Riv. Mat. Univ. Parma (4) 6 (1980), 211-215

J. ACHARI (*)

On fixed points of operators (**)

1 – In recent years many extensions and generalizations of Banach's fixed point theorem had been done by many authors. But in all the cases the mapping under consideration contains only two points of the space. Until recently Pittnauer [3]$_1$ and also Rhoades [4] studied contractive type mappings involving three points of the space. Pittnauer [3]$_2$ also studied contractive type mappings involving four points of the space.

The aim of this paper is to establish a fixed point theorem for contractive type mapping involving six points of the space. We have then extended the result to family of mappings. Finally we have shown that our result contains as special cases that of Hardy and Rogers [1], Reich [5] and Kannar [2].

2 – Let (X, d) be a complete metric space. Let $\psi_i\colon \overline{P} \to [0, \infty)$ $(i = 1, 2, \ldots, 5)$ (P is the range of d and $\overline{P}$ is the closure of P) be upper semi-continuous functions from the right on $\overline{P}$ and satisfy the condition

$$\psi_i(t) < t/5 \qquad \text{for } t > 0 \quad \text{and} \quad \psi_i(0) = 0 \quad (i = 1, 2, 3, 4, 5)\,. \tag{1}$$

Also let f be a mapping of X into itself such that $(u_1, u_2, u_3, u_4, u_5, u_6) \in X$.

$$\begin{aligned} d(fu_1, fu_2) \leqslant \psi_1[d(u_1, u_2)] &+ \psi_2[d(u_1, fu_3)] \\ &+ \psi_3[d(u_2, fu_4)] + \psi_4[d(u_5, fu_1)] + \psi_5[d(u_6, fu_2)]\,, \end{aligned} \tag{2}$$

(*) Indirizzo: Munshifdanga, P. O. Raghunathpur (Pin: 723133), Dist. Purulia (W.B.), India.

(**) This work was carried out under the C.N.R. (Italy) fellowship. — Ricevuto: 23-XI-1978.

Theorem 1. *If f be a mapping of X into itself satisfying* (2), *then f has a unique fixed point.*

Proof. Let $x, y \in X$ and we define

$$u_1 = fx\,, \quad u_2 = fy\,, \quad u_3 = y\,, \quad u_4 = x\,, \quad u_5 = f^2x\,, \quad u_6 = f^2y\,.$$

Then the inequality (2) takes the form

$$d(f^2x, f^2y) \leqslant \psi_1[d(fx, fy)] + \psi_2[d(fx, fy)] + \psi_3[d(fx, fy)]\,. \tag{3}$$

Let $x_0 \in X$ be an arbitrary point. We shall show that the iterated sequence $\{x_n\} = (x_n = f^n x_0,\ n = 0, 1, 2, \ldots)$ is Cauchy. Let us take $x = f^{n-2}x_0$, $y = f^{n-1}x_0$, then we have from (3)

$$d(f^n x_0, f^{n+1}x_0) \leqslant \psi_1[d(f^{n-1}x_0, f^n x_0)] + \psi_2[d(f^{n-1}x_0, f^n x_0)] + \psi_3[d(f^{n-1}x_0, f^n x_0)]. \tag{4}$$

We denote by $\beta_{n+1} = d(f^n x_0, f^{n+1}x_0)$. We have from (4)

$$\beta_{n+1} = d(f^n x_0, f^{n+1}x_0) \leqslant \psi_1(\beta_n) + \psi_2(\beta_n) + \psi_3(\beta_n)\,. \tag{5}$$

From (5) it is clear that β_n decreases with n and hence $\beta_n \to \beta$ say as $n \to \infty$. If possible, let $\beta > 0$. Then ψ_i is upper semi-continuous, we obtain, as $n \to \infty$, $\beta \leqslant \psi_1(\beta) + \psi_2(\beta) + \psi_3(\beta) < (3/5)\beta$, which is impossible unless $\beta = 0$.

