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Exactness in the symmetrizations of a left exact category (**)

Introduction

Partitions, i.e. disjoint distributive unions in semilattices were used in $[4]_3$ to characterize the exactness of short sequences and functors in the context of distributive exact categories, i.e. exact categories in which the subobjects of any object form a distributive lattice.

The context is justified by the fact that subquotients in distributive exact categories behave « well », i.e. canonical isomorphisms between subquotients are composable. Then in the inverse symmetrization $[4]_2$ of the category, where isomorphism classes of subquotients are a semilattice of subobjects, in the short exact sequence $0 \to A \to B \to C \to 0$, A and C, taken as subobjects of B, supply a partition of B.

The fact is true for a wider class of categories, the « orthoquaternary » ones, $[4]_2$, [3] while a study of short exact sequences can be brought on also in left (or right) exact categories (the ones endowed with zero, kernels and cok-monic factorizations [5]).

These ones (of which distributive exact categories are a particular case) have two outstanding symmetrizations, the « quaternary » one, whose subobjects are the subquotients of the former, and the « inverse » one, in which subobjects are the isomorphism classes of the subquotients.

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In the inverse symmetrization of distributive exact categories, distributive unions appear as the good notion of exactness (in the sense outlined before); but this is no more true in the wider context of left exact orthoquaternary categories where partitions must be replaced by the weaker notion of « decompositions ».

Moreover, decompositions can be used not only in inverse symmetrizations, but also in quaternary ones, where subobjects of an object are no more a semilattice, but only a band (an idempotent semigroup).

Paragraph **1** introduces the notions of bands and semilattices with decompositions. Paragraph **2** introduces the notion of categories with decompositions. Paragraph **3** shows that quaternary and inverse symmetrizations of an orthoquaternary left exact category C are categories with decompositions. Relations between exact sequences in C and decompositions in its symmetrizations are stated; so it is shown that the « good » notion of exactness is « decomposition ». Paragraph **4** shows that with adjunctive hypotheses decompositions and partitions are the same. Paragraph **5** shows some examples.

Preliminary definitions

If S is a semigroup and E the set of its idempotent elements, S is called: *regular* iff for any $x \in S$ there exists an $y \in S$ such that $xyx = x$, $yxy = y$; *inverse* iff moreover such y is unique; *orthodox* iff it is regular and E is a subsemigroup.

If S is an orthodox semigroup it is called: *left-inverse* iff for any $x, y \in E$ $xyx = xy$ [**8**]; *right-inverse* iff for any $x, y \in E$ $xyx = yx$ [**8**]; *quasi-inverse* iff for any $x, y, z \in E$ $xyxzx = xyzx$ [**9**].

A band (idempotent semigroup) will be called a *left-inverse* (resp. *right-inverse*, *quasi-inverse*) *band* iff it is a left-inverse (resp. right-inverse, quasi-inverse) semigroup; in usual terminology it is called left-regular (resp. right-regular, regular).

An involution on a category (i.e. a contravariant endofunctor identical on the objects and autoinverse on the morphism; if x is a morphism $x \to \tilde{x}$ will denote the involution) is called *regular* iff for any morphism x $x\tilde{x}x = x$.

An involution category is called *regular involution category* iff its involution is regular; *factorizing* iff it has unique-up-to-isomorphisms epic-monic factorizations.

M. Grandis (in a series of works named *Orthodox Categories*, for which see the bibliography in [**4**]$_2$; in them one can also find proofs of the statements in **1.1** and **1.2** below) has extended semigroup definitions to regular involution categories: a regular involution category is *orthodox* (resp. *quasi-inverse*,

inverse) iff the semigroup of endomorphisms of any object is an orthodox (resp. quasi-inverse, inverse) semigroup.

A category is called *left-exact* [**5**] iff it has zero, kernels and unique-up-to-isomorphisms cok-monic factorizations.

In a left-exact category a monic is called *subnormal* iff it is the composition of (finitely many) normal monics.

1 - Bands and semilattices with decompositions

1.1 – Let E be a band, i.e. an idempotent semigroup. Then:

(1) E has a canonical preorder, compatible with composition: $x \mathfrak{a} y$ iff $x = xyx$;

(2) if E has an identity 1, then 1 is the unique maximum for $\mathfrak{a}$;

(3) for any x, y, $xy \mathfrak{a} yx$, $xy \mathfrak{a} x$, $xy \mathfrak{a} y$;

(4) $N = \{x \in E$ such that for any y $x \mathfrak{a} y\}$ is void or an ideal, the *null ideal* of E; moreover if $x \in N$ and $y \mathfrak{a} x$, then $y \in N$;

(5) $\mathfrak{a}$ is an order iff E is commutative.

1.2 – (1) On E there is a congruence Φ: $x \Phi y$ iff $x \mathfrak{a} y$ and $y \mathfrak{a} x$;

(2) E is commutative iff Φ is the identical relation;

(3) if E has an identity, the only element in its congruence class is the identity itself;

(4) N is a congruence class.

