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On the structure of classical kinematics (**)

Frames of reference - Observed kinematics

Frames of reference

Here we study the absolute kinematics of a continuum, which, viewed as a frame of reference, determines positions, the splitting of event space into space-time and the consequent splitting of velocity space. We analyse the positions space and its structures as the time-depending metric, the time-depending affine connection and the Coriolis map. Finally we make a classification of frames.

1 - Frames and the representation of E

1.1 - *Frames. positions and adapted charts.* The basic elements of observed kinematics are frames, constituted by a reference continuum, whose particles determine positions on $\boldsymbol{E}$.

For semplicity of notations, we consider only global frames, leaving to the reader the obvious generalization to local frames.

Definition. A *frame* is a couple $\mathscr{P} \equiv \{\boldsymbol{P}, \{\boldsymbol{T}_q\}_{q\in \boldsymbol{P}}\}$ where $\boldsymbol{P}$ is a set and, $\forall q \in \boldsymbol{P}$, $\boldsymbol{T}_q$ is a world line, such that

(a) $\boldsymbol{E} = \bigcup_{q\in \boldsymbol{P}} \boldsymbol{T}_q$ is the disjoint union of $\{\boldsymbol{T}_q\}_{q\in \boldsymbol{P}}$;

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(b) $\forall e \in \boldsymbol{E}$, there exists a neighbourhood U of e and a C^∞ chart $x \equiv \{x^0, x^i\}\colon U \to \boldsymbol{R}\times\boldsymbol{R}^3$ adapted to the family of submanifolds $\{\boldsymbol{T}_q\}_{q\in\boldsymbol{P}}$.

$\boldsymbol{P}$ is the *position space*; each $q \in \boldsymbol{P}$ is a *position*; $p\colon \boldsymbol{E} \to \boldsymbol{P}$, $e \mapsto$ the unique $q \in \boldsymbol{P}$, such that $e \in \boldsymbol{T}_q$, is the *position map*.

Henceforth we assume a frame $\mathscr{P}$ to be given.

1.2 - Calculations develop in an easier way if performed with respect to a chart adapted to $\mathscr{P}$. For simplicity of notations, we consider only global charts, leaving to the reader the obvious generalization to local charts.

Definition. *A chart adapted to* $\mathscr{P}$ is a special chart $\{x^0, x^i\}\colon \boldsymbol{E} \to \boldsymbol{R}\times\boldsymbol{R}^3$, such that $\{x^i\}\colon \boldsymbol{E} \to \boldsymbol{R}^3$ factorizes through $p\colon \boldsymbol{E} \to \boldsymbol{P}$, by $\{\underset{\wedge}{x}^i\}\colon \boldsymbol{P} \to \boldsymbol{R}^3$.

Charts adapted to $\mathscr{P}$ exist by Definition 1.

Hencefort we assume a chart x adapted to $\mathscr{P}$ to be given.

1.3 - *Representation of the position space* $\boldsymbol{P}$. $\boldsymbol{P}$ results naturally into a C^∞ manifold.

Proposition. There is a unique C^∞ structure on $\boldsymbol{P}$, such that the map $p\colon \boldsymbol{E} \to \boldsymbol{P}$ is C^∞. Namely it is induced by the charts adapted to $\{\boldsymbol{T}_q\}_{q\in\boldsymbol{P}}$.

1.4 - One gets a first immediate representation of $\boldsymbol{P}$.

The frame $\mathscr{P}$ determines a partition of $\boldsymbol{E}$ into the equivalence classes $\{\boldsymbol{T}_q\}_{q\in\boldsymbol{P}}$.

Then we get the natural identification of $\boldsymbol{P}$ with the quotient space $\boldsymbol{E}/\mathscr{P}$ given by $[e] = q = [e'] \Leftrightarrow p(e) = q = p(e')$.

1.5 - Choicing a time $\tau \in \boldsymbol{T}$ and taking, for each equivalence class, its representative, at the time τ, one gets a second interesting representation of $\boldsymbol{P}$.

For this purpose, let us introduce three maps related with $\mathscr{P}$, $\forall \tau, \tau' \in \boldsymbol{T}$,

$$P_\tau\colon \boldsymbol{P} \to \boldsymbol{S}_\tau, \qquad q \mapsto \text{the unique } e \in \boldsymbol{S}_\tau \cap \boldsymbol{T}_q;$$

$$p_\tau \equiv p_{|S_\tau}\colon \boldsymbol{S}_\tau \to \boldsymbol{P}; \qquad \tilde{P}_{(\tau',\tau)} \equiv P_{\tau'}\circ p_\tau\colon \boldsymbol{S}_\tau \to \boldsymbol{S}_{\tau'}.$$

1.6 - Then we see that $\boldsymbol{P}$ is diffeomorphic (not canonically) to a 3-dimensional affine space, by means of $\tilde{P}_\tau\colon \boldsymbol{P} \to \boldsymbol{S}_\tau$, $p_\tau\colon \boldsymbol{S}_\tau \to \boldsymbol{P}$. Moreover one has

$$\tilde{P}_{(\tau'',\tau')}\circ\tilde{P}_{(\tau',\tau)} = P_{(\tau'',\tau)}, \qquad \tilde{P}_{(\tau,\tau)} = \mathrm{id}_{S_\tau}$$

hence $\tilde{P}_{(\tau',\tau)}$ is a C^∞ diffeomorphism.

1.7 - *Frame motion.* We need a further map given by the motions associated to the world lines of $\mathscr{P}$.

Definition. The *motion* of $\mathscr{P}$ is the map $P: \boldsymbol{T}\times \boldsymbol{P} \to \boldsymbol{E}$, $(\tau, q) \mapsto$ the unique $e \in \boldsymbol{S}_\tau \cap \boldsymbol{T}_q$, or the map $\tilde{P} \equiv P\circ(\mathrm{id}_{\boldsymbol{T}}\times p): \boldsymbol{T}\times\boldsymbol{E} \to \boldsymbol{E}$, $(\tau, e) \mapsto P(\tau, p(e))$.

Thus P is the union of the family of maps $\{P_\tau\}_{\tau\in \boldsymbol{T}}$ previously introduced; on the other hand, P is the union of the family of maps $\{P_q\}_{q\in\boldsymbol{P}}$, constituted by the motions associated with the world-lines of $\mathscr{P}$. The motion P (or $\tilde{P}$ equivalently) characterizes the frame $\mathscr{P}$.

1.8 - The following immediate formulas will be used in calculations.

Proposition.

$$t(\tilde{P}(\tau, e)) = \tau\,, \quad \tilde{P}(t(e), e) = e\,, \quad \tilde{P}(\tau, \tilde{P}(\sigma, e)) = \tilde{P}(\tau, e)\,, \quad x^0\circ\tilde{P} = x^0, \quad x^i\circ\tilde{P} = x^i.$$

1.9 - *Rappresentation of* $\boldsymbol{E}$. The frame $\mathscr{P}$ determines the splitting of the event space into space-time.

