X_1$ which satisfy: (a) f is analytic in S, (b) f is continuous and bounded on the closure of S, (c) for $j = 0, 1$, there exist constants k_j such that $\|f(j+it)\|_{X_j} \leqslant k_j$ for all real t. Then

(3) $H(X_0, X_1)$ is a Banach space under the norm

$$\|f\|_{H(X_0,X_1)} = \max_{j=0,1} \{\sup_t \|f(j+it)\|_{X_j}\} .$$

Let $0 \leqslant s \leqslant 1$ be fixed, then

$$(X_0, X_1)_s = \{x \colon x = f(s), \text{ for some } f \in H(X_0, X_1)\}$$

is a Banach space under the norm

$$\|x\|_{(X_0,X_1)_s} = \inf \{\|f\|_{H(X_0,X_1)}\} \qquad [f(s) = x, \ f \in H(X_0, X_1)] .$$

We now define the Lions-Peetre [**13**] « mean » interpolation space.

Let $\{X_0, X_1, \chi_0\}$ be a compatible triplet; $X_0 + X_1$ and $X_0 \cap X_1$ are as above.

Let $1 \leqslant P_0, \ P_1 \leqslant \infty$. Let E_0 and E_1 be real numbers such that $E_0 \cdot E_1 < 0$ (say, $E_0 > 0$ and $E_1 < 0$).

Let $W_X = W(P_0, E_0, X_0, P_1, E_1, X_1)$ be the set of all functions f with domain R, the set of reals, and have range values in $X_0 + X_1$ such that (a) $e^{E_0 t} f \in L^{P_0}(X_0)$, (b) $e^{E_1 t} f \in L^{P_1}(X_1)$. Then W_X is Banach under the norm

$$\|f\|_{W_X} = \max \ [\|e^{E_0 t} f\|_{L^{P_0}(X_0)}, \ \|e^{E_1 t} f\|_{L^{P_1}(X_1)}] .$$

For each $f \in W_X$, $\int_{-\infty}^{\infty} f(t)\, dt$ converges in $X_0 + X_1$.

Let $S_X = S(P_0, E_0, X_0, P_1, E_1, X_1) = \{x \in X_0 + X_1 \colon x = \int_{-\infty}^{\infty} f(t)\, dt, \text{ where } f \in W_X\}$.

S_X is Banach under the norm

$$\|x\|_{S_X} = \inf_{f \in W_X} \{\|f\|_{W_X} \colon x = \int_{-\infty}^{\infty} f(t)\, dt\} .$$

In the special case where $X_0 = X_1$, we write $S_{XX} = S_X$, and $W_{XX} = W_X$.

Now, we define the Schechter [**21**]$_3$ interpolation space.

Let $\{X_0, X_1, \chi_0\}$ be a compatible triplet. $X_0 + X_1$, $X_0 \cap X_1$, and $H(X_0, X_1)$ are defined as for the Calderon interpolation space. Let T be a two-dimensional distribution with compact support in the open strip S, and have range values in $X_0 + X_1$. Then

$$(X_0, X_1)_T = \{T(f) : f \in H(X_0, X_1)\}$$

is a Banach space under the norm

$$\|x\|_{(X_0,X_1)_T} = \inf\{\|f\|_{H(X_0,X_1)} : f \in H(X_0, X_1),\ x = Tf\} .$$

As defined above, $(X_0, X_1)_s$, S_X, and $(X_0, X_1)_T$ are the Calderon, Lions-Peetre, and Schechter interpolation spaces respectively.

Definition $[\mathbf{23}]_1$. $X_{(T,C)}$ is a *generalized interpolation space* in X (relative to T and C), that is, C and X are Banach spaces and $T : C \to X$ is a bounded linear operator, and $X_{(T,C)} = \{T(f) : f \in C\}$ is Banach under the norm

$$\|x\|_{X_{(T,C)}} = \inf \|f\|_C , \quad [Tf = x,\ f \in C] .$$

The norm on $X_{(T,C)}$ is referred to as the generalized interpolation norm (or for brevity, interpolation norm).