1.3 – Definition. A band is called a *N-band* iff it has an identity and N is non-void.

1.4 – If E is a N-band E/Φ is a (meet) semilattice with 0 (the class N) and 1 (the class $\{1_E\}$).

1.5 – Remark. Any meet semilattice is a commutative band, where intersection is the product.

1.6 – Definition. In a N-band the family $(y_0, \ldots, y_n)$ is a *decomposition* of y iff

(1) $y_i \mathfrak{a} y$ for any i;

(2) $y_i \ y_j \in N$ if $i \neq j$;

(3) if $z \mathfrak{a} y$ and $zy_i \in N$ whenever $i \neq i_0$, then $z \mathfrak{a} y_{i_0}$.

1.7 – Definition. Let E be a N-band; for any $x \in E$, let $I(x)$ the ideal of all y such that $y \,\mathfrak{a}\, x$.

Let D be a family, indexed on E, of subsets $D(x)$ of $I(x)$ such that: (D1) $x \in D(x)$; (D2) if $x \in D(y)$ and $y \in D(z)$, then $x \in D(z)$; (D3) if $x \in D(y)$, then $zx \in D(zy)$; (D4) for any $y_0 \in D(y)$, there exists a decomposition of y $(y_0, \ldots, y_n)$ where all y_i's are in $D(y)$ (called a *D-decomposition*).

Then (E, D) is called a *band with decompositions*, or a *D-band*.

1.8 – Remark. If E is a 0-semilattice we have in the same way the definition of *semilattice with decompositions or D-semilattice.*

1.9 – Proposition. *Let* $(y_0, \ldots, y_n)$ *be a decomposition of* y (resp. *a D-decomposition*) *in a N-band* (resp. *in a D-band*). *Then:*

(1) $(xy_0, \ldots, xy_n)$ *is a decomposition* (resp. *a D-decomposition*) *of* xy.

(2) *If for any* i $(x_{i1}, \ldots, x_{ik_i})$ *is a decomposition* (resp. *a D-decomposition*) *of* y_i, *then* $(x_{ij})_{i,j}$ *is a decomposition* (resp. *a D-decomposition*) *of* y.

Proof. (1) If $z \,\mathfrak{a}\, xy$ and $zxy_i \in N$ whenever $i \neq i_0$, then $zx \,\mathfrak{a}\, xyx \,\mathfrak{a}\, y$, so $zx \,\mathfrak{a}\, y_{i_0}$, but then $zx \,\mathfrak{a}\, y_{i_0}$ $x \,\mathfrak{a}\, xy_{i_0}$.

(2) If $z \,\mathfrak{a}\, y$ and $zx_{ij} \in N$ whenever $i \neq i_0$, $j \neq j_0$, then if $i \neq i_0$, $zx_{ij} \,\mathfrak{a}\, zy_i$ and $zx_{ij} \in N$ for any j; therefore $zy_i \,\Phi\, zx_{ij} \in N$.

Then, if $i \neq i_0$ $zy_i \in N$ and so $z \,\mathfrak{a}\, y_{i_0}$; as for any $j \neq j_0$ $zx_{i_0 j} \in N$, $z \,\mathfrak{a}\, x_{i_0 j_0}$.

1.10 – Definition. Let (E, D), (E', D') be D-bands. $f\colon (E, D) \to (E', D')$ is called *D-homomorphism* iff: (1) $f\colon E \to E'$ is a semigroup homomorphism; (2) for any $x \in E$, $f(D(x)) \subset D'(f(x))$; (3) $f(N_E) \subset N_{E'}$; (4) f preserves D-decompositions.

1.11 – A commutative band is left inverse.

1.12 – Proposition. *Let* (E, D), (E', D') *be left-inverse D-bands*, $f\colon E \to E'$ *a semigroup homomorphism verifying* (2) *and* (3) *of* **1.10**.

f *is a D-homomorphism iff for any* $y \in E$, *for any* $y_0 \in D(y)$, *there exists a D-decomposition* $(y_0, \ldots, y_n)$ *preserved by* f.

Proof. By induction. Unary decompositions are obviously preserved. So let f preserve n-ary D-decompositions and $(y_0, \ldots, y_n)$ be a D-decomposition of y. Let $z \,\mathfrak{a}\, f(y)$, $zf(y_i) \in N_E$, whenever $i \neq i_0$. To fix ideas let $i_0 = 0$.

There exists, by hypothesis, a D-decomposition of y, $(y_0, x_1, \ldots, x_k)$, preserved by f. For any i, $(x_i y_0, \dots, x_i y_n)$ is an n-ary D-decomposition of $x_i y = x_i$, hence preserved by f. So $(f(y_0), f(x_1 y_1), \dots, f(x_1 y_n), f(x_2 y_1), \dots, f(x_k y_n))$ is a D'-decomposition of $f(y)$.