Theorem *The maps* $(t, p): \boldsymbol{E} \to \boldsymbol{T}\times\boldsymbol{P}$ *and* $P: \boldsymbol{T}\times\boldsymbol{P} \to \boldsymbol{E}$ *are inverse* C^∞ *diffeomorphisms.*

Hence $(\boldsymbol{E}, p, \boldsymbol{P})$ results into a C^∞ bundle, with fiber $\boldsymbol{T}$.

Thus we have two bundle structures on $\boldsymbol{E}$, namely:

$\eta \equiv (\boldsymbol{E}, t, \boldsymbol{T})$, which has an absolute basis $\boldsymbol{T}$ and a non canonical fiber diffeomorfic to $\boldsymbol{P}$ or to $\boldsymbol{S}_\tau$, $\forall\tau \in \boldsymbol{T}$;

$\pi \equiv (\boldsymbol{E}, p, \boldsymbol{P})$, which has a frame depending basis $\boldsymbol{P}$, diffeomorfic to $\boldsymbol{S}_\tau$, $\forall \tau\in\boldsymbol{T}$, and an absolute fiber $\boldsymbol{T}$.

The frame bundle π characterizes the frame $\mathscr{P}$.

1.10 - *Physical description.* A frame $\mathscr{P}$ is a set $\boldsymbol{P}$ of particles, never meeting, filling, at each time $\tau\in\boldsymbol{T}$, the whole space $\boldsymbol{S}_\tau$, with a C^∞ flow. Hence first a frame is a continuum and we study the absolute kinematics of its particles.

Such a continuum can be viewed as a frame of reference. In fact it determines a partition of $\boldsymbol{E}$ in positions. Each position is the set of all events touched by the same frame particle. Under this aspect we can identify the set of positions with the set of particles $\boldsymbol{P}$.

2 - Frames and the representation of TE

In this section we are dealing with the first order derivatives of the frame and tangent spaces.

2.1 - *Frame velocity and jacobians.* The velocity of the frame is the vector field on $\boldsymbol{E}$ constituted by the velocities of the world-lines of the frame. Hence it is the first derivative of the notion with respect to time. On the orther hand, the jacobians are the first derivatives with respect to event. We consider only free entities, for simplicity of notations, leaving to the reader to write them in the complete form.

Definition. The *velocity* is $D_1\tilde{P}\colon \boldsymbol{T}\times\boldsymbol{E}\to\bar{\boldsymbol{E}}$ or $\bar{P}\equiv D_1\tilde{P}\circ j\colon \boldsymbol{E}\to\bar{\boldsymbol{E}}$.
The *Jacobian* is $D_2\tilde{P}\colon \boldsymbol{T}\times\boldsymbol{E}\to\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{E}}$ or $\hat{P}\equiv D_2\tilde{P}\circ j\colon \boldsymbol{E}\to\bar{\boldsymbol{E}}^*\times\bar{\boldsymbol{E}}$.
The *spatial Jacobian* is $\check{P}\equiv\check{D}_2\tilde{P}\colon \boldsymbol{T}\times\boldsymbol{E}\to\bar{\boldsymbol{S}}^*\otimes\bar{\boldsymbol{E}}$ or $\check{P}_{(\tau',\tau)}\equiv D\tilde{P}_{(\tau',\tau)}\colon \boldsymbol{S}_\tau\to\bar{\boldsymbol{S}}^*\otimes\bar{\boldsymbol{S}}$.
We will write $\check{u}_p\equiv\hat{P}(e)(u)$, $\forall x\in T_e\boldsymbol{E}$.

2.2 - We get immediate important properties of these maps.

Proposition. One has $\underline{t}\circ D_1\tilde{P}=1$, $\underline{t}\circ D_2\tilde{P}=0$.

Hence we can write

$$D_1\tilde{P}\colon \boldsymbol{T}\times\boldsymbol{E}\to\boldsymbol{U}\,,\qquad D_2\tilde{P}\colon \boldsymbol{T}\times\boldsymbol{E}\to\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{S}}\,,$$

$$\bar{P}\colon \boldsymbol{E}\to\boldsymbol{U}\,,\qquad \check{P}\colon \boldsymbol{E}\to\bar{\boldsymbol{S}}^*\otimes\bar{\boldsymbol{S}}\,,\qquad \hat{P}\colon \boldsymbol{E}\to\boldsymbol{E}^*\otimes\bar{\boldsymbol{S}}\,.$$

Moreover, all the previous maps are expressible by $\tilde{P}$, $\bar{P}$ and $\check{P}$

$$D_1\tilde{P}=\bar{P}\circ\tilde{P};\qquad \hat{P}=\mathrm{id}_{\bar{E}}-\underline{t}\otimes\bar{P}\,.$$

We have also the group properties

$$(\check{P}_{(\tau'',\tau')}\circ\check{P}_{(\tau',\tau)})\circ\check{P}_{(\tau',\tau)}=\check{P}_{(\tau'',\tau)}\,,\qquad \check{P}_{(\tau,\tau)}=\mathrm{id}_{\bar{S}}\,,$$

hence $\check{P}_{(\tau',\tau)}$ preserves the orientation of $\bar{\boldsymbol{S}}$, i.e. $\det\check{P}_{(\tau',\tau)}>0$. One has $\bar{P}=\delta x_0$. $\hat{P}=Dx^i\otimes\delta x_i$.

Proof. It follows by derivation of formulas (II.**1**.**8**).

2.3 - *Representation of* $\boldsymbol{TP}$. In order to get the space $\boldsymbol{TP}$ handy, it is useful to regard it as a quotient $\boldsymbol{TE}_{/\mathscr{P}}$.

Proposition. Let $v\in\boldsymbol{TP}$. Then $\boldsymbol{C}_v\equiv\check{T}p^{-1}(v)=(T_2P)_v(\boldsymbol{T})\hookrightarrow\check{T}\boldsymbol{E}$ is a C^∞ submanifold. Then we get a partition $\check{T}\boldsymbol{E}=\bigcup_{v\in TP}\boldsymbol{C}_v$, and a quotient space

$\check{T}\boldsymbol{E}_{/\mathscr{P}}$, which has a natural C^∞ structure and whose equivalence classes are characterized by

$$[e, u] = [e', u'] \Leftrightarrow \{p(e) = p(e'),\ \check{P}(t(e'), e)(u) = u'\}\ .$$

We get a natural C^∞ diffeomorphism between $T\boldsymbol{P}$ and $\check{T}\boldsymbol{E}_{/\mathscr{P}}$.

We will often make the identification $T\boldsymbol{P} = \check{T}\boldsymbol{E}_{/\mathscr{P}}$.

2.4 - Choicing a time $\tau \in \boldsymbol{T}$ and taking, for each equivalence class, its representative at the time τ, we get a second interesting representation of $T\boldsymbol{P}$, given by the inverse C^∞ diffeomorphism $T\tilde{P}_\tau$: $T\boldsymbol{P} \to T\boldsymbol{S}_\tau$, Tp_τ: $T\boldsymbol{S}_\tau \to T\boldsymbol{P}$.