We shall now show that the sequence $\{x_n\}$ is Cauchy. Let us assume that it is not so. Then there exists an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$, $\{n(k)\}$ with $m(k) > n(k) \geqslant k$ such that

$$C_k = d(x_{m(k)}, x_{n(k)}) \geqslant \varepsilon \qquad (k = 1, 2, 3, \ldots)\,. \tag{6}$$

If $m(k)$ is the smallest integer exceeding $n(k)$ for which (6) holds, then from the well ordering principle, we have

$$d(x_{m(k)-1}, x_{n(k)}) < \varepsilon\,. \tag{7}$$

and whence we derive that

$$C_k \leqslant d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) < \beta_{m(k)} + \varepsilon < \beta_k + \varepsilon\,.$$

This implies $C_k \to \varepsilon$ as $k \to \infty$. Also we have

$$C_k = d(x_m, x_n) \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_n, x_{n+1})$$
$$\leqslant \beta_{m+1} + \beta_{n+1} + d(f^{m+1}x_0, f^{n+1}x_0) \leqslant \beta_{m+1} + \beta_{n+1} + \psi_1[d(f^m x_0, f^n x_0)]$$
$$+ \psi_2[d(f^m x_0, f^m x_0)] + \psi_3[d(f^n x_0, f^n x_0)]$$
$$+ \psi_4[d(f^{m+2}x_0, f^{m+1}x_0)] + \psi_5[d(f^{n+2}x_0, f^{n+1}x_0)],$$

(by putting $u_1 = f^m x_0$, $u_2 = f^n x_0$, $u_3 = f^{m-1} x_0$, $u_4 = f^{n-1} x_0$, $u_5 = f^{m+2} x_0$, $u_6 = f^{n+2}x_0$ in (2))

$$C_k \leqslant \beta_{m+1} + \beta_{n+1} + \psi_1(C_k) + \psi_4(\beta_{m+2}) + \psi_5(\beta_{n+2}).$$

Letting $k \to \infty$ in the above inequality we derive

$$\varepsilon \leqslant \psi_1(\varepsilon) < \varepsilon/5, \tag{8}$$

which is impossible. Thus the sequence $\{x_n\}$ is Cauchy and since X is complete so $\lim_n x_n = z \in X$. We shall now show that z is the fixed point f. Putting $u_1 = f^n x_0$, $u_2 = z$, $u_3 = f^{n-1}x_0$, $u_4 = f^{n+1}x_0$, $u_5 = f^{n-2}x_0$, $u_6 = f^{n+2}x_0$ in (2) we get

$$d(f^{n+1}x_0, fz) \leqslant \psi_1[d(f^n x_0, z)] + \psi_2[d(f^n x_0, f^n x_0)] + \psi_3[d(z, f^{n+2}x_0)]$$
$$+ \psi_4[d(f^{n-1}x_0, f^{n+1}x_0)] + \psi_5[d(f^{n+2}x_0, fz)]$$
$$\leqslant \psi_1[d(f^n x_0, z)] + \psi_3[d(z, f^{n+2}x_0)] + \psi_4[d(f^{n-1}x_0, f^{n+1}x_0)] + \psi_5[d(f^{n+2}x_0, fz)],$$

letting $n \to \infty$ in the above inequality we get

$$d(z, fz) \leqslant \psi_5[d(z, fz)] < d(z, fz)/5,$$

which is a contradiction. Hence $z = fz$. Next we shall show that z is the unique fixed point of f. Let z and ω be fixed points of f and $z \neq \omega$. Then putting $u_5 = u_4 = u_1 = z$, $u_6 = u_3 = u_2 = \omega$ in (2) we get

$$d(z, \omega) = d(fz, f\omega) \leqslant \psi_1[d(z, \omega)] + \psi_2[d(z, \omega)] + \psi_3[d(z, \omega)] < (3/5)\, d(z, \omega),$$

which is a contradiction and hence $z = \omega$. This completes the proof of the theorem.

Note. It is remarkable to note that in establishing this theorem we have considered the functions $\psi_i(t)$, upper semi-continuous on the right instead of continuous on the right used by Pittnauer [**3**]$_2$.