By $[\mathbf{23}]_1$, the Calderon, Lions-Peetre, and Schechter interpolation spaces are all generalized interpolation spaces, that is, $(X_0, X_1)_s = (X_0 + X_1)_{(T_1, H(X_0,X_1))}$, $S_X = (X_0 + X_1)_{(T_2, W_X)}$, and $(X_0, X_1)_T = (X_0 + X_1)_{(T_3, H(X_0,X_1))}$, where $T_1 : H(X_0, X_1) \to X_0 + X_1$, $T_2 : W_X \to X_0 + X_1$, and $T_3 : H(X_0, X_1) \to X_0 + X_1$ are all bounded linear from one Banach space to another Banach space (see above remarks), and they are defined by $T_1(f) = f(s)$, $T_2(g) = \int_{-\infty}^{\infty} g(x)\,dx$, and $T_3(h) = T(h)$ for $f \in H(X_0, X_1)$, $g \in W_X$, and $h \in H(X_0, X_1)$.

Other notation is standard: if X and Y are normed linear spaces, then $B(X, Y)$ is the set of all bounded linear operators from X to Y. If $X = Y$, we write $B(X)$.

Throughout this paper, when we speak of a two-dimensional distribution with compact support in the open strip S contained in the complex plane which has values in $X_0 + X_1$ and $Y_0 + Y_1$, we assume that T is of the form

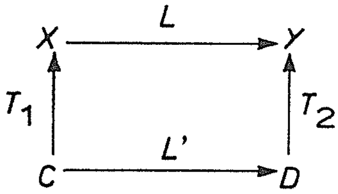
$$T = \sum_{k=1}^{n} \sum_{i=0}^{m_k} a_{k_i} \sigma^{(i)}(z_k) ,$$

where $z_1, z_2, \ldots, z_n$ are prescribed points in the open strip S and the a_{ki} are fixed complex constants.

The following theorem generalizes the generalized interpolation theorem for linear operators $[\mathbf{23}]_1$.

Theorem 1. (Weak generalized interpolation theorem for bounded nonlinear operators). *Let* ${}_wX_{(T_1,C)}$ *and* ${}_wY_{(T_2,D)}$ *be weak generalized interpolation spaces. If* $L'\colon C \to D$ *is bounded (not necessarily linear) such that* $LT_1 = T_2L'$ *then* $L\colon {}_wX_{(T_1,C)} \to {}_wY_{(T_2,D)}$ *is bounded, with* $\|L\| \leqslant \|L'\|$.

Proof of Theorem 1. We have diagram



$LT_1 = T_2L'$, thus $L\colon {}_wX_{(T_1,C)} \to {}_wY_{(T_2,D)}$.

By the definition of the interpolation norm on ${}_wX_{(T_1,C)}$, for every $x \in {}_wX_{(T_1,C)}$ and for every $\varepsilon > 0$, there exists $f \in C$ such that $T_1(f) = x$ and $\|f\|_C < \|x\|_{{}_wX_{(T_1,C)}} + \varepsilon$. Thus,

$$\|Lx\|_{{}_wY_{(T_2,D)}} = \|LT_1 f\|_{{}_wY_{(T_2,D)}} = \|T_2L'f\|_{{}_wY_{(T_2,D)}} \leqslant \|L'f\|_D \leqslant \|L'\| \cdot \|f\|_C$$

$$< \|L'\|\left(\|x\|_{{}_\omega X_{(T_1,C)}} + \varepsilon\right). \qquad \varepsilon > 0 \text{ is arbitrary, thus}$$

$$\|L_x\|_{{}_wX_{(T_2,D)}} \leqslant \|L'\| \cdot \|x\|_{{}_wX_{(T_1,C)}} \qquad \text{for every } x \in {}_wX_{(T_1,C)}.$$

Thus, L is bounded and $\|L\| \leqslant \|L'\|$, and Theorem 1 holds.

Remark. (1) It is clear that if X and Y are Banach spaces, and if L and L' are linear in Theorem 1, then we get the generalized interpolation theorem for linear operators in **[23]**$_1$.

We now have a definition.

Definition. Let X and Y be normed linear spaces, $L\colon X \to Y$ is *compact* (not necessarily linear), if for every bounded sequence $\{x_n\} \in X$, the sequence $\{Lx_n\}$ has a convergent subsequence in Y.