2.5 - Taking into account the identification $T\boldsymbol{P} \cong T\boldsymbol{E}_{/\mathscr{P}}$, one has the following expression of Tp and TP

$$Tp(e, u) = [e, \hat{P}(e)(u)]\ ,$$

$$TP(\tau, \lambda; [e, u]) = \big(\tilde{P}(\tau, e),\ \lambda\overline{P}(\tilde{P}(\tau, e) + \check{P}(\tau, e)(u)\big)\ .$$

2.6 - *Frame vertical and horizontal spaces.* The bundle $\Pi \equiv (\boldsymbol{E}, p, \boldsymbol{P})$ induces two useful spaces.

Definition. The *frame vertical tangent spaces* is $\check{T}_{\mathscr{P}}\boldsymbol{E} \equiv \operatorname{Ker} Tp \hookrightarrow T\boldsymbol{E}$. The *frame horizontal tangent space*, or *frame phase space* is $\mathring{T}_{\mathscr{P}}\boldsymbol{E} \equiv T\boldsymbol{E}/\check{T}_{\mathscr{P}}\boldsymbol{E}$

2.7 - *Representation of* $T\boldsymbol{E}$.

Theorem. *The maps* $T\boldsymbol{E} \to \mathring{T}_{\mathscr{P}}\boldsymbol{E} \times_{\boldsymbol{E}} \mathring{T}\boldsymbol{E}$ *given by the natural projections*, $\check{T}\boldsymbol{E} \oplus_{\boldsymbol{E}} \check{T}_{\mathscr{P}}\boldsymbol{E} \to T\boldsymbol{E}$, *given by the natural inclusions, and*

$$T(t, p)\colon T\boldsymbol{E} \to T\boldsymbol{T} \times T\boldsymbol{P}\ , \qquad TP\colon T\boldsymbol{T} \times T\boldsymbol{P} \to T\boldsymbol{E}$$

are C^∞ *diffeomorphisms and* $T(t, p)$ *is the inverse of* TP.

Moreover one has the C^∞ diffeomorphisms

$$\mathring{T}_{\mathscr{P}}\boldsymbol{E} \to \boldsymbol{T} \times T\boldsymbol{P} \to \check{T}\boldsymbol{E} \qquad \text{and} \qquad \mathring{T}\boldsymbol{E} \to T\boldsymbol{T} \times \boldsymbol{P} \to \check{T}_{\mathscr{P}}\boldsymbol{E}\ .$$

Relation among the previous three representations of $T\boldsymbol{E}$ can be found in a natural way.

The maps $T\boldsymbol{E} \to \check{T}\boldsymbol{E} \to \boldsymbol{T} \times T\boldsymbol{P} \to \mathring{T}_{\mathscr{P}}\boldsymbol{E}$ are given by $(e, u) \mapsto (e, \hat{P}(e)(u)) \mapsto (t(e), [e, \hat{P}(e)(u)]) \mapsto [e, u]$.

The maps $TE \to \check{T}_{\mathscr{P}}E \to TT \times P \to \mathring{T}E$ are given by $(e, u) \mapsto (e, u^0\overline{P}(e)) \mapsto (t(e), u^0, [e]) \mapsto (e, u^0)$.

The choice of the most convenient representation depends on circumstances; we will generally use the identifications $TP \cong \check{T}E_{/\mathscr{P}}$, and $T \times TP \cong T_{\mathscr{P}}E$.

Let us remark that in the decomposition of the vector $x \in \overline{E}$

$$x = x^0 \overline{P}(e) + \check{x}_{\mathscr{P}}(e)$$

the component x^0 is absolute, but the space $\check{T}_{\mathscr{P}}E$ is frame depending, and the space $\check{T}E$ is absolute, but the component $\check{x}_{\mathscr{P}}$ is frame depending.

2.8 - *Frame metric function.* Wet get a «time depending» Riemannian structure on P, induced by the family of diffeomorphisms $TP \to TS_\tau$.

Definition. The *frame time depending metric function* is the function g_P: $T \times TP \to R$ given by the composition

$$T \times TP \to \check{T}E \xrightarrow{g} R\,, \qquad g_{\mathscr{P}}(\tau, [e, u]) \equiv \tfrac{1}{2}(\check{P}(\tau, e)(u))^2.$$

One has $g_{\mathscr{P}} = \frac{1}{2}\, g_{ij}\, \dot{x}^i \dot{x}^j$.

3 - Frames and the representation of T^2E

In this section we are dealing with the second order derivatives of the frame and tangent spaces.

3.1 - *Frame acceleration, second jacobians, strain and spin.* The acceleration of the frame is the vector field on E constituted by the acceleration of the world-lines of the frame. Hence it is the second derivative of the motion with respect to time. On the other hand, the second and mixed jacobians are the second derivatives with respect to event-event and time-event. We consider only free entities, for simplicity of notations, leaving to the reader to write them in the complete form.

Definition. The *acceleration* is $D_1^2\tilde{P}$: $T \times E \to \overline{E}$, or $\overline{\overline{P}} \equiv D_1^2\tilde{P} \circ j$: $E \to \overline{E}$.

The *second Jacobian* is $D_2^2\tilde{P}$: $T \times E \to \overline{E}^* \otimes \overline{E}^* \otimes \overline{E}$, or $\hat{\hat{P}} \equiv D_2^2\tilde{P} \circ j$: $E \to \overline{E}^* \otimes \overline{E}^* \otimes \overline{E}$.

The *spatial second Jacobian* is $\check{\check{P}} \equiv \check{D}_2^2\tilde{P}$: $T \times E \to \overline{S}^* \otimes \overline{S}^* \otimes \overline{S}$ or $\check{\check{P}}_{(\tau',\tau)} \equiv D^2 P_{(\tau',\tau)}$: $S_\tau \to \overline{S}^* \otimes \overline{S}^* \otimes \overline{S}$.

The *mixed second Jacobian* is $D_2 D_1 \tilde{P}\colon \boldsymbol{T}\times\boldsymbol{E}\to\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{E}}$ or $\hat{\bar{P}}\equiv D_2 D_1\tilde{P}\circ i\colon \boldsymbol{E}\to\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{E}}$.

The *mixed spatial second Jacobian* is $\check{\bar{P}}\equiv\check{D}_2 D_1\tilde{P}\circ j\colon \boldsymbol{E}\to\bar{\boldsymbol{S}}^*\otimes\bar{\boldsymbol{E}}$.

The *strain* is $\varepsilon_{\mathscr{P}}\equiv S\circ\check{\bar{P}}\colon \boldsymbol{E}\to\boldsymbol{S}\otimes\bar{\boldsymbol{S}}$.

The *spin* is $\omega_{\mathscr{P}}\equiv A/2\circ\check{\bar{P}}\colon \boldsymbol{E}\to\bar{\boldsymbol{S}}^*\otimes\bar{\boldsymbol{S}}$.

