Riv. Mat. Univ. Parma (4) 6 (1980), 333-345

REMIGIO RUSSO (*)

Continuous dependence for the viscous compressible fluids in unbounded domains (**)

1 - Introduction

In a previous paper [7], by using the weight function method [6], we established a uniqueness theorem for classical solutions of the equations governing the motion of a viscous compressible fluid in unbounded domains, removing some « classical » conditions of convergence at large spatial distance, wich are *a priori* artificial both from the physical and mathematical point of view. More precisely, denoting respectively by $\boldsymbol{v}$, ϱ, T, p, $\boldsymbol{f}$, Ω, τ, r, velocity, density, temperature, pressure, body force, the region of motion, a positive number and the spatial distance from some fixed point, and putting

$$I_1 = \{\boldsymbol{v},\ \nabla\boldsymbol{v},\ \nabla\cdot\boldsymbol{v},\ \frac{1}{\sqrt{\varrho}}\,\frac{\partial\boldsymbol{v}}{\partial t},\ \frac{\boldsymbol{v}\cdot\nabla\boldsymbol{v}}{\sqrt{\varrho}},\ \frac{\boldsymbol{v}\cdot\nabla\varrho}{\varrho},\ \frac{\nabla\varrho}{\sqrt{\varrho}},\ \frac{1}{\varrho}\,\frac{\partial\varrho}{\partial t},\ \nabla T,$$

$$\frac{\boldsymbol{v}\cdot\nabla T}{\sqrt{\varrho}},\ \frac{1}{\varrho},\ \frac{1}{\sqrt{\varrho}}\,\frac{\partial\boldsymbol{f}}{\partial\varrho},\ \frac{1}{\sqrt{\varrho}}\,\frac{\partial p}{\partial\varrho}\nabla\cdot\boldsymbol{v},\ \frac{1}{\sqrt{\varrho}}\,\frac{\partial p}{\partial T}\nabla\cdot\boldsymbol{v},\ \frac{1}{\sqrt{\varrho}}\,\frac{\partial p}{\partial\varrho},\ \frac{1}{\varrho}\,\frac{\partial p}{\partial T}\}\,,$$

$$I_2 = \{\frac{\partial p}{\partial\varrho},\ \frac{1}{\sqrt{\varrho}}\,\frac{\partial p}{\partial T},\ \frac{p}{\sqrt{\varrho}},\ \frac{\nabla\cdot\boldsymbol{v}}{\sqrt{\varrho}},\ \frac{\nabla\boldsymbol{v}}{\sqrt{\varrho}},\ \varrho\}\,,$$

we prove that, in order to get uniqueness, it is sufficient to require that

---

(*) Indirizzo: Istituto di Matematica, Università, Via Mezzocannone 8, 80134 Napoli, Italy.

(**) Work performed under the auspices of G.N.F.M. (C.N.R.). — Ricevuto: 19-III-1979.

(I) $|F| \leq Mr^n$, $M = \text{const} > 0$, in $\Omega_\tau = \Omega \times [0, \tau]$, with $F \in I_1 U I_2$ and

$$n = \begin{cases} 1/2\,, & F \in I_1 \\ 1/4\,, & F \in I_2\,. \end{cases}$$

I should stress that such assumptions are much weaker than those of classical Graffi's paper [2]$_1$, where it is required $n = 0$ for $F \in I_1 U I_2 - \{1/\varrho\}$ and $\inf \varrho > 0$ in Ω_τ. This last condition appears physically immotivated, as the same Author admits.

In the present paper we investigate the question in the more general context of continuous dependence upon the initial data, boundary data and body force (1) for solutions of the equations of viscous compressible fluids. Precisely, after a brief section (n. **2**) devoted to preliminaires and to statement of the problem, in section **3**, following the methods outlined in [**1**]$_{1,2,3}$ we obtain a continuous dependence theorem with respect to the weighted norm

$$\text{(II)} \qquad \int_\Omega g\{\varrho u^2 + \sigma^2 + \varrho\theta^2\}\, \mathrm{d}\Omega$$

where $\boldsymbol{u}$, σ, θ are, respectively, the velocity, density and temperature perturbations, $g = \exp\,[\alpha(t + t_0)^\gamma r^k]$, α and γ are positive constants and $k \in (0, 2 - \varepsilon)$ with $\varepsilon \in (0, 2)$. The statement is proved under the assumptions that (I) holds with

$$n = \begin{cases} k/2\,, & F \in I_1 - \{1/\varrho\} \\ k/4\,, & F \in I_2\,. \end{cases}$$

$1/\varrho \leq Mr^\varepsilon$, and for solutions belonging to $L^2(\Omega, g)$ (2), together with their first derivatives and data.

Successively (sect. **4**), from the theorem proved in section **3**, we get a continuous dependence theorem with respect to a suitable nonweighted metric, related to continuous dependence of Hölder type.

---

(1) Continuous dependence theorems for classical solutions of the equations of the viscous compressible fluids in bounded domains are given in [**4**], [**6**]. Till now, however, we know nothing about similar theorems in unbounded domains. In R^n and for inviscid fluids see [**4**].

(2) The space $L^2(\Omega, g)$ is defined as the completion in the norm $\int_\Omega g u^2\, \mathrm{d}\Omega$ of the functions which are finite in Ω [**8**].

Furthermore, in section 4, we shall give a continuous dependence theorem in L^2-norm under the hypothesis that the initial data and body force belong to $L^2(\Omega)$.