Note. X and Y are not necessarily Banach spaces, and we do not assume that L is continuous.

Clearly, if L is compact then it takes bounded sets in X to precompact (relatively compact) sets in Y.

Next, we give an example of a nonlinear compact operator which is also continuous.

This example was given by Roger Nussbaum in his « Nonlinear Functional Analysis » 511 course during the fall of 1973 at Rutgers University.

We consider the Urysohn-integral operator (one of its many variants).

Let $X =$ the Banach space of continuous real valued functions on $[0, 1]$, $C[0, 1]$ with the usual sup norm. Let $f\colon [0,1]\times[0, 1]\times R^1 \to R^1$ be a continuous map. Define: $F\colon X \to X$ by $(F_x)(s) = \int_0^1 f(s, t, x(t))\, dt$, then F is a nonlinear continuous compact operator.

The following theorem generalizes a generalized compactness proposition in [**9**].

Theorem 2. (Weak generalized compactness interpolation theorem for nonlinear operators). *Let* ${}_wX_{(T_1,C)}$ *and* ${}_wY_{(T_2,D)}$ *be weak generalized interpolation spaces. Let* $L\colon X \to Y$ *(not necessarily linear), and let* $L'\colon C \to D$ *(not necessarily linear) be compact and such that* $LT_1 = T_2L'$, *then* $L\colon {}_wX_{(T_1,C)} \to {}_wY_{(T_2,D)}$ *is compact.*

Proof. $LT_1 = T_2L'$, thus, $L\colon {}_wX_{(T_1,C)} \to {}_wY_{(T,D)}$. Let $\{x_n\} \in {}_wX_{(T_1,C)}$ be an arbitrary bounded sequence. We show $\{Lx_n\}$ has a convergent subsequence in ${}_wY_{(T_2,D)}$.

There exists $M \geqslant 0$ such that

$$\|x_n\|_{{}_wX_{(T_1,C)}} = \inf \|f_n\|_C \leqslant M \qquad \text{for } n = 1, 2, \ldots \qquad [T_1 f_n = x_n,\ f_n \in C]\,.$$

Claim 1. There exists a sequence $\{f_n\} \in C$ such that $T_1(f_n) = x_n$ for every positive integer n, and $\|f_n\|_C \leqslant M + 1$, for $n = 1, 2, \ldots$.

Proof of Claim 1. By a property of infimum, for $\varepsilon = 1 > 0$, and for every positive integer n, there exists $f_n \in C$ such that $T_1 f_n = x_n$, and

$$\|f_n\|_C \leqslant \|x_n\|_{{}_wX_{(T_1,C)}} + 1 \leqslant M + 1\,, \qquad \text{for } n = 1, 2, \ldots.$$

Thus, $\|f_n\|_C \leqslant M + 1$, for $n = 1, 2, \ldots$, and Claim 1 holds.

L' is compact, by hypothesis, thus, the sequence $\{L'f_n\}$ has a convergent subsequence, $\{L'f_v\}$, in D. Thus $\{L'f_v\}$ is Cauchy in D.

Claim 2. $\{Lx_v\} = \{LT_1f_v\}$ is Cauchy in ${}_wY_{(T_2,D)}$.

Proof. By construction,

$$\|Lx_v - Lx_{v'}\|_{{}_wY_{(T_2,D)}} = \|LT_1f_v - LT_1f_{v'}\|$$

$$= \|T_2L'f_v - T_2L'f_{v'}\|_{{}_wY_{(T_2,D)}} = \|T_2[L'f_v - L'f_{v'}]\|_{{}_wY_{(T_2,D)}}$$

$$\leqslant \|L'f_v - L'f_{v'}\|_D \to 0 \qquad \text{as } v, v' \to \infty$$

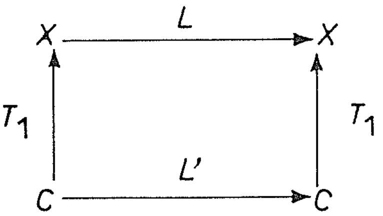
(by definition of ${}_wY_{(T_2,D)}$ norm), since $\{L'f_v\}$ is Cauchy in D.