As $zf(y_i) \in N_{E'}$ if $i \neq 0$, $zf(x_j y_i) \in N_{E'}$ for any j, for any $i \neq 0$. Therefore $z \,\mathbf{\alpha}\, f(y_0)$ and $(f(y_0), \dots, f(y_n))$ is a D'-decomposition of $f(y)$.

1.13 – Lemma. *Let* (E, D) *be a left inverse* D*-band. Let* $L = E/\Phi$ *and* $f\colon E \to L$ *the homomorphism associated to* Φ. *If* $f(x) = f(y)$, i.e. $x \,\Phi\, y$, *then* $f(D(x)) = f(D(y))$.

Proof. Let $\alpha \in f(D(x))$, $\alpha = f(a)$, $a \in D(x)$. Then $ya \in D(yx) = D(y)$ and $f(ya) = f(a) = \alpha$ as $(ya)a = ya$ and $a(ya) = a$, for $a \,\mathbf{\alpha}\, x \,\mathbf{\alpha}\, y$.

1.14 – Proposition. *With the same hypotheses and notations of* **1.13**, *let* $D_\Phi(\alpha) = f(D(x))$, *where* x *is such that* $\alpha = f(x)$; $D_\Phi = (D_\Phi(\alpha))_{\alpha \in L}$. *Then* (L, D_Φ) *is a* D*-semilattice and* f *is a* D*-homomorphism.*

Proof. (D1) For any $\alpha \in L$, $\alpha \in D_\Phi(\alpha)$, as for any $x \in E$, $x \in D(x)$.

(D2) Let $\alpha \in D_\Phi(\beta)$, $\beta \in D_\Phi(\gamma)$; $x, y \in E$ such that $\alpha = f(x)$, $\beta = f(y)$, $x \in D(y)$; $z, t \in E$ such that $\beta = f(z)$, $\gamma = f(t)$, $z \in D(t)$. Then $zx \in D(zy) = D(z)$; so $zx \in D(t)$. $zx \,\Phi\, x$ implies $\alpha = f(zx) \in f(D(t)) = D_\Phi(\gamma)$.

(D3) Let $\alpha \in D_\Phi(\beta)$, $\gamma \in L$; $x, y \in E$ such that $\alpha = f(x)$, $\beta = f(y)$, $x \in D(y)$; $z \in E$ such that $\gamma = f(z)$. $zx \in D(zy)$; so $\gamma\alpha = f(zx) \in f(D(zy)) = D_\Phi(\gamma\beta)$.

(D4) Let $\alpha_0 \in D_\Phi(\alpha)$; $y_0, y \in E$ such that $\alpha_0 = f(y_0)$, $\alpha = f(y)$, $y_0 \in D(y)$; let $(y_0, \dots, y_n)$ be a D-decomposition of y. $y_i \,\mathbf{\alpha}\, y$ implies $f(y_i) \leqslant f(y)$ for any i; $y_i y_j \in N$ implies $f(y_i) f(y_j) = 0$. Let $\gamma \leqslant \alpha$, $z \in E$ such that $\gamma = f(z)$. Then $z \,\mathbf{\alpha}\, y$. If $\gamma f(y_i) = 0$ whenever $i \neq i_0$, then $f(zy_i) = 0$, $zy_i \in N$. So $z \,\mathbf{\alpha}\, y_{i_0}$ and $\gamma = f(z) \leqslant f(y_{i_0})$. Therefore $(f(y_0), \dots, f(y_n))$ is a D_Φ-decomposition of α.

1.15 – Proposition. *Let* S *be a quasi-inverse semigroup. Let* $P(S) = \{a \in S$ *such that* $a = aa = \tilde{a}\}$.

(1) *If* $a, b \in P(S)$, *then* ab *is idempotent*; (2) *if* $a, b \in P(S)$, *then* $aba \in P(S)$; (3) *if we define on* $P(S)$ *the product* $a \,\Box\, b = aba$, *then* $(P(S), \Box)$ *is a left-inverse semigroup*; (4) *if* S *is commutative,* $P(S)$ *is commutative and* $a \,\Box\, b = ab = ba$.

2 - Categories with decompositions

2.1 – Let H be a factorizing, quasi-inverse involution category, A an object of H. $H(A)$ be the semigroup of endomorphisms of A; $H_1(A)$ the (quasi-inverse) subsemigroup of idempotent endomorphisms of A; $H_0(A)$ the set of projections (symmetric idempotent endomorphisms) of A, which is a semigroup for the composition $\Box$ of **1.15**; $M(A)$ the set of subobjects of A.

Then there exists a biunivocal correspondence between $H_0(A)$ and $M(A)$ which allows to define an induced structure of left-inverse band on $M(A)$. If moreover H is inverse then $H_0(A)$ and $M(A)$ are semilattices with 1.

2.2 – Definition. $O \in Ob(H)$ is called *pseudozero* iff for any object A there exists $x\colon O \to A$ and $H(O) = \{1_0\}$.