Theorem 2. *Let f_k $(k = 1, 2, \ldots, n)$ be a family of mappings of X into itself. If $\{f_k\}_{k=1}^n$ satisfy the conditions*

$$f_k f_l = f_l f_k \qquad (l, k = 1, 2, \ldots, n), \tag{9}$$

there exists a system of positive integers $m_1, m_2, \ldots, m_n$ such that

$$d(f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n} u_1, f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n} u_2) \leqslant \psi_1[d(u_1, u_2)] + \psi_2[d(u_1, f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n} u_3)]$$
$$+ \psi_3[d(u_2, f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n} u_4)] + \psi_4[d(u_5, f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n} u_1)] + \psi_5[d(u_6, f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n} u_2)]\,. \tag{10}$$

Proof. Let $f = f_1^{m_1} f_2^{m_2} f_3^{m_3} \ldots f_n^{m_n}$, then (10) takes the form

$$d(fu_1, fu_2) \leqslant \psi_1[d(u_1, u_2)] + \psi_2[d(u_1, fu_3)]$$
$$+ \psi_3[d(u_2, fu_4)] + \psi_4[d(u_5, fu_1)] + \psi_5[d(u_6, fu_2)]\,. \tag{11}$$

By Theorem 1, f has a unique fixed point z. Then

$$f_k(fz) = f_k z\,, \qquad (k = 1, 2, \ldots, n)\,. \tag{12}$$

By commutativity of $\{f_k\}$, (12) implies $f(f_k z) = f_k z$ $(k = 1, 2, \ldots, n)$. Since f has a unique fixed point z, we get $f_k z = z$, $k = 1, 2, \ldots, n$. Hence z is a common fixed point of the family $\{f_k\}$. Let z, ω be common fixed points of the family $\{f_k\}$ with $z \neq \omega$. Then putting $u_5 = u_4 = u_1 = z$, $u_6 = u_3 = u_2 = \omega$ in (11) we get

$$d(z, \omega) = d(fz, f\omega) \leqslant \psi_1[d(z, \omega)] + \psi_2[d(z, \omega)] + \psi_3[d(z, \omega)] < (3/5)\, d(z, \omega)$$

which is a contradiction. Hence $z = \omega$. This completes the proof of the theorem.

We shall now show that our result contains some well known fixed point theorems as special cases.

If we define the functions $\psi_i(t)$ by $\psi_1(t) = a_1 t$, $\psi_2(t) = a_2 t$, $\psi_3(t) = a_3 t$, $\psi_4(t) = a_4 t$, $\psi_5(t) = a_5 t$ with $0 < a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then we have the following theorems as special cases.

(a) If we put $u_6 = u_3 = u_1$ and $u_5 = u_4 = u_2$ in Theorem 1, we get the results of Hardy and Rogers [**1**].

(b) If we put $u_3 = u_1$ and $u_4 = u_2$, $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = a_5 = 0$ in Theorem 1, we have the results of Reich [**5**].

(c) If we put $u_3 = u_1$ and $u_4 = u_2$, $a_2 = a_3 = a$, $a_1 = a_4 = a_5 = 0$ in Theorem 1, we get the results of Kannan [**2**].

References

[**1**] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16** (1973), 201-206.

[**2**] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71-76.

[**3**] F. Pittnauer: $[\bullet]_1$ *Ein fixpunktsatz in metrischen Raumen*, Arch. Math. **26** (1975), 421-426; $[\bullet]_2$ *A fixed point theorem in complete metric spaces*, Period. Math. Hungar. (to appear).

[**4**] B. E. Rhoades, *A fixed point theorem in metric spaces* (to appear).

[**5**] S. Reich, *Kannan's fixed point theorems*, Boll. Un. Mat. Ital. **4** (1971), 1-11.

Abstract

In this paper fixed point theorems for contractive type mappings involving six points of the space have been studied. This result includes many well-known fixed point theorems as special cases. As a consequence this can be applied to various discontinuous mappings.

* * *