2 - Statement of the problem

In the theory of viscous compressible flow a fluid filling a domain Ω (3) is governed by the equations

$$(1)_1 \qquad \varrho\left\{\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right\} = -\nabla p + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{v} + \mu\Delta\boldsymbol{v} + \boldsymbol{f}(p, \varrho, t)\,,$$

$$(1)_2 \qquad \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{v}) = 0\,,$$

$$(1)_3 \qquad \varrho\left\{\frac{\partial E}{\partial t} + \boldsymbol{v}\cdot\nabla E\right\} = (-p + \lambda\nabla\cdot\boldsymbol{v})\boldsymbol{I}\cdot\boldsymbol{D} + 2\mu\boldsymbol{D}\cdot\boldsymbol{D} - \nabla\boldsymbol{q}\,,$$

$$(1)_{4,5,6} \qquad \boldsymbol{q} = -k\nabla T\,, \qquad p = p(\varrho, T)\,, \qquad E = E(\varrho, T)\,,$$

where the symbols have their usual meaning [9]$_2$.

For the sake of formal simplicity; we shall suppose that equation $(1)_6$ has the special form

$$E = c_V T\,, \qquad c_V = \text{const} > 0\,.$$

We can however get the same results on the more general assumption $c_V = (\partial E/\partial T)_\varrho$ be positive [7].

Let $\mathscr{T}$ be the class of regular solutions $(\boldsymbol{v}, \varrho, T)$ of (1) such that $(\tau > 0)$:

(i) if $(\boldsymbol{v}, \varrho, T)$, $(\boldsymbol{v}+\boldsymbol{u}, \varrho+\sigma, T+\theta)$ belong to $\mathscr{T}$, then $\boldsymbol{u}$, σ, θ, $\boldsymbol{f}$, p, belong to $L^2(\Omega, \exp[\alpha(t+t_0)^\gamma r^k)$, with $k\in(0, 2-\varepsilon)$ and $\varepsilon\in(0,2)$, together with their first derivatives;

(3) The boundary $\partial\Omega$ is assumed as smooth as required by the validity of the divergence theorem. If Ω is unbounded we assume that Ω is the exterior of a closed fixed region Ω_0 ($\subset R^3$), containing the unit sphere. This assumption on Ω (unbounded) is made for the sake of simplicity [1]$_1$.

(ii) $\exists M > 0$: $1/\varrho \leqslant M r^{\varepsilon}$ and $|F| \leqslant M r^{n}$, for $F \in I_1 - \{1/\varrho\} \cup I_2$ and

$$n = \begin{cases} k/2\,, & F \in I_1 - \{1/\varrho\} \\ k/4\,, & F \in I_2 \end{cases} \qquad \text{in } \Omega_\tau;$$

(iii) f and p are as smooth as required by the mean theorem.

Let $V = (v, \varrho, T)$, $V + U = (v + u, \varrho + \sigma, T + \theta)$ two elements of $\mathscr{T}$. Denoting by $((v_0, \varrho_0, T_0), (v_0 + u_0, \varrho_0 + \sigma_0, T_0 + \theta_0))$, $((v_\Sigma, \varrho_\Sigma, T_\Sigma), (v_\Sigma + u_\Sigma, \varrho_\Sigma + \sigma_\Sigma, T + \theta_\Sigma))$ [4], $(f, f + \varphi)$ the initial data, boundary data and body force, respectively, corresponding to the above solutions, we ask if, provided that the perturbation U_0 of the data associated to V is « small », the perturbation U is itself sufficiently « small », that is the motion $V + U$ is not much different from V. It is clear that the meaning of the word « small » is related to a suitable measure for the perturbations of the solutions and the data, that is to metrics [**5**]. In the sequel we shall consider weighted as well as non-weighted metrics. Precisely, we shall prove continuous dependence with respect to the metrics (II), L^2 and continuous dependence of the type (Hölder) [5]

$$\sup\left(|U_0| + |U_\Sigma| + |\varphi|\right) < \delta \;\Rightarrow \int_{\Omega_R} e\,\mathrm{d}\Omega_R + \int_0^\tau \mathrm{d}t \int_{\Omega_R} \{(\nabla u)^2 + (\nabla\theta)^2\}\,\mathrm{d}\Omega_R < \delta^p,$$

$$\text{(III)} \qquad \Omega_R = S_R/\Omega_0\,, \quad \forall R \geqslant R_0\,, \quad \forall t \in [0, T]\,, \quad R_0 = \inf\{R : S_R \supseteq \Omega_0\}$$

$$|U_0| = |\sqrt{\varrho}\,u_0| + |\sigma_0| + |\sqrt{\varrho}\,\theta_0|, \quad |U_\Sigma| = |u_\Sigma| + |\sigma_\Sigma| + |\theta_\Sigma|,$$

where $e = \varrho u^2 + \sigma^2 + \varrho\theta^2$ and S_R denote the ball of radius R centered in $\mathring{\Omega}$.

Now, we collect some relations wich will be frequentelly used in the sequel.

Let A, B, C, D be vector fields and f, g, E scalar functions with $E > 0$. Then [6]

$$(2)_1 \qquad \nabla\cdot(fA) = A\cdot\nabla f + f\nabla\cdot A\,,$$

$$(2)_2 \qquad f\nabla(\nabla\cdot A) = \nabla\cdot[f(\nabla\cdot A)A] - f(\nabla\cdot A)^2 + (\nabla\cdot A)A\cdot\nabla f\,,$$

$$(2)_3 \qquad fA\cdot\Delta A = \nabla\cdot[f\nabla A\cdot A - A^2\nabla f] - f(\nabla A)^2 + A^2\Delta f\,,$$

---

(4) ϱ is assigned $\forall(P, t) \in \partial\Omega\times[0, \tau]$: $v\cdot n \leqslant 0$ [**7**].