The *angular velocity* is $\Omega_P\equiv\check{*}A/2\circ\check{\bar{P}}\colon \boldsymbol{E}\to\bar{\boldsymbol{S}}$.

3.2 - We get immediate important properties of these maps.

Proposition. $0=\underline{t}\circ D_1^2\tilde{P}=\underline{t}\circ D_2^2\tilde{P}=\underline{t}\circ D_2 D_1\tilde{P}$, hence we can write

$$D_1^2\tilde{P}\colon \boldsymbol{T}\times E\to\bar{\boldsymbol{S}}\,,\qquad D_2^2\tilde{P}\colon \boldsymbol{T}\times\boldsymbol{E}\to\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{E}}^*\times\bar{\boldsymbol{S}}\,,\qquad D_2 D_1\tilde{P}\colon \boldsymbol{T}\times E\to\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{S}}\,,$$

$$\bar{\bar{P}}\colon \boldsymbol{E}\to\bar{\boldsymbol{S}}\,,\qquad \hat{\hat{P}}\colon \boldsymbol{E}\to\boldsymbol{E}^*\otimes\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{S}}\,,\qquad \hat{\bar{P}}\colon \boldsymbol{E}\to\bar{\boldsymbol{E}}^*\otimes\bar{\boldsymbol{S}}\,,$$

$$\check{\bar{P}}\colon \boldsymbol{E}\to\bar{\boldsymbol{S}}^*\otimes\bar{\boldsymbol{S}}\,.$$

Moreover all the previous maps are expressible by $\tilde{P}$, $\bar{P}$, DP and $\check{\check{P}}$

$$D_1^2\tilde{P}=\bar{\bar{P}}\circ\tilde{P}\,,$$

$$\hat{\hat{P}}=-\bar{\bar{P}}\otimes\underline{t}\otimes\underline{t}-(\check{D}\bar{P}\circ\hat{P})\otimes\underline{t}-t\otimes(\check{D}\bar{P}\circ\hat{P})\,,$$

$$\hat{\bar{P}}=\check{D}\bar{P}\circ\hat{P}\,,\qquad \bar{\bar{P}}=D\bar{P}(\bar{P})\,,\qquad \check{\bar{P}}=\check{D}\bar{P}\,,$$

$$(D_2^2\tilde{P})_{\tau'|S_\tau}=\check{\check{P}}_{(\tau',\tau)}\circ\hat{P}_{|S_\tau}\,,\qquad (D_1\check{P})\circ j=\check{D}\bar{P}\,.$$

If $u\equiv u^0\bar{P}(e)+\check{u}_{\mathscr{P}}(e)\in\bar{\boldsymbol{E}}\to\bar{\boldsymbol{E}}$, then $D\bar{P}(u)=u^0\,\bar{\bar{P}}+\frac{1}{2}\varepsilon_{\mathscr{P}}(\check{u}_{\mathscr{P}})+\Omega_{\mathscr{P}}\times\check{u}_{\mathscr{P}}$. One has $\varepsilon_{\mathscr{P}}=L_{\bar{P}}\,g$. One has

$$\bar{\bar{P}}=\Gamma^i_{00}\,\delta x_i\,,\qquad \hat{\bar{P}}=\Gamma^k_{i0}\,Dx^i\otimes\delta x_k\,,$$

$$\hat{\hat{P}}=-\Gamma^k_{00}\,Dx^0\otimes Dx^0-\Gamma^k_{i0}(Dx^i\otimes Dx^0+Dx^0\otimes Dx^i)\otimes\delta x_k\,,$$

$$\varepsilon_{\mathscr{P}} = (\Gamma_{j,0i} + \Gamma_{i,0j}) Dx_i \otimes Dx^j = \partial_0 g_{ij} Dx^i \otimes Dx^j, \quad \omega_{\mathscr{P}} = \tfrac{1}{2}(\Gamma_{j,0i} - \Gamma_{i,0j}) Dx^i \otimes Dx^j,$$

$$\Omega_{\mathscr{P}} = \tfrac{1}{2}\sqrt{\det(g^{ij})}\, \varepsilon^{kij} \Gamma_{j,0i}\, \delta x_k .$$

Proof. It follows by derivation of formulas (II.**1**.**8**).

3.3 - *Representation of $T^2\boldsymbol{P}$ and $vT^2\boldsymbol{P}$.* In order to get the space $T^2\boldsymbol{P}$ handy, it is useful to regard it as a quotient $\check{T}^2\boldsymbol{E}_{/\mathscr{P}}$.

Proposition. Let $v \in T^2\boldsymbol{P}$. Then $\boldsymbol{C}_v \equiv T^2 p^{-1}(v) = (T^2_2 P)_v(\boldsymbol{T}) \to T^2 \boldsymbol{E}$ is a C^∞ submanifold.

Then we get a partition $T^2\boldsymbol{E} = \bigcup_{v \in T^2\boldsymbol{P}} \boldsymbol{C}_v$, and a quotient space $\check{T}^2\boldsymbol{E}_{/\mathscr{P}}$, which has a natural C^∞ structure and whose equivalence classes are characterized by

$$[e, u, v, w) = [e', u', v', w'] \Leftrightarrow \{p(e) = p(e'),\ \check{P}(t(e'), e)(u) = u',$$
$$\check{P}(t(e'), e)(v) = v',\ \check{\check{P}}(t(e'), e)(u, v) + \check{P}(t(e'), e)(w) = w'\} .$$

We get a natural C^∞ diffeomorphism between $T^2\boldsymbol{P}$ and $\check{T}^2\boldsymbol{E}_{/\mathscr{P}}$.

3.4 - Choicing a time $\tau \in T$ and taking, for each equivalence class, its representative at the time τ, we get a second interesting representation of $T^2\boldsymbol{P}$, given by inverse C^∞ diffeomorphisms

$$T^2 \tilde{P}_\tau \colon T^2 P \to T^2\boldsymbol{S}_\tau , \qquad T^2 p_\tau \colon T^2\boldsymbol{S}_\tau \to T^2\boldsymbol{P} .$$

3.5 - The previous representations of $T^2\boldsymbol{P}$ reduce to analogous representation of $vT^2\boldsymbol{P}$.

3.6 - Taking into account the identification $T^2\boldsymbol{P} \cong \check{T}^2\boldsymbol{E}_{/\mathscr{P}}$, we get the following expression of T^2p:

$$T^2p(e, u, v, w) = [e, P(e)(u), \hat{P}(e)(v), \hat{\hat{P}}(e)(u, v) + \hat{P}(e)(w)] .$$

3.7 - *Frame connection and Coriolis map.* For each $\tau \in \boldsymbol{T}$, we can view $\boldsymbol{P}$ as an affine space, depending on τ, taking into account the isomorphism $\boldsymbol{T} \times T\boldsymbol{P} \to \check{T}\boldsymbol{E}$. Hence we get a « time depending » affine connection on $\boldsymbol{P}$.