Thus, $\{Lx_v\}$ is Cauchy in the Banach space ${}_wY_{(T_2,D)}$, and Claim 2 holds. Thus, Lx_v is a subsequence of Lx_n which converges in ${}_wY_{(T_2,D)}$. Thus, the theorem holds.

Next we consider fixed point theorems.

Proposition 3. (Nonlinear fixed point proposition for weak generalized interpolation spaces). *Let ${}_wX_{(T_1,C)}$ be a weak generalized interpolation space, where $C \neq \{0\}$. Suppose T_1 is 1-1, and $L: X \to X$ (not necessarily linear), and $L': C \to C$ (not necessarily linear) is such that $LT_1 = T_1L'$, and L' has a nonzero fixed point, then $L: {}_wX_{(T_1,C)} \to {}_wX_{(T_1,C)}$ has a nonzero fixed point.*

Proof. We have diagram



$LT_1 = T_1L'$, thus, $L: {}_wX_{(T_1,C)} \to {}_wX_{(T_1,C)}$.

By hypothesis, there exists $f \neq 0 \in C$ such that $L'(f) = f$. Now, $T_1L'f = T_1(L'f) = T_1(f) = x_1 \in {}_wX_{(T_1,C)}$, for some point $x_1 \in {}_wX_{(T_1,C)}$. Now, $x_1 \neq 0$ as an element of ${}_wX_{(T_1,C)}$, since

$$\|x_1\|_{{}_wX_{(T_1,C)}} = \inf \|f\|_C , \quad [T_1f = x_1,\ f \in C] \quad \text{(since } T_1 \text{ is 1-1)}$$

$$= \|f\|_C \neq 0 .$$

Thus, $L(x_1) = L(T_1 f) = LT_1 f = T_1 L' f = T_1(L' f) = T_1 f = x_1$, and, $x_1 \in {}_w X_{(T_1, C)}$ is the nonzero fixed point of L, and Proposition 3 holds.

A corollary of Proposition 3 is now stated with three parts which correspond to the Calderon, Lions-Peetre and Schechter spaces respectively.

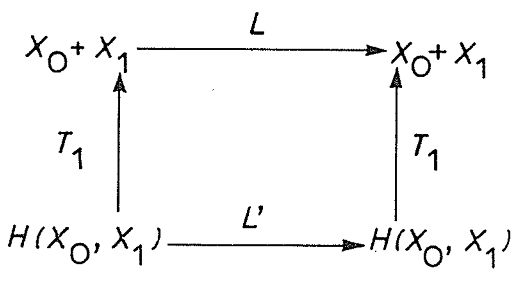
Corollary 3. *Let $\{X_0, X_1, \chi_0\}$ be a compatible triplet. Suppose $L: X_0 + X_1 \to X_0 + X_1$ is linear, and $L \in B(X_i, X_i)$, $i = 0, 1$.*

(1) *Let $0 \leqslant s \leqslant 1$ be fixed. Suppose $H(X_0, X_1) \neq \{0\}$, and $T_1: H(X_0, X_1) \to X_0 + X_1$ is 1-1, where T_1 is defined by $T_1(f) = f(s)$. Suppose $L': H(X_0, X_1) \to H(X_0, X_1)$ has a nonzero fixed point, where L' is defined by $L'(f) = L \circ f$, then $L: (X_0, X_1)_s \to (X_0, X_1)_s$ has a nonzero fixed point.*

(2) *Suppose $W_x \neq \{0\}$, and $T_1: W_x \to X_0 + X_1$ is 1-1, where T_1 is defined by $T_1(f) = \int_{-\infty}^{\infty} f(x)\,dx$. Suppose $L': W_X \to W_X$ has a nonzero fixed point, where L' is defined by $L'(f) = L \circ f$, then $L: S_X \to S_X$ has a nonzero fixed point.*