(5) We shall successively observe this continuous dependence is given by a suitable metric [**1**]$_3$.

(6) Of course such vector fields and scalar functions are assumed smooth enough and the product $ABCD$ suitably carried out.

$$(2)_4 \qquad fl\Delta l = \nabla\cdot[fl\nabla l - l^2\nabla f] - f(\nabla l)^2 + l^2\Delta f\,,$$

$$(3) \qquad g\boldsymbol{A}\cdot\boldsymbol{B}\boldsymbol{C}\cdot\boldsymbol{D} \ll g\frac{A^2}{2j} + \frac{jr^k}{2}\,gB^2\,, \qquad |CD| \ll r^{k/2}\,, \quad j>0\,,$$

$$(4)_1 \qquad g\boldsymbol{A}\boldsymbol{B}\cdot\boldsymbol{C} \leqslant gE\frac{A^2B^2}{2jE} + jg\frac{C^2}{2} \ll g\frac{EA^2}{8j^2\delta} + \frac{\delta}{2}r^kEA^2 + \frac{jgC^2}{2}\,,$$

$$(4)_2 \qquad B/\sqrt{E} \ll r^{k/4}\,, \qquad j,\delta>0\,.$$

We shall finally note that the sign $\ll$ in the formula $F\ll G$, where F and G depend on a function $u(x)$, means that $F\leqslant cG$, where the constant $c>0$ does not depend on $u(x)$.

3 - Continuous dependence with respect to a weighted-norm

As can be easily checked, the difference motion $(\boldsymbol{u}, \sigma, \theta)$ obeys the equations [7]

$$(5)_1 \qquad \frac{1}{2}\frac{\partial}{\partial t}(g\varrho u^2) = \frac{1}{2}gu^2\frac{\partial\varrho}{\partial t} + \frac{1}{2}\varrho u^2\frac{\partial g}{\partial t} - g\sigma\boldsymbol{u}\cdot\frac{\partial}{\partial t}(\boldsymbol{v}+\boldsymbol{u})$$

$$- g\varrho\boldsymbol{u}\cdot\nabla(\boldsymbol{v}+\boldsymbol{u})\cdot\boldsymbol{u} - g\sigma(\boldsymbol{v}+\boldsymbol{u})\cdot\nabla(\boldsymbol{v}+\boldsymbol{u})\cdot\boldsymbol{u} + g\boldsymbol{\psi}\cdot\boldsymbol{u}$$

$$- g\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u} + g\pi\nabla\cdot\boldsymbol{u} + \pi\boldsymbol{u}\cdot\nabla g - (\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{u}\cdot\nabla g$$

$$- (\lambda+\mu)g(\nabla\cdot\boldsymbol{u})^2 - \mu g(\nabla\boldsymbol{u})^2 + u^2\Delta g + g\boldsymbol{\varphi}(x, \varrho+\sigma, t)\cdot\boldsymbol{u}$$

$$+ \nabla\cdot[(\lambda+\mu)g(\nabla\cdot\boldsymbol{u})\boldsymbol{u} + \mu g\nabla\boldsymbol{u}\cdot\boldsymbol{u} - \mu u^2\nabla g - g\pi\boldsymbol{u}]\,,$$

$$(5)_2 \qquad \frac{1}{2}\frac{\partial}{\partial t}(g\sigma^2) = \frac{1}{2}\sigma^2\frac{\partial g}{\partial t} - g\varrho\sigma\nabla\cdot\boldsymbol{u} - g\sigma\boldsymbol{u}\cdot\nabla\varrho - \frac{1}{2}g\sigma^2\nabla\cdot(\boldsymbol{v}+\boldsymbol{u})$$

$$+ \frac{1}{2}\sigma^2(\boldsymbol{v}+\boldsymbol{u})\cdot\nabla g - \frac{1}{2}\nabla\cdot[g\sigma^2(\boldsymbol{v}+\boldsymbol{u})]\,,$$

---

[7] In (5), without loss of generality, we have set $c_V = 1$.

$$(5)_3 \qquad \frac{1}{2}\frac{\partial}{\partial t}(g\varrho\theta^2) = \frac{1}{2}\varrho\theta^2\frac{\partial g}{\partial t} + \frac{1}{2}g\theta^2\frac{\partial \varrho}{\partial t} - g\theta\sigma(\boldsymbol{v}+\boldsymbol{u})\cdot\nabla(T+\theta)$$

$$- g\varrho\theta\boldsymbol{u}\cdot\nabla(T+\theta) - g\theta\pi\nabla\cdot\boldsymbol{v} - g\theta p(\varrho+\sigma, T+\theta)\nabla\cdot\boldsymbol{u}$$

$$+ \lambda g\theta\nabla\cdot(\boldsymbol{v}+\boldsymbol{u})\nabla\cdot\boldsymbol{u} + \lambda g\theta\nabla\cdot\boldsymbol{v}\,\nabla\cdot\boldsymbol{u} + 2\mu g\boldsymbol{d}\cdot(\boldsymbol{D}+\boldsymbol{d})$$