Theorem. *There is a unique map $\check{\Gamma}_{\mathscr{P}} \colon \boldsymbol{T} \times sT^2\boldsymbol{P} \to vT^2\boldsymbol{P}$, such that $T^2p \circ \Gamma = \check{\Gamma}_{\mathscr{P}} \circ (t, T^2p) \colon T^2\boldsymbol{E} \to vT^2\boldsymbol{P}$, namely*

$$\check{\Gamma}_{\mathscr{P}} = T^2p \circ \Gamma \circ (T^2P)_{(0,0,0)}, \quad or \quad \check{\Gamma}_P(t(e), [e, u, u, w]) = [e, u, 0, w].$$

3.8 - Then we can introduce the following map, that will be used (III,1) to define the covariant derivative of maps $\boldsymbol{T} \to \boldsymbol{TP}$, hence the acceleration of observed motions.

Definition. The *frame time depending affine connection* is

$$\check{\Gamma}_P\colon \boldsymbol{T} \times sT^2\boldsymbol{P} \to vT^2\boldsymbol{P}\,.$$

3.9 - The time depending affine connection $\check{\Gamma}_{\mathscr{P}}$ does not sufficies for kinematics. Coriolis theorem (III,1) requires a further map, which is obtained taking into account the isomorphism $T \times \boldsymbol{TP} \to \overset{|}{T}\boldsymbol{E}$.

Theorem. *There is a unique map* $\overset{|}{\Gamma}_{\mathscr{P}}\colon \boldsymbol{T} \times sT^2\boldsymbol{P} \to vT^2\boldsymbol{P}$ *such that* $T^2p \circ \overset{|}{\Gamma} = \overset{|}{\Gamma}_{\mathscr{P}} \circ (t, T^2p)\colon \overset{|}{T}{}^2\boldsymbol{E} \to vT^2\boldsymbol{P}$ *namely*

$$\overset{|}{\Gamma}_{\mathscr{P}} = T^2\, p \circ \overset{|}{\Gamma} \circ (T^2P)_{(1,1,0)} \quad \text{or} \quad \overset{|}{\Gamma}_{\mathscr{P}}(t(e), [e, u, u, w]) = [e, u, o, w + 2\check{\overline{P}}(e) + \overline{\overline{P}}(e)].$$

Thus one has $\overset{|}{\Gamma}_{\mathscr{P}} = \check{\Gamma}_{\mathscr{P}} + \check{C}_{\mathscr{P}} + \check{\check{D}}_{\mathscr{P}}$, *where* $C_{\mathscr{P}}\colon \boldsymbol{T} \times \boldsymbol{TP} \to \boldsymbol{TP}$ *and* $D_{\mathscr{P}}\colon \boldsymbol{T} \times \boldsymbol{P} \to \boldsymbol{TP}$ *are given by* $C_{\mathscr{P}}(\tau, [e, u]) \equiv [\tilde{P}(\tau, e), 2\check{\check{\overline{P}}}(P(\tau, e))(u)]$ *and* $D_{\mathscr{P}}(\tau, e) \equiv [\tilde{P}(\tau, e), \overline{\overline{P}}(\tilde{P}(\tau, e))]$.

3.10 - Then we can give the following definition.

Definition. The *inertial connection* is $\overset{|}{\Gamma}_P\colon \boldsymbol{T} \times sT^2\boldsymbol{P} \to vT^2\boldsymbol{P}$.

The *frame Coriolis map* is $C_{\mathscr{P}}\colon \boldsymbol{T} \times \boldsymbol{TP} \to \boldsymbol{TP}$.

The *frame dragging map* is $D_{\mathscr{P}}\colon \boldsymbol{T} \times \boldsymbol{P} \to \boldsymbol{TP}$.

4 - Special frames

A classification of the most important types of frames can be performed taking into account the vanishing of quantities occurring, in $D\overline{P}$. So we get a chain of four types, characterized by a more and more rich structure of the position space $\boldsymbol{P}$.

4.1 - *Affine frames.*

Definition. The frame $\mathscr{P}$ is *affine* if $\check{D}^2\overline{P} = 0$.

4.2 - We have interesting characterizations of affine frames.

Proposition. The following conditions are equivalent.

(a) $\mathscr{P}$ is affine. (b) $\check{D}\overline{P}$ depends only on time. (c) $\check{\check{P}} = 0$. (d) $\check{P}$ depends only on time. (e) If $t(e) = t(e')$, then $\tilde{P}_{e'}(\tau) = \tilde{P}_e(\tau) + \check{P}_{(\tau,\sigma)}(e' - e)$. (f) If $t(e) = t(e')$, then $\overline{P}(e') = \overline{P}(e) + \frac{1}{2}\varepsilon_{\mathscr{P}}(\tau)(e' - e) + \Omega_{\mathscr{P}}(\tau) \times (e' - e)$.

Hence the motion of an affine frame $\mathscr{P}$ is characterized by the motion of one of its particles P_q: $\boldsymbol{T} \to \boldsymbol{E}$ and by $\varepsilon_{\mathscr{P}}$: $\boldsymbol{T} \to \bar{\boldsymbol{S}}^* \otimes \bar{\boldsymbol{S}}$, $\bar{\Omega}$: $\boldsymbol{T} \to \bar{\boldsymbol{S}}$.

4.3 - Let $\mathscr{P}$ be affine. Since $\check{P}$ depends only on time, we can get a reduction of the representation of $\boldsymbol{TP}$ by $\check{T}\boldsymbol{E}_{/\mathscr{P}}$, writing

$$(\boldsymbol{E} \times \bar{\boldsymbol{S}})_{/\mathscr{P}} \cong (\boldsymbol{P} \times \boldsymbol{T} \times \bar{\boldsymbol{S}})_{/\mathscr{P}} = \boldsymbol{P} \times (\boldsymbol{T} \times \bar{\boldsymbol{S}})_{/\mathscr{P}} .$$

Theorem. (a) *Let* $\overline{\boldsymbol{P}} \equiv (\boldsymbol{T} \times \bar{\boldsymbol{S}})_{/\mathscr{P}}$, *be the quotient space given by*

$$[\tau, u] = [\tau', u'] \Leftrightarrow u' = \check{P}_{(\tau',\tau)}(u) .$$

Then $\overline{\boldsymbol{P}}$ *results into a vector space, putting*

$$\lambda[\tau, u] \equiv [\tau, \lambda u] , \qquad [\tau, u] + [\tau', u'] \equiv [\tau, u + \check{P}_{(\tau,\tau')}(u)] .$$

For each $\tau \in \boldsymbol{T}$, *the map* $\overline{\boldsymbol{P}} \to \bar{\boldsymbol{S}}$, $[\tau', u] \mapsto \check{P}_{(\tau,\tau')}(u)$, *is an isomorphism.*