(3) *Let T be a two-dimensional distribution with compact support in the open strip $S \subset ¢$, and have range values in $X_0 + X_1$. Suppose $H(X_0, X_1) \neq \{0\}$, and $T_1: H(X_0, X_1) \to X_0 + X_1$ is 1-1, where T_1 is defined by $T_1(f) = T(f)$. Suppose $L': H(X_0, X_1) \to H(X_0, X_1)$ has a nonzero fixed point, where L' is defined by $L'(f) = L \circ f$, then $L: (X_0, X_1)_T \to (X_0, X_1)_T$ has a nonzero fixed point.*

Proof of Part 1. Consider diagram



$(X_0, X_1)_s = (X_0 + X_1)_{(T_1, H(X_0, X_1))}$ is a generalized interpolation space, and thus a weak generalized interpolation space. By V. Williams [**23**]$_1$, $LT_1 = T_1 L'$, thus, the hypothesis of Proposition 3 holds, and therefore, $L: (X_0, X_1)_s \to (X_0, X_1)_s$ has a nonzero fixed point, and Part 1 of Corollary 3 holds.
In a similar fashion, Parts 2 and 3 of Corollary 3 hold.
Finally, we consider nonexpansive maps.

Definition. Let (X, d) be a metric space, let Y be a nonempty subset of X, $f\colon Y \to X$, then f is *nonexpansive* if for every pair of elements $y_0, y_1 \in Y$, $d(f(y_0), f(y_1)) \leqslant d(y_0, y_1)$.

Note. In other words, nonexpansive maps are Lipschitzian with Lipschitz constant 1.

Contractive maps are nonexpansive. However, a nonexpansive map is not always a contractive map. Isometries are nonexpansive maps.

Consider C_0, the Banach space consisting of convergent sequences of scalars with limit 0. C_0 has the usual sup norm. $f\colon C_0 \to C_0$ defined by $f(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$ is a nonexpansive map.

The analogue of the Contraction Mapping Theorem (Principle) which says that every contraction on a complete metric space has a unique fixed point, does not hold for nonexpansive maps.

The map f above, is nonexpansive from C_0 to C_0, but f has no fixed point, since $f(x) = f((x_1, x_2, \ldots)) = (1, x_1, x_2, \ldots)$ would mean $x_1 = x_2 = \ldots = 1$, if f had a fixed point, but sequence $(1, 1, \ldots)$ is not in C_0.

If X is a normed linear space, and if $L\colon X \to X$ is bounded linear with $\|L\| \leqslant 1$, then L is nonexpansive.

It is also significant to note that all the bounded algebra homomorphisms listed by the author in [**9**], from one interpolation Banach algebra to another interpolation Banach algebra, have norm $\leqslant 1$, and thus are nonexpansive operators when the domain and range are the same.

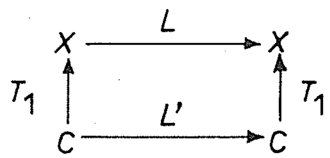
A nonexpansive proposition is now stated.

Proposition 4. (Nonexpansive proposition for weak generalized interpolation spaces). *Let* ${}_wX_{(T_1,C)}$ *be a weak generalized interpolation space. Let* $L\colon X \to X$ *(not necessarily linear), and let* $L'\colon C \to C$ *(not necessarily linear) be nonexpansive and such that* $LT_1 = T_1L'$.

(a) *If* T_1 *is* 1-1, *then* $L\colon {}_wX_{(T_1,C)} \to {}_wX_{(T_1,C)}$ *is nonexpansive.*

(b) *If* L *is linear and* $L'(0) = 0$, *then* $L\colon {}_wX_{(T_1,C)} \to {}_wX_{(T_1,C)}$ *is linear and nonexpansive.*

Proof of Part (a). We have diagram



$LT_1 = T_1L'$, thus $L: {}_wX_{(T_1,C)} \to {}_wX_{(T_1,C)}$. Let $D \subset {}_wX_{(T_1,C)}$, where D is any nonempty set, then we have to show that $\|Lx - Ly\|_{{}_wX_{(T_1,C)}} \leqslant \|x - y\|_{{}_wX_{(T_1,C)}}$ for every $x, y \in D$.