$$+ 2\mu g\theta\boldsymbol{D}\cdot\boldsymbol{d} - g\varrho\theta\boldsymbol{v}\cdot\nabla\theta - \chi g(\nabla\theta)^2 + \chi\theta^2\Delta g + \chi\nabla\cdot[g\theta\nabla\theta - \theta^2\nabla g]\,,$$

where $g = \exp[\alpha(t+t_0)^\gamma r^k]$, and the initial and boundary data

$$(6)\quad \begin{aligned} &\boldsymbol{u}(P,t) = \boldsymbol{u}_\Sigma(P,t)\,, \quad \theta(P,t) = \theta_\Sigma(P,t)\,, \quad \forall(P,t)\in\partial\Omega\times[0,\tau]\,,\\ &\sigma(P,t) = \sigma_\Sigma(P,t)\,, \quad \forall(P,t)\in\partial\Omega\times[0,\tau]:\ \boldsymbol{v}\cdot\boldsymbol{n}\leqslant 0 \end{aligned}$$

$$(7)\quad \boldsymbol{u}(P,0) = \boldsymbol{u}_0(P)\,, \quad \sigma(P,0) = \sigma_0(P)\,, \quad \theta(P,0) = \theta_0(P)\,, \quad \forall P\in\Omega\,.$$

In (5) we have set ([8])

$$\pi = p(\varrho+\sigma, T+\theta) - p(\varrho,\theta)\,,$$

$$\boldsymbol{\psi} = \boldsymbol{f}(x, \varrho+\sigma, t) - \boldsymbol{f}(x,\varrho,t)\,,$$

$$\boldsymbol{d} = [\nabla(\boldsymbol{v}+\boldsymbol{u})]^s - [\nabla\boldsymbol{v}]^s\,.$$

By (iii) it turns out that

$$\pi = \sigma\frac{\partial p}{\partial\varrho}(\varrho + h_1\sigma, T) + \theta\frac{\partial p}{\partial T}(\varrho, T+h_2\theta) = a\sigma + b\theta \qquad (h_1, h_2 > 0)\,,$$

$$\boldsymbol{\psi}\cdot\boldsymbol{u} = \sum_{j=1}^{3}\frac{\partial\psi^j}{\partial\varrho}(x, \varrho + l_j\sigma, t)u_j\sigma \leqslant 2\left|\frac{\partial\boldsymbol{\psi}}{\partial\varrho}\right||\sigma||\boldsymbol{u}| \qquad (l_j > 0)\,.$$

By the properties of the weight function g we get

$$(8)\quad \begin{aligned} &\nabla g = -\alpha(t+t_0)^\gamma\, k r^{k-1}\,\boldsymbol{e}_r\,,\\ &\frac{\partial g}{\partial t} = -\alpha\gamma(t+t_0)^{\gamma-1}r^k g\,. \end{aligned}$$

([8]) $\boldsymbol{A}^s$ is the symmetric part of a second order tensor $\boldsymbol{A}$.

Now, exploiting the inequality (3), we increase the following terms of (5)

$$\text{(I)}\quad \begin{gathered} \frac{1}{2} g\varrho u^2 \Big[\frac{1}{\varrho}\frac{\partial \varrho}{\partial t}\Big], \quad -g\sigma\sqrt{\varrho}\,\boldsymbol{u}\cdot\Big[\frac{1}{\sqrt{\varrho}}\frac{\partial}{\partial t}(\boldsymbol{v}+\boldsymbol{u})\Big], \qquad -g\varrho\boldsymbol{u}\cdot[\nabla(\boldsymbol{v}+\boldsymbol{u})]\cdot\boldsymbol{u}, \\ -g\sigma\Big[\frac{1}{\sqrt{\varrho}}(\boldsymbol{v}+\boldsymbol{u})\cdot\nabla(\boldsymbol{v}+\boldsymbol{u})\Big]\cdot\sqrt{\varrho}\,\boldsymbol{u}, \quad -g\sigma\sqrt{\varrho}\,\boldsymbol{u}\cdot\Big[\frac{1}{\sqrt{\varrho}}\nabla\varrho\Big], \\ 2g\sigma\sqrt{\varrho}\,\Big[\frac{1}{\sqrt{\varrho}}\sum_{j=1}^{3}\frac{\partial\psi^j}{\partial\varrho}\Big]; \end{gathered}$$

$$\text{(II)}\quad -\frac{1}{2} g\sigma^2[\nabla\cdot(\boldsymbol{v}+\boldsymbol{u})];$$

$$\text{(III)}\quad \begin{gathered} \frac{1}{2} g\varrho\theta^2\Big[\frac{1}{\varrho}\frac{\partial\varrho}{\partial t}\Big], \quad -g\sigma\sqrt{\varrho}\,\theta\Big[\frac{1}{\sqrt{\varrho}}(\boldsymbol{v}+\boldsymbol{u})\cdot\nabla(T+\theta)\Big], \quad -g\varrho\theta\boldsymbol{u}\cdot[\nabla(T+\theta)], \\ -g\sigma\sqrt{\varrho}\,\theta\Big[\frac{a}{\sqrt{\varrho}}\nabla\cdot\boldsymbol{v}\Big], \qquad -g\varrho\theta^2\Big[\frac{b}{\varrho}\nabla\cdot\boldsymbol{v}\Big], \end{gathered}$$

where the first factor and [·] correspond respectively to A and CD of (3). In (I), (II), (III) the positive constant j is expressed respectively by ξ, β, ν.