2.3 – Remark. If O and O' are pseudozeros, there exists one and only one morphism from O to O' and it is an isomorphism. If $x \in H(O, A)$, then x is a monic. If H is inverse, a pseudozero is a zero. $N(A) = \{\alpha \in H_0(A)$ such that $Im(\alpha) = O\}$ is the (non-void) *null-ideal* of $H_0(A)$.

2.4 – Definition. Let H be a factorizing quasi-inverse category with pseudo-zero. Let M be a subcategory of monics in H such that:

(M1) Iso $(H) \subset M$;

(M2) if μ, ν are monics with the same codomain and $\tilde{\nu}\mu = \lambda\xi$ is a canonical factorization, then if $\mu \in M$ also $\lambda \in M$;

(M3) for any object A, for any $\mu_0 \in M(A) = \{\mu \in M$ such that $\operatorname{Cod}\mu = A\}$, there exist $\mu_1, \ldots, \mu_n$, $\mu_i \in M(A)$ such that: (1) Im $\tilde{\mu}_j\mu_i = O$ whenever $i \neq j$; (2) for any $\nu \in M(A)$, if $\operatorname{Im}(\tilde{\mu}_i\nu) = O$ whenever $i \neq i_0$, then $\nu \subset \mu_{i_0}$, i.e. $\nu = \nu \,\Box\, \mu_{i_0}$.

Then (H, M) is called a *category with decompositions* or a D-category. $(\mu_0, \ldots, \mu_n)$ verifying (M3) is called an M-*decomposition* of A.

2.5 – Remark. If H is inverse (M2) and (M3) can be so formulated:

(M2) if $[\lambda, \xi, \nu, \mu]$ is a pullback of monics and $\mu \in M$ also $\lambda \in M$;

(M3) for any object A, for any $\mu \in M(A)$ there exist $\mu_1, \ldots, \mu_n$ such that: (1) $\mu_i \cap \mu_j = O$ whenever $i \neq j$; (2) if $\nu \in M(A)$ and $\nu \cap \mu_i = O$ whenever $i \neq i_0$, then $\nu = \mu_{i_0}\lambda$.

2.6 – Proposition. *Let H be a factorizing quasi-inverse category with pseudo-zero. M be a subcategory of monics containing all the isomorphisms. For any $\mu \in M(A)$ let*

$$D_M(\mu) = \{\mu\lambda\tilde{\lambda}\tilde{\mu} \text{ where } \lambda \in M(\operatorname{Dom}(\mu))\}\,, \qquad D_M(A) = \big(D_M(\mu)\big)_{\mu \in M(A)}\,.$$

Then (H, M) *is a D-category iff for any object* A, $(H_0(A), D_M(A))$ *is a D-band.*

Proof. (D1) is obvious.

(D2) Let $\lambda, \mu, \nu \in M(A)$, $\lambda\tilde{\lambda} \in D_M(\mu)$, $\mu\tilde{\mu} \in D_M(\nu)$. Then $\lambda = \mu\xi$, $\mu = \nu\zeta$, $\xi, \zeta \in M$. So $\lambda = \nu(\zeta\xi)$, $\zeta\xi \in M$. So $\lambda\tilde{\lambda} \in D_M(\nu)$.

(D3) Let $\lambda, \mu, \nu \in M(A)$, $\mu\tilde{\mu} \in D_M(\nu)$. Then $\mu = \nu\xi$, $\xi \in M$. Let $\tilde{\nu}\lambda = \lambda_1\tilde{\nu}_1$, $\xi\tilde{\lambda}_1 = \lambda_2\tilde{\xi}_1$ be canonical factorizations, $\xi_1 \in M$, $\lambda\tilde{\lambda}\nu\tilde{\nu}\lambda\tilde{\lambda} = \lambda\nu_1\tilde{\nu}_1\tilde{\lambda}$ and $\lambda\tilde{\lambda}\mu\tilde{\mu}\lambda\tilde{\lambda} = \lambda\nu_1\xi_1\tilde{\xi}_1\tilde{\nu}_1\tilde{\lambda}$; therefore $\lambda\tilde{\lambda} \square \mu\tilde{\mu} \in D_M(\lambda \square \nu)$.

(D4) Let $\mu \in M(A)$, $\mu_0\tilde{\mu}_0 \in D(\mu)$, i.e. $\mu_0 = \mu\nu_0$, $\nu_0 \in M$. Let $B = \text{Dom}(\mu)$. There exists $(\nu_0, \dots, \nu_n)$, an M-decomposition of B. Let $\alpha = \lambda\tilde{\lambda} \in H_0(A)$; $\alpha \,\mathbb{C}\, \mu\tilde{\mu}$ means that $\tilde{\mu}\lambda$ is a monic whose codomain is B. $\alpha \,\mathbb{C}\, \mu\nu_i\tilde{\nu}_i\tilde{\mu} \in N$ whenever $i \neq i_0$ means $\text{Im}(\tilde{\nu}_i\tilde{\mu}\lambda) = O$ whenever $i \neq i_0$, in which case $\tilde{\mu}\lambda \,\mathbb{C}\, \nu_{i_0}$, i.e. $\tilde{\nu}_{i_0}\tilde{\mu}\lambda$ is a monic. But then $\alpha \,\mathbb{C}\, \mu\nu_{i_0}\tilde{\nu}_{i_0}\tilde{\mu}$. We have so proved that $(\mu\nu_0\tilde{\nu}_0\tilde{\mu}, \dots, \mu\nu_n\tilde{\nu}_n\tilde{\mu})$ is a $D_M(A)$-decomposition of $\mu\tilde{\mu}$.