(b) *Let* $\sigma_{\mathscr{P}}$: $\boldsymbol{P} \times \overline{\boldsymbol{P}} \to \boldsymbol{P}$, *be the map, given by* $(q, [\tau, u]) \mapsto p(P(\tau, q) + u)$. *Then the triple* $(\boldsymbol{P}, \overline{\boldsymbol{P}}, \sigma_{\mathscr{P}})$ *is a three dimensional affine space.*

(c) *For each* $\tau \in \boldsymbol{T}$, *the maps* p_τ: $\boldsymbol{S}_\tau \to \boldsymbol{P}$ *and* P_τ: $\boldsymbol{P} \to \boldsymbol{S}_\tau$ *are affine isomorphisms.*

(d) $\check{\Gamma}_{\mathscr{P}}$ *results to be time independent and it is the affine connection of* $\boldsymbol{P}$

$$\check{\Gamma}_{\mathscr{P}}: sT^2\boldsymbol{P} \to \nu T^2\boldsymbol{P} , \qquad (q, [\tau, u], [\tau, u], [\tau, w]) \mapsto (q, [\tau, u], 0, [\tau, w]) .$$

4.4 - *Rigid frames.*

Definition. The frame $\mathscr{P}$ is *rigid if it is affine and* $\varepsilon_{\mathscr{P}} = 0$.

4.5 - We have interesting characterizations of rigid frames.

Proposition. The following conditions are equivalent.

(a) $\mathscr{P}$ is rigid.

(b) If $t(e) = t(e')$, then $\|\tilde{P}_{(\tau,\sigma)}(e) - \tilde{P}_{(\tau,\sigma)}(e')\| = \|e - e'\|$.

(c) If $t(e) = t(e')$, then $\overline{P}(e') = \overline{P}(e) + \Omega_{\mathscr{P}}(\sigma) \times (e' - e)$.

(d) $\check{\check{P}} = 0$ and $\check{P}$: $\boldsymbol{T} \times \boldsymbol{E} \to SU(\bar{\mathbf{S}})$.

Hence the motion of a rigid frame $\mathscr{P}$ is characterized by the motion of one of its particles P_q: $\boldsymbol{T} \to \boldsymbol{E}$ and by $\Omega_{\mathscr{P}}$: $\boldsymbol{T} \to \bar{\mathbf{S}}$.

4.6 - Let $\mathscr{P}$ be rigid.

Theorem. ***P*** *results into an affine euclidean space. In fact* $g_{\mathscr{P}}$ *results to be time independent and we can define the map* $g_{\mathscr{P}}$: $\overline{\mathbf{P}} \to \boldsymbol{R}$, $[\tau, u] \to \frac{1}{2} u^2$.
The affine connection $\check{\Gamma}_{\mathscr{P}}$ *results into the Riemannian connection of* $\mathscr{P}$.

4.7 - *Translating frames.*

Definition. A frame $\mathscr{P}$ is *translating* if it is rigid and $\Omega_{\mathscr{P}} = 0$.

4.8 - We have interesting characterizations of translating frames.

Proposition. The following conditions are equivalent.

(a) $\mathscr{P}$ is translating.

(b) If $t(e) = t(e')$, then $\tilde{P}_{e'}(\tau) = \tilde{P}_e(\tau) + (e' - e)$.

(c) If $t(e) = t(e')$, then $\overline{P}(e') = \overline{P}(e)$.

Hence the motion of a translating frame is characterized by the motion of one of its particles P_q: $\boldsymbol{T} \to \boldsymbol{E}$.

4.9 - Let $\mathscr{P}$ be translating. Since $\check{\boldsymbol{P}} = \mathrm{id}_{\bar{\mathbf{S}}}$, we can get a further reduction of the representation of $\boldsymbol{TP}$ by $\check{T}\boldsymbol{E}_{/\mathscr{P}}$, writing $(\boldsymbol{E} \times \bar{\mathbf{S}})_{/\mathscr{P}} \cong (\boldsymbol{P} \times \boldsymbol{T} \times \bar{\mathbf{S}})_{/\mathscr{P}} = \boldsymbol{P} \times \bar{\mathbf{S}}$.

Theorem. *Let* $\mathscr{P}$ *be translating.*

(a) *The map* $\boldsymbol{P} \to \bar{\mathbf{S}}$, $[\tau, u] \mapsto u$ *is well defined and it is an isomorfism. Then* *the map* $\sigma_{\mathscr{P}}: \boldsymbol{P}\times\overline{\boldsymbol{S}} \to \boldsymbol{P}$, $(q, u) \to p(P(\tau, q) + u)$, *does not depend on the choice of* $\tau \in \boldsymbol{T}$.

(b) *The triple* $(\boldsymbol{P}, \overline{\boldsymbol{S}}, \sigma_{\mathscr{P}})$ *is an affine euclidean space.*

4.10 - *Inertial frames.*

Definition. A frame $\mathscr{P}$ is *inertial* if it is translating and $\overline{\overline{P}} = 0$.

4.11 - Proposition. The following conditions are equivalent.

(a) $\mathscr{P}$ is inertial,

(b) $\mathscr{P}$ is translating and $D\overline{P} = 0$,

(c) $\tilde{P}(\tau, e) = e + \overline{P}(\tau - t(e))$, with $\overline{P} \in \boldsymbol{U}$,

(d) $\overline{P}: \boldsymbol{E} \to \boldsymbol{U}$ is a constant map.

Hence an inertial frame is characterized by its constant velocity.

4.12 - Proposition. $\overset{\scriptscriptstyle|}{\Gamma}_P$ results time independent and $\overset{\scriptscriptstyle|}{\Gamma}_{\mathscr{P}} = \check{\Gamma}_{\mathscr{P}}$.

Observed Kinematics

Here we analyse the one-body kinematics in terms of the positions determined by a frame, introducing the observed motion and its velocity and acceleration. By comparison between the absolute and the observed motion we get the « absolute » velocity addition and Coriolis theorem. Finally we make the comparison between the observed motions relative to two frames, getting the velocity addition and Coriolis theorem.

1 - Observed kinematics

Let $\mathscr{P}$ a fixed frame and let M be a fixed motion. We analyse M as viewed by $\mathscr{P}$.

1.1 - We first introduce useful notations. Let $f: \boldsymbol{T} \to \boldsymbol{P}$ be a C^∞ map.

(a) We put $\check{f} \equiv (id_{\boldsymbol{T}}, f): \boldsymbol{T} \to \boldsymbol{T}\times\boldsymbol{P}$, $\check{d}f \equiv (id_{\boldsymbol{T}}, df): \boldsymbol{T} \to \boldsymbol{T}\times T\boldsymbol{P}$, $\check{d}^2f \equiv (id_{\boldsymbol{T}}, d^2f): \boldsymbol{T} \to \boldsymbol{T}\times T^2\boldsymbol{P}$.