In the same way as before we apply the Cauchy inequality (4) to the terms of (5)

(I)′ $-g\theta[p]\nabla\cdot\boldsymbol{u}$, $\lambda g\theta[\nabla\cdot(\boldsymbol{v}+\boldsymbol{u})]\nabla\cdot\boldsymbol{u}$, $\lambda g\theta[\nabla\cdot\boldsymbol{v}]\nabla\cdot\boldsymbol{u}$, $-g[\boldsymbol{b}]\theta\nabla\cdot\boldsymbol{u}$;

(II)′ $2\mu g\theta\boldsymbol{d}\cdot[\boldsymbol{D}+\boldsymbol{d}]$, $2\mu g\theta[\boldsymbol{D}]\cdot\boldsymbol{d}$;

where θ, [·] correspond respectively to A, B of (4) and $E=\varrho>0$. In (I)′, (II)′, the positive constants j and δ are expressed respectively by (ζ,ν) and (η,ν).

To give an idea of as (3)-(4) are applied, we carry out the (3) to $-g\sigma\boldsymbol{u}\cdot(\partial/\partial t)(\boldsymbol{v}+\boldsymbol{u})$ and (4) to $2\mu g\theta\boldsymbol{D}\cdot\boldsymbol{d}$

$$-g\sigma\boldsymbol{u}\cdot\frac{\partial}{\partial t}(\boldsymbol{v}+\boldsymbol{u}) = -g\sigma\sqrt{\varrho}\,\boldsymbol{u}\cdot\Big[\frac{1}{\sqrt{\varrho}}\frac{\partial}{\partial t}(\boldsymbol{v}+\boldsymbol{u})\Big] \leqslant g\frac{\sigma^2}{2\xi}+\frac{\xi}{2}r^k g\varrho u^2,$$

since $\dfrac{1}{\sqrt{\varrho}}\Big|\dfrac{\partial}{\partial t}(\boldsymbol{v}+\boldsymbol{u})\Big|\leqslant r^{k/2}$ (for (ii));

$$2\mu g\theta\boldsymbol{D}\cdot\boldsymbol{d}\leqslant g\mu\frac{\theta^2D^2}{\eta}+\eta\mu d^2\leqslant\frac{g\mu\varrho\theta^2}{2\eta^2\gamma}+\frac{\nu\mu r^k g\theta^2}{2}+\eta\mu(\nabla\boldsymbol{u})^2,$$

since $|d|\leqslant|\nabla\boldsymbol{u}|$ and $\dfrac{1}{\sqrt{\varrho}}|\nabla\boldsymbol{v}|\leqslant r^{k/4}$ (for (ii)).

Now, taking into account (ii) and Cauchy inequality, we increase the other terms of (3)

(a) $$-g\varrho\, \boldsymbol{v}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u} \leqslant \frac{g\varrho^2 v^2 u^2}{2\eta} + \frac{\eta g(\nabla\boldsymbol{u})^2}{2} \ll \frac{g\varrho u^2}{8\eta^2\xi} + \frac{\xi r^k \varrho g u^2}{2} + \frac{\eta g(\nabla\boldsymbol{u})^2}{2},$$

(b) $$-g\sigma\varrho\,\nabla\cdot\boldsymbol{u} \leqslant \frac{g\varrho^2\sigma^2}{2\zeta} + \frac{\zeta g(\nabla\cdot\boldsymbol{u})^2}{2} \ll \frac{g\sigma^2}{8\zeta^2\beta} + \frac{\beta r^k g\sigma^2}{2} + \frac{\zeta g(\nabla\cdot\boldsymbol{u})^2}{2},$$

(c) $$-g\varrho\theta\boldsymbol{v}\cdot\nabla\theta \leqslant \frac{g\theta^2\varrho^2 v^2}{2\psi} + \frac{\psi g(\nabla\theta)^2}{2} \ll \frac{g\varrho\theta^2}{8\psi^2\nu} + \frac{\nu r^k g\varrho\theta^2}{2} + \frac{\psi g(\nabla\theta)^2}{2},$$

(d) $$ga\sigma(\nabla\cdot\boldsymbol{u}) \leqslant \frac{ga^2\sigma^2}{2\zeta} + \frac{\zeta g(\nabla\cdot\boldsymbol{u})^2}{2} \ll \frac{g\sigma^2}{8\zeta^2\beta} + \frac{\beta r^k g\sigma^2}{2} + \frac{\zeta g(\nabla\cdot\boldsymbol{u})^2}{2}.$$

Putting $h = \alpha(\tau + t_0)^\gamma k$ and proceeding in the same way as before, we get

(e) $$a\sigma\boldsymbol{u}\cdot\nabla g = hr^{k-1} ga\sigma u_r \leqslant \frac{ga^2\sigma^2}{2\eta} + \frac{\eta h^2 r^{2(k-1)} g u^2}{2}$$
$$\ll \frac{g\sigma^2}{8\eta^2\beta} + \frac{\beta r^k g\sigma^2}{2} + \frac{\eta h^2 r^{2(k-1)} g u^2}{2},$$

(f) $$b\theta\boldsymbol{u}\cdot\nabla g \leqslant \frac{gb^2\theta}{2\eta} + \frac{\eta h^2 r^{2(k-1)} g u^2}{2} \ll \frac{g\varrho\theta^2}{8\eta^2\nu} + \frac{\nu r^k g\varrho\theta^2}{2} + \frac{\eta h^2 r^{2(k-1)} g\boldsymbol{u}^2}{2},$$

(g) $$-(\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{u}\cdot\nabla g \leqslant \frac{\zeta(\lambda+\mu)g(\nabla\cdot\boldsymbol{u})^2}{2} + \frac{(\lambda+\mu)h^2 r^{2(k-1)} g\boldsymbol{u}^2}{2\zeta}.$$

To deduce (a)-(f) we have used the relations

$$\sqrt{\varrho}\,|\boldsymbol{v}|,\qquad \varrho,\qquad |a|,\qquad |b|/\sqrt{\varrho} \ll r^{k/4},$$

wich are contained in (ii).