(M2) Let $\mu, \nu \in M(A)$, $\tilde{\nu}\mu = \lambda\tilde{\xi}$ be a canonical factorization, $\mu \in M$. In $H_0(A)$ $\mu\tilde{\mu} \in D_M(1_A)$ so $\nu\tilde{\nu} \square \mu\tilde{\mu} \in D_M(\nu)$. But $\nu\tilde{\nu} \square \mu\tilde{\mu} = \nu\lambda\tilde{\lambda}\tilde{\nu}$ so $\lambda \in M$.

(M3) Let $\mu_0 \in M(A)$, $\mu_0\tilde{\mu}_0 \in D_M(1_A)$; then there exist $\mu_1, \dots, \mu_n \in M(A)$, such that $(\mu_0\tilde{\mu}_0, \dots, \mu_n\tilde{\mu}_n)$ is a $D_M(A)$-decomposition of 1_A. We then easily infer that $(\mu_0, \dots, \mu_n)$ is an M-decomposition of A.

2.7 – Definition and Proposition. Let (H, M) and (K, N) be D- categories. A pseudozero preserving functor $f\colon H \to K$ is called a *D-functor* iff it satisfies the equivalent conditions: (1) $f(M) \subset N$ and f preserves M-decompositions. (2) $f(M) \subset N$ and for any A, object of H, for any $\mu_0 \in M(A)$ there exists an M-decomposition $(\mu_0, \dots, \mu_n)$ preserved by f. (3) For any object A the band homomorphism $f_A\colon H_0(A) \to K_0(f(A))$ associated to f is a D-homomorphism.

2.8 – Proposition. *Let* (H, M) *be a D-category.* H/Φ *be the inverse category associated to* H, f_H *the canonical functor. Then* $(H/\Phi, f_H(M))$ *is an inverse D-category. Let* $g\colon (H, M) \to (K, M)$ *be a D-functor,* $\bar{g}\colon H/\Phi \to K/\Phi$ *the functor associated to* g; $\bar{g}\colon (H/\Phi, f_H(M)) \to (K/\Phi, f_K(N))$ *is a D-functor.*

3 - Left exact orthoquaternary categories and D categories

3.1 – Theorem. *Let* C *be a left exact* [**5**] *orthoquaternary* [**3**] *category.* C^W *be its quaternary symmetrization* [**4**]$_1$, *which is quasi-inverse* [**4**]$_2$ *with* 0_C *as pseudozero. Let* $M_C = \{m\tilde{p}$, p *conormal epic,* m *subnormal* [**5**] *monic*$\}$. *Then* (C^W, M_C) *is a D-category.*

Proof. M_C is a subcategory for if $[n, q, p, m]$ is a pullback in C where p is a conormal epic and m a normal monic, then n is a normal monic and q a conormal epic.

(M2) We need only to observe that if $[n_1, m_1, m, n]$ is a pullback of monics in C and m is normal also m_1 is such; and that if $pm = nq$ in C where m and n are monics, p and q conormal epics, if m is normal, n is such.

(M3) Let $\mu_1 = m\tilde{p} \in M_C$. Then $m = m_k \dots m_2$, where all the m_i's are normal; let $m_1 = \ker(p)$. Let us now define $\mu_0 = m_k \dots m_1$; $\mu_i = m_k \dots m_{i+1} \cdot (\operatorname{cok} m_i)^{\sim}$, for $i = 1 \dots k-1$; $\mu_k = (\operatorname{cok} m_k)^{\sim}$.

We claim that $(\mu_0, \dots, \mu_k)$ is a decomposition. In fact, let $\nu = n\tilde{q}$ be a monic in C^W with the same codomain as μ_1. Inductively let $n_k = n$; $q_k = q$; $[n_i, l_{i+1}, m_{i+1}, n_{i+1}]$ be a pullback; $q_{i+1}l_{i+1} = s_i q_i$ and $(\operatorname{cok} m_i)n_i = t_i \operatorname{cok}(l_i)$ be factorizations (by [**3**]), $[q_i, \operatorname{cok}(l_i), v_i, z_i]$ be a pushout, for $i = k-1, \dots, 1$.