Let $\varepsilon > 0$ be given; for any $x, y \in D \subset {}_wX_{(T_1,C)}$, by the definition of ${}_wX_{(T_1,C)}$, there exists $f, g \in C$ such that $T_1f = x$, $T_1g = y$, and thus

$$T_1f - T_1g = T_1(f - g) = x - y .$$

Now, $(x - y) \in {}_wX_{(T_1,C)}$ so, there exists $h \in C$ such that $T_1(h) = (x - y)$, and $\|h\|_C \leqslant \|x - y\|_{{}_wX_{(T_1,C)}} + \varepsilon$. By hypothesis, T_1 is 1-1, thus, $h = f - g$. Now

$$\begin{aligned}
\|Lx - Ly\|_{{}_wX_{(T_1,C)}} &= \|LT_1f - LT_1g\|_{{}_wX_{(T_1,C)}} \\
&= \|T_1L'f - T_1L'g\|_{{}_wX_{(T_1,C)}} \\
&= \|T_1(L'f - L'g)\|_{{}_wX_{(T_1,C)}} \\
&\leqslant \|L'f - L'g\|_C \quad \text{(by definition of } {}_wX_{(T_1,C)} \text{ norm)} \\
&\leqslant \|f - g\|_C = \|h\|_C < \|x - y\|_{{}_wX_{(T_1,C)}} + \varepsilon \text{ (since } L' \text{ is nonexpansive)},
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Thus

$$\|Lx - Ly\|_{{}_wX_{(T_1,C)}} \leqslant \|x - y\|_{{}_wX_{(T_1,C)}} \quad \text{for } x, y \in D.$$

Thus, L is nonexpansive.

Proof of Part (b). Since the restriction of a linear map is linear, and $LT_1 = T_1L'$, then $L: {}_wX_{(T_1,C)} \to {}_wX_{(T_1,C)}$ is linear.

Now we show L is nonexpansive. Let $D \subset {}_wX_{(T_1,C)}$ be any nonempty set, and let $x, y \in D$ be arbitrary; $(x - y) \in {}_wX_{(T_1,C)}$, and thus, for every $\varepsilon > 0$, there exists $h \in C$ such that $T_1(h) = x - y$, and $\|h\|_C < \|x - y\|_{{}_wX_{(T_1,C)}} + \varepsilon$. Now

$$\begin{aligned}
\|Lx - Ly\|_{{}_wX_{(T_1,C)}} &= \|L(x - y)\|_{{}_wX_{(T_1,C)}} \\
&= \|LT_1h\|_{{}_wX_{(T_1,C)}} = \|T_1L'h\|_{{}_wX_{(T_1,C)}} \\
&\leqslant \|L'h\|_C \quad \text{(by definition of } {}_wX_{(T_1,C)} \text{ norm)} \\
&= \|L'h - 0\|_C = \|L'h - L'0\| \quad \text{(by hypothesis)} \\
&\leqslant \|h - 0\|_C \quad \text{(since } L' \text{ is nonexpansive)} \\
&= \|h\|_C < \|x - y\|_{{}_wX_{(T_1,C)}} + \varepsilon .
\end{aligned}$$

Thus, L is nonexpansive.

Remark. A nonexpansive corollary for Proposition 4 holds for the Calderon, Lions-Peetre, and Schechter interpolation spaces just as Corollary 3 holds for Proposition 3.

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Abstract

This paper deals with nonlinear interpolation, where all the operators involved are no. necessarily linear. We also consider the case where all spaces are not necessarily Banacht Boundedness and compactness are considered for nonlinear operators. This serves as a germ for the definition of a weak generalized interpolation space. Afterwhich, a theorem is stated which generalizes the interpolation theorem of V. Williams, and thus the interpolation theorems of A. P. Calderon, Lions-Peetre, and M. Schechter. It can be shown from the Ph. D. Dissertation of the author that the aforementioned theorem also generalizes the classical Riesz Interpolation Theorem, the Hausdorff-Young Theorem, and Young's Inequality. Finally, sufficient conditions are given for an operator on a weak generalized interpolation space to have a nonzero fixed point, and for the operator to be nonexpansive. There are fixed point and nonexpansive corollaries for the Calderon, Lions-Peetre and Schechter interpolation spaces.

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