Now, we notice that

(a)′ $$\frac{1}{2}\sigma^2(\boldsymbol{v}+\boldsymbol{u})\cdot\nabla g + \frac{1}{4}\frac{\sigma^2\,\partial g}{\partial t} \leqslant \frac{1}{2}\alpha(t+t_0)^{\gamma-1} g\sigma^2\left[k(\tau+t_0)(\boldsymbol{v}+\boldsymbol{u})_r - \frac{\gamma r}{2}\right],$$

(b)′ $$\mu u^2 \Delta g + \frac{(\lambda+\mu)}{2\zeta} h^2 g r^{2(k-1)} u^2 + \eta h^2\, r^{2(k-1)} g u^2 + \frac{1}{4}\varrho u^2 \frac{\partial g}{\partial t}$$
$$\ll h^2\left(\mu + \frac{\lambda+\mu}{2\zeta} + \eta\right) r^{2(k-1)+\varepsilon} g\varrho u^2 - \frac{\alpha\gamma t_0 r^k g\varrho u^2}{4},$$

(c)' $$\chi\theta^2\Delta g + \frac{1}{4}\varrho\theta^2\frac{\partial g}{\partial t} \ll \chi h^2 r^{2(k-1)+\varepsilon} g\varrho\theta^2 - \frac{\alpha\gamma t_0 r^k g\varrho\theta^2}{4},$$

from wich, choosing

$$\gamma = \max\left\{2k(\tau + t_0),\ \frac{4h^2}{\alpha t_0}(\mu + \lambda + \mu/2\zeta + \eta),\ 4\chi h^2/\alpha t_0\right\}$$

and taking account of the fact that $r^k \geqslant r^{2(k-1)+\varepsilon}$ (for (ii)), it follows that the sums in (a)', (b)', (c)' are nonpositive.

Finally, we increase the term $g\boldsymbol{\varphi}(x, \varrho + \sigma, t)\cdot\boldsymbol{u}$ by (3) and (ii)

(h) $$g\boldsymbol{\varphi}\cdot\boldsymbol{u} \leqslant g\frac{r^\varepsilon\varphi^2}{2} + g\frac{\varrho u^2}{2}.$$

The next step is to integrate over Ω_R and to apply the divergence theorem. Let's note, by the way, that the surface integrals extended over $\partial\Omega_R$, in the limit $R \to \infty$, tend to zero.

Choosing the constant γ as before, from (3)-(4), (a)-(h), (a)'-(c)', the reader should have no trouble in getting the following

(9) $$\frac{d\varepsilon}{dt} \leqslant \left(\frac{3}{\xi} + \frac{1}{8\eta^2\xi} + \frac{1}{2\nu} + \frac{1}{2}\right)\int_\Omega g\varrho u^2\,d\Omega + \left(\frac{2}{\xi} + \frac{1}{4\zeta^2\beta} + \frac{1}{\nu} + \frac{1}{8\eta^2\beta}\right.$$
$$\left. + \frac{1}{4\beta}\right)\int_\Omega g\sigma^2\,d\Omega + \left(\frac{1}{2\zeta^2\nu} + \frac{9}{8\eta^2\nu} + \frac{3}{4\nu} + \frac{1}{8\psi^2\nu}\right)\int_\Omega g\varrho\theta^2\,d\Omega$$
$$+ \left(\frac{13}{\xi} - \frac{\alpha\gamma t_0}{4}\right)\int_\Omega r^k g\varrho u^2\,d\Omega + \left(\frac{7}{4}\beta - \frac{\alpha\gamma t_0}{4}\right)\int_\Omega r^k g\sigma^2\,d\Omega$$
$$+ \left(\frac{25}{4}\nu - \frac{\alpha\gamma t_0}{4}\right)\int_\Omega r^k g\varrho\theta^2\,d\Omega + \left[\eta\left(\frac{1}{2} + 2\mu^2\right) - \omega\mu\right]\int_\Omega g(\nabla u)^2\,d\Omega$$
$$+ \left[\left(2 + \lambda^2 + \frac{\lambda+\mu}{2}\right)\zeta - (\lambda + \mu)\right]\int_\Omega g(\nabla\cdot\boldsymbol{u})^2\,d\Omega + \left(\frac{\psi}{2} - \omega\chi\right)\int_\Omega g(\nabla\theta)^2\,d\Omega$$
$$- \mu(1-\omega) + \int_\Omega g(\nabla\boldsymbol{u})^2\,d\Omega - \chi(1-\omega)\int_\Omega g(\nabla\theta)^2\,d\Omega$$
$$+ \int_\Omega g r^\varepsilon\varphi^2(x, \varrho + \sigma, t)\,d\Omega + \int_\Sigma \mathscr{F}(x, t)\cdot\boldsymbol{n}\,d\Sigma,$$

where $\omega \in (0,1)$ and

$$\mathscr{F}(x,t) = (\lambda+\mu)g(\nabla\cdot\boldsymbol{u})\boldsymbol{u} + \mu g\nabla\boldsymbol{u}\cdot\boldsymbol{u} - \mu u^2\nabla g - g\pi\boldsymbol{u} - \frac{(\boldsymbol{v}+\boldsymbol{u})g\sigma^2}{2} + \chi[g\theta\nabla\theta - \theta^2\nabla g]\,.$$

Now, by the Clausius-Duhem inequality, is $\mu > 0$, $3\lambda + 2\mu > 0$ **[9]**$_2$, and so $\lambda + \mu > 0$.