(b) df and d^2f being functions on $\boldsymbol{T}$, we can choose a natural representative of the equivalence classes of $T\boldsymbol{P}$ and $T^2\boldsymbol{P}$. So we write

$$df \equiv [f, D_{\mathscr{P}} f], \qquad d^2f \equiv [f, D_{\mathscr{P}}f, D_{\mathscr{P}} f, D^2_{\mathscr{P}} f],$$

where

$$D_{\mathscr{P}} f\colon \boldsymbol{T} \to \bar{\boldsymbol{S}} \qquad \text{and} \qquad D^2_{\mathscr{P}} f\colon \boldsymbol{T} \to \bar{\boldsymbol{S}}\,.$$

(c) We put $\check{\nabla}_{\mathscr{P}} df \equiv \coprod_{\mathscr{P}} \circ \check{\Gamma}_{\mathscr{P}} \circ \check{d}f\colon \boldsymbol{T} \to T\boldsymbol{P}$, $\overset{\shortmid}{\nabla}_{\mathscr{P}} df \equiv \coprod_{\mathscr{P}} \circ \overset{\shortmid}{\Gamma}_{\mathscr{P}} \circ \check{d}^2 f\colon \boldsymbol{T} \to T\boldsymbol{P}$

1.2 - *Observed motion and absolute velocity addition and Coriolis theorem.*

The basic definition of observed kinematics is the following.

Definition. The *motion of M observed by $\mathscr{P}$* is $M_{\mathscr{P}} \equiv p \circ M\colon \boldsymbol{T} \to \boldsymbol{P}$.

The *velocity of M observed by $\mathscr{P}$* is $(dM)_{\mathscr{P}} \equiv Tp \circ dM\colon \boldsymbol{T} \to T\boldsymbol{P}$.
The *velocity of the observed motion* $M_{\mathscr{P}}$ is $dM_{\mathscr{P}}\colon \boldsymbol{T} \to T\boldsymbol{P}$.
The *acceleration of M observed by $\mathscr{P}$* is $(\nabla dM)_{\mathscr{P}} \equiv Tp \circ \nabla dM\colon \boldsymbol{T} \to T\boldsymbol{P}$.

The *acceleration of the observed motion* M_P is $\check{\nabla}_{\mathscr{P}} \check{d}M_{\mathscr{P}} \equiv \coprod_{\mathscr{P}} \circ \check{\Gamma}_{\mathscr{P}} \circ \check{d}^2 M_{\mathscr{P}}$: $\boldsymbol{T} \to T\boldsymbol{P}$.

1.3 - We can make the comparison between the observed entities and the entities of the observed motion. One gets simplified formulas by means of the identification $\boldsymbol{E} \cong \boldsymbol{T} \times \boldsymbol{P}$.

Theorem. *Absolute velocity addition and Coriolis theorem. One has*

$$M \cong \check{M}_{\mathscr{P}}\,, \qquad DM - \overline{P} \circ M = D_{\mathscr{P}} M_{\mathscr{P}}\,,$$

$$D^2 M = D^2_{\mathscr{P}} M_{\mathscr{P}} + (\varepsilon_{\mathscr{P}} \circ \check{M}_{\mathscr{P}})(D_{\mathscr{P}} M_{\mathscr{P}}) + 2(\Omega_{\mathscr{P}} \circ \check{M}_{\mathscr{P}}) \times D_{\mathscr{P}} M_{\mathscr{P}} + \overline{\overline{P}} \circ \check{M}_{\mathscr{P}}\,.$$

and

$$(dM)_{\mathscr{P}} = \check{d}M_{\mathscr{P}} = [M, D_{\mathscr{P}} M_{\mathscr{P}}]\,,$$

$$(\nabla\, dM)_{\mathscr{P}} = \overset{\shortmid}{\nabla}_{\mathscr{P}} \check{d}M_{\mathscr{P}} = \check{\nabla}_{\mathscr{P}} \check{d}M_{\mathscr{P}} + C_{\mathscr{P}} \circ \check{d}M_{\mathscr{P}} + D_{\mathscr{P}} \circ \check{M}_{\mathscr{P}}\,,$$

$$\check{\nabla}_{\mathscr{P}} \check{d}M^2_{\mathscr{P}} = [M, D^2_{\mathscr{P}} M_{\mathscr{P}}]\,.$$

Moreover

$$x^k \circ M_{\mathscr{P}} = M^k \equiv x^k \circ M\,, \qquad \underset{\frown}{\dot{x}}{}^k \circ dM_{\mathscr{P}} = DM^k,$$

$$\underset{\frown}{\dot{x}}{}^k \circ \check{\nabla}_{\mathscr{P}} \check{d}M_{\mathscr{P}} = D^2 M^k + (\Gamma^k_{ij} \circ \check{M}_{\mathscr{P}})\, DM^i DM^j,$$

$$\underset{\frown}{\dot{x}}{}^k \circ \overset{\shortmid}{\nabla}_{\mathscr{P}} \check{d}M_{\mathscr{P}} = D^2 M^k + (\Gamma^k_{ij} \circ \check{M}_{\mathscr{P}})\, DM^i DM^j + (2\Gamma^k_{0j} \circ \check{M}_{\mathscr{P}})\, DM^j + \Gamma^k_{00} \circ \check{M}_{\mathscr{P}}\,.$$

1.4 - Corollary.

If $\mathscr{P}$ *is affine, then* $D^2 M = D^2_{\mathscr{P}} M_{\mathscr{P}} + \varepsilon_{\mathscr{P}}(D_{\mathscr{P}} M_{\mathscr{P}}) + 2\Omega_{\mathscr{P}} \times D_{\mathscr{P}} M_{\mathscr{P}} + \overline{\overline{P}} \circ \check{M}_{\mathscr{P}}$.

If $\mathscr{P}$ *is rigid, then* $D^2 M = D^2_{\mathscr{P}} M_{\mathscr{P}} + 2\Omega_{\mathscr{P}} \times D_{\mathscr{P}} M_{\mathscr{P}} + \overline{\overline{P}} \circ \check{M}_{\mathscr{P}}$.

If $\mathscr{P}$ *is translating, then* $D^2 M = D^2_{\mathscr{P}} M_{\mathscr{P}} + \overline{\overline{P}}$.

If $\mathscr{P}$ *is inertial, then* $D^2 M = D^2_{\mathscr{P}} M_{\mathscr{P}}$.

2 - Relative kinematics

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two fixed frames and let the subfixes « 1 » and « 2 » denote quantities relative to $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. Let M be a fixed motion. We make a comparison between the kinematics observed by $\mathscr{P}_1$ and $\mathscr{P}_2$.

2.1 - *Motion of a frame observed by a frame.* If we consider $\mathscr{P}_1$ as a set of world-lines and $\mathscr{P}_2$ as observing $\mathscr{P}_1$, we are led naturally to the following definition by (III,**1**,**2**).

We consider only free velocity and acceleration for simplicity of notations, leaving to the reader to write them in the complete form.