Then $\tilde{\mu}_i\nu = t_i\tilde{z}_i v_i(s_{k-1} \dots s_i)^{\sim}$ if $1 \leqslant i \leqslant k-1$; $\tilde{\mu}_0\nu = n_0\,\tilde{q}_0(s_{k-1} \dots s_0)$; $\tilde{\mu}_k\nu = t_k\tilde{z}_k v_k$. Let us observe that $s_{i-1} = \ker(v_i)$ so if $\operatorname{Im}(\tilde{\mu}_i\nu) = 0$, then s_{i-1} is an isomorphism and $q_i l_i \simeq q_{i-1}$. If $\ker(q_i)$ is an isomorphism and $\operatorname{Im}(\tilde{\mu}_{i+1} \cdot\nu) = 0$ then $q_i = 0$, $q_i = q_{i+1}l_{i+1}$, so $q_{i+1} = 0$ and $\ker(q_{i+1})$ is an isomorphism.

Now, let us suppose that $\operatorname{Im}(\tilde{\mu}_i\nu) = 0$ if $i > 0$; then $v_i = 0$ if $i > 0$, s_i is an isomorphism if $i \geqslant 0$. So $\tilde{\mu}_0\nu = n_0\tilde{q}_0(s_{k-1} \dots s_0)^{\sim} \simeq n_0\tilde{q}_0$ is a monic, i.e. $\nu \,\mathbf{\complement}\, \mu_0$.

Let $j \neq 1, k$ and $\operatorname{Im}(\tilde{\mu}_i\nu) = 0$ whenever $i \neq j$. $\operatorname{Im}(\tilde{\mu}_0\nu) = 0$ implies $q_0 = 0$ so inductively $q_j l_j = 0$ and v_j is an isomorphism. Also, s_i is an isomorphism for any $i \geqslant j$. Then $\tilde{\mu}_j\nu = t_j\tilde{z}_j v_j(s_{k-1} \dots s_j)^{\sim} \simeq t_j\tilde{z}_j$ is a monic, i.e. $\nu \,\mathbf{\complement}\, \mu_j$.

If $\operatorname{Im}(\tilde{\mu}_j\nu) = 0$ if $j \neq k$, v_k is an isomorphism; $\tilde{\mu}_k\nu = t_k\tilde{z}_k$ is a monic and $\nu \,\mathbf{\complement}\, \mu_k$. So the proof is complete.

3.2 – Theorem. *Let C be a left-exact $C\theta$-category* ([**6**], *whose axioms are weaker than those in* [**1**]$_1$); *C^0 be its $S\theta$-symmetrization* [**1**]$_1$, [**6**], *which is inverse and with 0_C as zero.*

Let C moreover verify axiom A6 *of* [**2**]; *«If $pm = nq$, where p, q are conormal epics, m, n monics and m is normal, also n is such».*

Let $M_C = \{m\tilde{p},\ p$ conormal epic, m subnormal monic$\}$. Then (C^0, M_C) is an inverse D-category.

Proof. It is only slightly different from the one of Theorem **3.1.**

3.3 – Proposition. *C be a left-exact orthoquaternary category* (resp. *a left-exact $C\theta$-category verifying* A6); *p a conormal epic, m a monic. Then $m = \ker(p)$ in C iff $(m, \tilde{p})$ is a decomposition in (C^W, M_C)* (resp. *in (C^0, M_C)*).

Proof. An implication follows from Theorem **3.1** (resp. Theorem **3.2**).

As for the other one, if $(m, \tilde{p})$ is a decomposition and $n = \ker(p)$, $\operatorname{Im}(pn) = 0$ implies $n \leqslant m$. But also $(n, \tilde{p})$ is a decomposition and $\operatorname{Im}(pm) = 0$, so $m \leqslant n$. Therefore $m = \ker(p)$.

3.4 – Corollary. *Let C, D be left-exact orthoquaternary categories* (resp. *left-exact $C\Theta$-categories verifying* A6), *$f: C \to D$ a W-functor* (resp. *a θ-functor*). *f is left-exact iff f^W* (resp. f^Θ) *is a D-functor.*

Proof. Let p conormal, $m = \ker(p)$; then $(m, \tilde{p})$ is a decomposition, $(f(m), f(p))^\sim$ is a decomposition, $f(m) = \ker(f(p))$. Conversely, let $m\tilde{p} \in M_C$, $m = m_k \dots m_2$, all m_i's normal monics, $m_1 = \ker(p)$. $(\mu_0, \dots, \mu_k)$ taken as in the proof of Theorem **3.1** is a decomposition. But $f(m_i)$ is normal and $f(\operatorname{cok} m_i) = \operatorname{cok}(f(m_i))$. So, by Theorem **3.1** (resp. Theorem **3.2**) also $(f^W(\mu_0), \dots, f^W(\mu_k))$ (resp. with f^Θ) is a decomposition. Then f^W (resp. f^Θ) is a D-functor.

4 - Partition semilattices and categories

4.1 – Definition and Proposition. *Let (L, D) be a D-semilattice.*

The following conditions are equivalent.