Choosing

$$\xi\,,\quad \beta\,,\quad \nu \leqslant \frac{4\gamma t_0}{7}\,,\quad \eta \leqslant \frac{2\omega\mu}{1+4\mu^2}\,,\quad \psi \leqslant 2\varepsilon\chi\,,\quad \zeta \leqslant \frac{2(\lambda+\mu)}{4+2\lambda^2+\lambda+\mu}$$

and putting

$$k = \max\Big\{\frac{3}{\xi} + \frac{1}{8\eta^2\xi} + \frac{1}{\nu}\,,\ \frac{2}{\xi} + \frac{1}{4\zeta^2\beta} + \frac{1}{\nu} + \frac{1}{8\eta^2\beta} + \frac{1}{4\beta}\,,\ \frac{1}{2\zeta^2\nu} + \frac{9}{8\eta^2\nu} + \frac{3}{4\nu} + \frac{1}{8\psi^2\nu}\Big\}\,,$$

by (9) we have

$$\frac{d\varepsilon}{dt} + (1-\omega)\int_\Omega g[(\nabla\boldsymbol{u})^2 + (\nabla\theta)^2]\,d\Omega \ll \varepsilon + \int_\Omega g r^\varepsilon \varphi^2\,d\Omega + \int_\Sigma \mathscr{F}(x,t)\cdot\boldsymbol{n}\,d\Sigma\,. \tag{10}$$

From (10), by the use of the well known Gronwall's lemma, it follows continuous dependence for the solutions of (1) belonging to $\mathscr{T}$ and such that

$$\sup_\Sigma\{|\nabla\boldsymbol{u}|,\, |\nabla\cdot\boldsymbol{u}|,\, \pi,\, |\nabla\theta|\} \leqslant H \qquad (H>0)\,, \tag{11}$$

with respect to the norms

$$L^2(\Omega, g)\,,\qquad L^2(\Omega, r^\varepsilon g)\,,\qquad \int_\Omega g e\,d\Omega + (1-\omega)\int_{\Omega_\tau} g[(\nabla\boldsymbol{u})^2 + (\nabla\theta)^2]\,d\Omega_\tau\,,$$

of initial data, body force and solutions, respectively.

By (10), as a consequence, we get the following *uniqueness theorem*.

Theorem 1. *If in $\mathscr{T}$ there is a flow $(\boldsymbol{v}, \varrho, T)$ with initial data $V_0 = (\boldsymbol{v}_0, \varrho_0, T_0)$, boundary data $(\boldsymbol{v}_\Sigma, \varrho_\Sigma, T_\Sigma)$ and body force $\boldsymbol{f}(P, \varrho, t)$, it is unique.*

4 - Continuous dependence with respect to non weighted metrics

Concerning the above type of continuous dependence, we should remark that it have not much meaning from the physical point of view, since the weighted-norm ε is not a suitable measure of the perturbation $\boldsymbol{U}$. Nevertheless, from (10) we can get continuous dependence upon data for solutions of (1) belonging to $\mathscr{T}$ in the sense of (III), and in L^2-norm. Precisely, the following theorems hold.

Theorem 2. *If a solution of* (1) *with initial data* V_0 *and boundary data* V_Σ *belongs to* $\mathscr{T}$ *with* $\varepsilon \in (1, 2)$ *and* (11) *holds, then it depends continuously on the initial data, boundary data and body force in the sense of* (III).

Theorem 3. *Let* V, *and* $V + U$ *be two solutions of* (1) *belonging to* $\mathscr{T}$. *If* $\boldsymbol{u}_0, \varrho_0, T_0 \in L^2(\Omega)$ *and* $\boldsymbol{\varphi} \in L^2(\Omega_\tau)$, *then*

$$\boldsymbol{u}, \sigma, \theta \in L^\infty(0, \tau; L^2(\Omega)) , \qquad \nabla \boldsymbol{u} , \qquad \nabla\theta \in L^2(\Omega_\tau)$$

$$\lim_{R\to\infty} R^2 \int_{\Sigma_R} (u^2 + \theta^2)\, \mathrm{d}\Sigma_R = 0 .$$

If $\varepsilon \in (1, 2)$ we increase the term $g\boldsymbol{\varphi}\cdot\boldsymbol{u}$ of (5) by using (3)

$$g\boldsymbol{\varphi}\cdot\boldsymbol{u} \leqslant \frac{g\varphi^2}{2c} + \frac{cgr^\varepsilon\varrho u^2}{2} .$$

Now, if we add $(cgr^\varepsilon\varrho u^2)/2$ to $\varrho(u^2/4)(\partial g/\partial t)$ and choose a suitable c, we get a nonpositive term since $r^k \geqslant r^\varepsilon$.