Here $D_{1\mathscr{P}_2}$ and $D^2_{1\mathscr{P}_2}$ are the derivative in the sence of (III,**1**,**1**,b) with respect to $\mathscr{P}_2$ and the suffix 1 denote partial derivative with respect to the first variable, i.e. the time.

Definition. The *motion of* $\mathscr{P}_1$ *observed by* $\mathscr{P}_2$ is $\tilde{P}_{12} \equiv p \circ \tilde{P}_1 \colon \boldsymbol{T} \times \boldsymbol{E} \to \boldsymbol{P}_2$.

The *mutual motion of* $(\mathscr{P}_1, \mathscr{P}_2)$ is $\tilde{P}_{(1,2)} \equiv \tilde{P}_1 - \tilde{P}_2 \colon \boldsymbol{T} \times \boldsymbol{E} \to \bar{\boldsymbol{S}}$.

The *velocity of the observed motion* $\tilde{P}_{12}$ is $\overline{P}_{12} \equiv (D_{1\mathscr{P}_2} \tilde{P}_{12}) \circ j \colon \boldsymbol{E} \to \bar{\boldsymbol{S}}$.

The *velocity of* $\mathscr{P}_1$ *observed by* $\mathscr{P}_2$ is $\overline{P}_{12} \equiv \hat{P}_2 \circ \overline{P}_1 \colon \boldsymbol{E} \to \bar{\boldsymbol{S}}$.

The *velocity of the mutual motion* $\tilde{P}_{(1,2)}$ is $\overline{P}_{(1,2)} \equiv D_1 \tilde{P}_{(1,2)} \circ j \colon \boldsymbol{E} \to \bar{\boldsymbol{S}}$.

The *acceleration of the observed motion* $\tilde{P}_{12}$ is $\overline{\overline{P}}_{12} \equiv (D^2_{1\mathscr{P}_2} \tilde{P}_{12}) \circ j \colon \boldsymbol{E} \to \bar{\boldsymbol{S}}$.

The *acceleration of* $\mathscr{P}_1$ *observed by* $\mathscr{P}_2$ is $\overline{\overline{P}}_{1,2} \equiv \hat{P}_2 \circ \overline{\overline{P}}_1 \colon \boldsymbol{E} \to \bar{\boldsymbol{S}}$.

The *acceleration of the mutual motion* $\tilde{P}_{(1,2)}$ is $\overline{\overline{P}}_{(1,2)} = D^2_1 \tilde{P}_{(1,2)} \circ j \colon \boldsymbol{E} \to \bar{\boldsymbol{S}}$.

The *strain of the observed motion* $\tilde{P}_{12}$ is $\varepsilon_{12} = S \check{D} \overline{P}_{12} \colon \boldsymbol{E} \to \bar{\boldsymbol{S}}^* \otimes \bar{\boldsymbol{S}}$.

The *spin of the observed motion* $\tilde{P}_{12}$ is $\omega_{12} = (A/2) \check{D} \overline{P}_{12} \colon \boldsymbol{E} \to \boldsymbol{S}^* \otimes \bar{\boldsymbol{S}}$.

The *angular velocity of the observed motion* $\tilde{P}_{12}$ is $\Omega_{12} = \check{*} (A/2) \check{D} \overline{P}_{12}$.

2.2 - We can make the comparison between the observed entities and the entities of the observed motion, as shown by (III,1,3).

Proposition.

$$\tilde{P}_1 = (\pi_1, \tilde{P}_{12}) \colon \boldsymbol{T}\times\boldsymbol{E} \to \boldsymbol{T}\times\boldsymbol{P}_2 \simeq \boldsymbol{E}\,,$$

$$\overline{P}_{(1,2)} = \overline{P}_{1,2} = \overline{P}_1 - \overline{P}_2 = \overline{P}_{12}\,, \qquad \overline{\overline{P}}_{1,2} = \overline{\overline{P}}_1 = \overline{\overline{P}}_{12} + \varepsilon_{\mathscr{P}_2}(\overline{P}_{12}) + 2\Omega_{\mathscr{P}_2}\times\overline{P}_{12} + \overline{\overline{P}}_2\,,$$

$$\overline{\overline{P}}_{(1,2)} = \overline{\overline{P}}_1 - \overline{\overline{P}}_2\,, \qquad \varepsilon_{12} = \varepsilon_1 - \varepsilon_2\,, \qquad \omega_{12} = \omega_1 - \omega_2\,, \qquad \Omega_{12} = \Omega_1 - \Omega_2\,.$$

2.3 - Immediate comparison between the quantities « 12 » and « 21 » is obtained.

Corollary.

$$\tilde{P}_{(1,2)} = -\tilde{P}_{(2,1)}\,, \qquad \overline{P}_{(1,2)} = -\overline{P}_{(2,1)}\,, \qquad \overline{\overline{P}}_{(1,2)} = -\overline{\overline{P}}_{(2,1)}\,,$$

$$\varepsilon_{12} = -\varepsilon_{21}\,, \qquad \omega_{12} = -\omega_{21}\,, \qquad \Omega_{12} = -\Omega_{21}\,, \qquad \overline{P}_{11} = \varepsilon_{11} = \omega_{11} = \Omega_{11} = 0\,.$$

2.4 - One has time depending diffeomorphism between spaces concerning $\mathscr{P}_1$ and $\mathscr{P}_2$.

Proposition. Let $\tau\in T$. The maps

$$p_{12\tau} \equiv p_2\circ P_{1_\tau} \colon \boldsymbol{P}_1 \to \boldsymbol{P}_2\,, \qquad [e]_1 \to [p_1(\tau, e)]_2\,,$$

and

$$Tp_{12\tau} \colon \boldsymbol{T}P_1 \to \boldsymbol{T}P_2\,, \qquad [e, u]_1 \to [\tilde{P}_1(\tau, e), \check{P}_1(\tau, e)(u)]_2\,,$$

are C^∞ diffeomorphisms.

2.5 - *Velocity addition and generalized Coriolis theorems.* As conclusion, we get the comparison between velocity and acceleration of the motion M observed by $\mathscr{P}_1$ and $\mathscr{P}_2$.

Theorem. *Velocity addition and generalized Coriolis theorems.*

(a) $M_{\mathscr{P}_2} = p_2\circ M_{\mathscr{P}_1}$.

(b) $D_{\mathscr{P}_2} M_{\mathscr{P}_2} = D_{\mathscr{P}_1} M_{\mathscr{P}_1} + \overline{P}_{12}\circ M$.

(c) $D^2_{\mathscr{P}_2} M_{\mathscr{P}_2} = D^2_{\mathscr{P}_1} M_{\mathscr{P}_1} + \varepsilon_{12}\circ M(D_{\mathscr{P}_1} M_{\mathscr{P}_1}) + 2\Omega_{12}\circ M\times D_{\mathscr{P}_1} M_{\mathscr{P}_1} + \overline{\overline{P}}_{12}\circ M$.

If $\mathscr{P}_2$ is inertial and $\mathscr{P}_1$ is rigid, then one obtains the usual formulas.