(1) *For any $(y_1, \dots, y_n)$, D-decomposition of y, for any z, $z \cap y = \bigcup_{i=1}^{n} z \cap y_i$.*

(2) *For any $(y_1, \dots, y_n)$, D-decomposition of y, $y = \bigcup_i y_i$.*

(3) *For any $y_1 \in D(y)$ there exists a D-decomposition $(y_1, \dots, y_n)$ of y such that for any z, $z \cap y = \bigcup_i z \cap y_i$.*

(4) *For any $y_1 \in D(y)$ there exists a D-decomposition $(y_1, \dots, y_n)$ of y such that $y = \bigcup_i y_i$.*

(5) *For any $y_1 \in D(y)$ there exist $y_2, \dots, y_n \in D(y)$ such that $y_i \cap y_j = 0$ if $i \neq j$ and for any z, $z \cap y = \bigcup_i z \cap y_i$.*

When they are satisfied (L, D) is called a «partition semilattice».

Proof. We have only to prove (5) ⇒ (4) ⇒ (2) ⇒ (1).

(5) ⇒ (4). $(y_1, \dots, y_n)$ is a D-decomposition of y as, if $z \leqslant y$ and $z \cap y_i = 0$ whenever $i \neq i_0$, then $z = z \cap y = \bigcup_i z \cap y_i = z \cap y_{i_0}$, i.e. $z \leqslant y_{i_0}$.

(4) ⇒ (2). By induction. Unary decomposition satisfy the thesis.

Then let it be satisfied by n-ary decompositions and $(y_0, \dots, y_n)$ be a decomposition of y. There exists a decomposition $(x_0, \dots, x_k)$ of y such that $x_0 = y_0$ and $y = \cup x_i$.

Let $z \geqslant y_i$ for any i. Then $z \geqslant x_i \cap y_j$ for any i, j. $(x_i \cap y_1, \dots, x_i \cap y_n)$ is an n-ary decomposition of x_i $(i = 1, \dots, k)$; so $x_i = \bigcup_j x_i \cap y_j$ and $z \geqslant x_i$ for any i. But then $z \geqslant y$.

(2) $\Rightarrow$ (1). We need only to observe that $(z \cap y_1, \dots, z \cap y_n)$ is a decomposition of $z \cap y$.

4.2 – Definition. Let (H, M) be an inverse D-category.

(H, M) is called a *partition category* iff for any A object of H, for any $\mu_0 \in M(A)$, there exist $\mu_1, \dots, \mu_n \in M(A)$ such that $\mu_i \cap \mu_j = 0$ if $i \neq j$ and for any $\nu \in M(A)$, $\nu = \bigcup_i \nu \cap \mu_i$, or, equivalently, iff for any A $(H_0(A), D_M(A))$ is a partition semilattice.

4.3 – Lemma. *Let C be a left-exact $C\Theta$-category. The following conditions are equivalent.*

(1) (A7 of [**2**]) *if* $m = \ker(p)$, $n = \ker(q)$, $px = q$, $m = xn$, p, q *are conormal epics, and x is monic, then x is an isomorphism.*

(2) *If* $pm = q$, p, q *conormal epics,* m *a monic, then* $m \cup \ker(p) = 1$.

Proof. (2) $\Rightarrow$ (1). $(1) = x \cup m = x$ as $x \geqslant m$.

(1) $\Rightarrow$ (2). Let n be a monic with the same codomain of m and such that $n \geqslant m$, $n \geqslant \ker(p)$; then there exist n_1, n_2 monics such that $nn_1 = m$, $nn_2 = \ker(p)$. As $pm = q$, pn is a conormal epic; moreover $n \ker(pn) = n \cap \ker(p) = \ker(p)$. So, by the hypothesis, n is an isomorphism.

4.4 – Theorem. *Let C be a left-exact $C\Theta$-category verifying* A6. *Then* (C^{Θ}, M_C) *is a partition category iff C verifies the equivalent conditions of lemma* **4.3.**

Proof. If $m = \ker(p)$, $n = \ker(q)$, $px = q$, $m = xn$, x a monic, p, q conormal epics, then $x \cap m = m$, $x \cap p = p$; $(m, \tilde{p})$ is a decomposition hence a partition, therefore $x \simeq 1$.

Conversely, with the same notations as in the proof of Theorem **3.1**, let μ_1 be in M_C, so $(\mu_0, \dots, \mu_n)$ is a decomposition. Let us show that $1 = \bigcup \mu_i$.

For, let ν be a monic with the same codomain of μ_1 and $\nu \geqslant \mu_i$ for any i; $\nu \geqslant \mu_0$ implies n_0 is an isomorphism and $n \geqslant (m_k \dots m_1)$; $\nu \geqslant \mu_i$, $1 \leqslant i < k$, implies

t_i is an isomorphism and $m_k \dots m_{i+1} = ((m_k \dots m_{i+1}) \cap n) \cup (m_k \dots m_i)$, so inductively, $n \geqslant m_k \dots m_i$ for all $i \leqslant k$ and in particular $n \geqslant m_k$; $\nu \geqslant \mu_k$ implies that t_k is an isomorphism. We have then $m_k = n l_k$ and $\mathrm{cok}\,(m_k) = \mathrm{cok}\,(l_k) n$; n then is an isomorphism.