Therefore the relation (10) implies

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \int_\Omega g[(\nabla u)^2 + (\nabla\theta)^2]\, \mathrm{d}\Omega \leqslant \varepsilon + \int_\Sigma \mathscr{F}(x, t)\cdot\boldsymbol{n}\, \mathrm{d}\sigma + \int_\Omega g\varphi^2\, \mathrm{d}\Omega . \tag{12}$$

Integrating (12) from $t = 0$ to $t = \tau$, we have

$$\int_\Omega ge\, \mathrm{d}\Omega + \int_{\Omega_\tau} g[(\nabla\boldsymbol{u})^2 + (\nabla\theta)^2]\, \mathrm{d}\Omega_\tau \leqslant \int_\Omega g(x, 0)e(x, 0)\, \mathrm{d}\Omega + \int_{\Omega_\tau} g\varphi^2\, \mathrm{d}\Omega_\tau + \int_{\Sigma_\tau} \mathscr{F}\cdot\boldsymbol{n}\, \mathrm{d}\Sigma_\tau .$$

Taking into account that $[\mathbf{1}]_2$

$$\int_\Omega g\, \mathrm{d}\Omega \leqslant 4\pi \int_0^{+\infty} \exp[-\alpha t_0^\gamma r^k] r^2\, \mathrm{d}r \leqslant \alpha^{-3/4} , \qquad \sup\{|\boldsymbol{u}_0| + |\boldsymbol{u}_\Sigma| + |\boldsymbol{\varphi}|\} < \delta' ,$$

for $\alpha = \bar{R}^{-k}$, $\bar{R} = \delta^s$, $s < 3/2$ we obtain

$$\int_{\Omega_{\bar{R}}} e \, d\Omega + \int_0^\tau dt \int_{\Omega_{\bar{R}}} [(\nabla \boldsymbol{u})^2 + (\nabla\theta)^2] \, d\Omega \ll [\delta^2 \bar{R}^3 + \delta] , \qquad \forall t \in [0, \tau] ;$$

from this relation we easily recover the Theorem 2.

We should remark that such type of continuous dependence may be reformulated in terms of metrics, by introducing suitable families of quasimetrics [**1**]$_3$.

In order to prove the Theorem 3, we notice that, after a simple integration, the inequality (12) yields

$$\int_{\Omega_R} e \, d\Omega + \int_0^\tau dt \int_{\Omega_R} [(\nabla \boldsymbol{u})^2 + (\nabla\theta)^2] \, d\Omega \ll \exp[\alpha R^k (t_0 + \tau)^\gamma] \int_\Omega e(x, 0) \, d\Omega + \int_{\Omega_\tau} \varphi^2 \, d\Omega_\tau + \sup |u_\Sigma| + \sup |\theta_\Sigma| + \sup |\sigma_\Sigma| . \tag{13}$$

Now, inequality (13) is similar to (14) of [**1**]$_3$ and so, proceeding as in [**1**]$_3$, the Theorem 3 is recovered ([9]).

In conclusion, the author expresses his sincere thanks to Professor S. Rionero for his guidance in his research.

References

[**1**] G. P. Galdi and S. Rionero: [•]$_1$ *Continuous dependence theorems for Navier-Stokes equations in unbounded domains by the weight function method*, Quart. J. Mech. App. Math., in press; [·]$_2$ *Continuous dependence theorems in linear elasticity on exterior domains*, Int. J. Engng. Sci., in press; [•]$_3$ *A priori estimates, continuous dependence and stability for solutions to Navier-Stokes equations on exterior domains*, Riv. Mat. Univ. Parma (4) **5** (1979), 533-556.

[**2**] D. Graffi: [•]$_1$ *Sul teorema di unicità per le equazioni del moto dei fluidi compressibili in un dominio illimitato*, Atti Acc. Sci. Bologna (11) **7** (1960), 1-8; [•]$_2$ *Il teorema di unicità nella dinamica dei fluidi compressibili*, J. Rational Mech. Anal. **2** (1953), 99-106.

([9]) On the other hand the Theorem 3 is immediately recovered letting in (13) first $\alpha \to 0$ and then $R \to \infty$.

[3] R. N. HILLS and R. J. KNOPS, *Continuous dependence for the compressible linearly viscous fluid,* Arch. Rational Mech. Anal. **51** (1973), 54.

[4] T. KATO, *Quasi linear equations of evolution, with application to partial differential equations,* Lecture notes in Mathematics, 488, Springer-Verlag, Berlin 1975.

[5] R. J. KNOPS and E. W. WILKES, *Theory of elastic stability,* Handbuch der Physik, Band VI a/3 1973, Springer-Verlag, Berlin 1975.

[6] S. RIONERO and G. P. GALDI, *The weight function approach to uniqueness of viscous flows in unbounded domains,* Arch. Rational Mech. Anal., in press.

[7] R. RUSSO, *On the uniqueness of viscous compressible fluid motions in unbounded domains,* Meccanica, in press.

[8] V. SEDOV, *Weight Spaces. An imbedding theorem,* Differential'nye Uravneniya **8** (1972).

[9] J. SERRIN: $[\bullet]_1$ *On the uniqueness of compressible fluid motions,* Arch. Rational Mech. Anal. **3** (1959), 271-288; $[\bullet]_2$ *Mathematical principles of classical fluid mechanics,* « Handbuch der Physik », Band VIII-1, Springer-Verlag, Berlin.

Sommario

Usando il metodo della funzione peso si dimostrano dei teoremi di dipendenza continua per le soluzioni regolari delle equazioni dei fluidi viscosi compressibili senza richiedere che la densità sia inferiormente limitata da una costante positiva e con velocità, temperatura, densità e loro derivate prime anche non limitate. Si migliora, inoltre un precedente teorema di unicità.

* * *