Moreover, $\nu \geqslant \mu_k$ implies that z_k is an isomorphism, i.e. $\ker\,(q_k) \leqslant l_k = m_k$. Also, $\nu \geqslant \mu_i$, $1 \leqslant i \leqslant k-1$, implies that z_i is an isomorphism and that $\ker\,(q_i) \leqslant m_i$. But $m_{i+1} \ker\,(q_i) = \ker\,(q_{i+1}) \cap m_{i+1} = \ker\,(q_{i+1})$, $0 \leqslant i \leqslant k-1$. So inductively $\ker\,(q_k) = m_k \dots m_1 \ker\,(q_0) \cdot \nu \geqslant \mu_0$ implies that q_0 is an isomorphism. So $\ker\,(q_0) = 0$, $\ker\,(q_k) = 0$, q is an isomorphism.

Therefore ν is an isomorphism and the thesis follows.

4.5 – Definition and Proposition. If (H, M) and (K, N) are partition categories, a zero-preserving functor $f\colon H \to K$ is called a *partition functor* iff it verifies the following equivalent conditions.

(1) $f(M) \subset N$ and f preserves partitions.

(2) $f(M) \subset N$ and for any A, for any $\mu_0 \in M(A)$, there exists a partition $(\mu_0, \dots, \mu_k)$ preserved *by* f.

(3) f is a D-functor.

4.6 – Proposition. *If C and D are left-exact $C\Theta$-categories verifying* A6 *and the conditions of Lemma* **4.3**, *and $f\colon C \to D$ is a Θ-functor, then f is left-exact iff f^{Θ} is a partition functor.*

4.7 – Proposition. *If (L, D) is a D-semilattice and for any x, $D(x) = I(x)$, then (L, D) is a partition semilattice.*

Proof. Let $(y_1, \dots, y_n)$ be a decomposition of y. Let x be such that $x \geqslant y_i$ for any i and $x \leqslant y$. Then there exists a decomposition of y $(x, x_1, \dots, x_k)$. For any i, j $x_i \cap y_j = x_i \cap x = 0$, so $x = y$ and $y = \cup\, y_i$.

4.8 – Corollary. *If (H, M) is an inverse D-category and M is the subcategory of its monics, then (H, M) is a partition category.*

5 - Examples

5.1 – Any category verifying axioms (D1)-(D5) of [**3**] (and in particular the « normodistributive expansions » studied in [**1**]$_2$) is a left-exact orthoquaternary (and $C\Theta$-) category.

5.2 – A left-exact $C\Theta$-category verifying A6 and not A7 is the category in [**2**] pag. 199.

5.3 – A left-exact $C\Theta$-category verifying A7 and not A6 is the following: non-zero objects are A, B, C, D, E, F, G, H; non-zero and non-identical maps are:

(1) monics: $m_1: A \to B$, $m_2: A \to C$, $m_3: B \to D$, $m_4: C \to D$, $m_5: A \to D$, $m_6: E \to H$, $m_7: F \to G$;

(2) conormal epics: $p_1: B \to E$, $p_2: C \to F$, $p_3: D \to G$, $p_4: D \to H$;

(3) other maps: $f_1: B \to H$, $f_2: C \to G$;

with the following non-zero compositions (and their consequences): $m_5 = m_3 \cdot m_1 = m_4 m_2$, $f_1 = p_4 m_3 = m_6 p_1$; $f_2 = p_3 m_4 = m_7 p_2$.

5.4 – A left exact $C\Theta$-category C such that in (C^0, M_C) there exist decompositions which don't have unions is the following.

Let I be the family of non-void open sets of the cofinite topology of $\boldsymbol{N}$ ordered by inclusion. Non-zero objects of C are H, L, $(K_i)_{i\in I}$; non-zero, non-identical maps are

(1) normal monics: $n_i: H \to K_i$, $i \in I$;

(2) other monics: $m_{ij}: K_i \to K_j$, $i, j \in I$, $i \leqslant j$;

(3) conormal epics: $p_i: K_i \to L$, $i \in I$;

with the following compositions:

$$m_{ij} n_i = n_j\,, \qquad p_i n_i = 0\,, \qquad m_{jk} m_{ij} = m_{ik}\,, \qquad p_j m_{ij} = p_i\,.$$

Let $i_0 = \boldsymbol{N}$. Then $(n_{i_0}, \tilde{p}_{i_0})$ is a decomposition of K_{i_0} in (C^0, M_C) but for any $j \in I$, $n_{i_0} < m_{ji_0}$, $\tilde{p}_{i_0} < m_{ji_0}$ so $n_{i_0} \cup \tilde{p}_{i_0}$ doesn't exist in K_{i_0}.

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Sunto

Decomposizioni in reticoli e semigruppi idempotenti sono definite in modo da fornire una « buona » definizione di esattezza di sequenze corte e funtori nel contesto delle categorie esatte sinistre dotate di simmetrizzazione ortoquaternaria o inversa.

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