Riv. Mat. Univ. Parma (4) 6 (1980), 347-361

CESARE DAVINI (*)

On some special theories in constrained oriented media (**)

Introduction

The introduction in Mechanics of a theory for oriented materials may be regarded both an a refinement of the description offered by the classical continuum model and as an attempt to cover such subjects as rods and shells unitarily. In some sense, it plays to the classical theory the same role as that did to the previous schemes ([1]). Its distinctive feature consists in accounting for some granular aspects of the matter keeping the theory all the analytical advantages of the continuum point of view. This is the course most of the applications have been developed along [**8**] and also the starting point of many approaches to consistent mechanical theories of oriented materials [**2**], [**9**], [**3**], [**1**]$_1$. Many approaches are different and consider different kinematical models. In [**1**]$_2$ Capriz - Podio Guidugli suggested a criterion to order the various contributions on the subject. Noticing that many proposed models can be deduced from a general one, when endowed with suitable internal constraints, they embedded some special cases into the general frame of [**1**]$_1$ and adapted to the purpose the theory of internal constraints given in [**6**]. Here

---

(*) Indirizzo: Istituto di Elaborazione della Informazione C.N.R. Pisa, Facoltà di Ingegneria, Università, 56100 Pisa, Italy.

(**) Ricevuto: 27-III-1979.

([1]) More explicitly, the classical continuum theory provided a finer description than rods and shells and was general enough to include them as special cases. To give some classical examples, we remind St. Venant's special solution for beams, Love's approach to plates and the asymptotic expansions proposed in order to obtain the shell theory from the three-dimensional equations of elasticity (see Green-Zerna [**5**], Chap. 16).

we carry on that idea and consider, among others, the cases of the materials of second grade [**12**], the liquid crystals [**2**] and the director theory of surfaces [**4**]. Besides unifying the matter, this way of looking at the subject helps a systematic criterium to formulate some of the basic axioms in those special theories. Here we refer to the peculiar forms of equations of micromomentum and generalized moment of momentum in those theories. Regarding them as constrained, these equations plainly follow from a version of the momentum equation and from constitutive assumptions that look familiar under the formal view when they are given for the three-dimensional oriented body (sect. **3**). The presence, in the constrained scheme, of reactions and their characterization offer a satisfying understanding of the indeterminacies and of the assumptions about them we sometimes meet in theories for kinematically simpler models, see for instance [**4**], (§ 7, sect. **5**e). As to the indeterminacies, the notion of null state of stress for materials with internal structure plays a special role. Its definition and peculiarities are discussed in section **3** and the algebraic characterization of the null state of stress for each of the examined cases is given in section **4**.

We deal with the subject in a purely mechanical context. The constraints considered here can be described in terms of a class of admissible configurations according to the usual ideas about holonomic constraints. However, because of the constraints, in many cases requirements of simmetry on the wrenching gradients come into the picture from the properties of the differential calculus and the ensuing restrictions on the fluxes elude the abstract approach to the holonomic constraints given in [**1**]$_2$. We leave that view as far as this point is concerned.

2 - Kinematics. Internal constraints

The description of an oriented continuous body may be done according to the general lines traced by W. Noll [**11**] for classical bodies.

Let $\mathcal{E}$ be the three-dimentional Euclidean space and $\mathcal{V}$ be its translation space endowed with the natural inner product. The structure of the oriented body depends on the choice of a class $\mathfrak{D}$ of smooth mappings (*displacements*) of $\mathcal{E}\times\{\mathcal{V}\}$ onto itself

$$\mathfrak{D} = \{\mathfrak{d} = (\mathbf{K}, \mathbf{G}) \colon \mathbf{K}\colon \mathcal{E} \to \mathcal{E} \text{ is a bijection, } \mathbf{G}\colon \mathcal{E} \to \operatorname{Inv}\{\mathcal{V}\}\} ,$$

on which we assume that suitable properties of closure are satisfied. Moreover we suppose that all the isometries of $\mathcal{E}\times\{\mathcal{V}\}$ belong to $\mathfrak{D}$, and, *by definition*, we name isometries the pairs $(\mathbf{K}, \mathbf{G})$ such that

(i) $\mathbf{K}: \mathcal{E} \to \mathcal{E}$ is an isometry for $\mathcal{E}$,

(ii) $\mathbf{G}: \mathcal{E} \to \mathrm{Orth}^+$ expresses the rotation associated with the rigid displacement $\mathbf{K}$.

Then let $\mathcal{B}$ and $\mathfrak{S}$ be a material body and a three-dimentional vector space. We say that $\mathcal{B}_s = \mathcal{B} \times \{\mathfrak{S}\}$ is a structured body of type $\mathfrak{D}$ and dimension $\mathbf{n}$ if there is given a non-empty class $\mathscr{P}$ of mappings (*placements*)

$$\mathscr{P} = \{\mathfrak{p} = (\boldsymbol{\chi}, \boldsymbol{\Gamma}) : \boldsymbol{\chi} : \mathcal{B} \to \mathcal{E},\ \boldsymbol{\Gamma} : \mathcal{B} \to \mathrm{Inv}\,(\mathfrak{S}, \mathcal{V})\} ,$$

such that

$(A)_1$ $\boldsymbol{\chi}$ are injective mappings onto n-dimensional open regions of $\mathcal{E}$,

$(A)_2$ $\mathscr{P} = \mathfrak{D} \circ \mathfrak{p}_0$ for some $\mathfrak{p}_0 \in \mathscr{P}$ (2).

We notice that, from $(A)_2$, $\mathfrak{D}$ defines all the admissible placements of $\mathcal{B}_s$ whenever some reference placement $\mathfrak{p}_0$ is given. Thus the choice of $\mathfrak{D}$ has a constitutive character and the assumptions that $\mathfrak{D}$ is closed and contains all the isometries guarantee to it a requirement of objectivity.

From the definition, the elements of $\mathcal{B}_s$ (material points of $\mathcal{B}_s$) are the pairs $(X, \mathfrak{S})$, where $X \in \mathcal{B}$, so that $\mathcal{B}_s$ consists of a body $\mathcal{B}$ to every point of which a three-dimentional affine body represented by $\mathfrak{S}$ is attached. According to Noll's idea, when the class $\mathscr{P}$ is prescribed we can define the tangent space τ_X at X and describe, approximately, the material points around X by $\tau_X \times \{\mathfrak{S}\}$ (*material neighborhood* of $(X, \mathfrak{S})$). For any placement $\mathfrak{p} = (\boldsymbol{\chi}, \boldsymbol{\Gamma})$, then, a *local placement* of X consists of the triplets $(\nabla\boldsymbol{\chi}/_X,\ \boldsymbol{\Gamma}/_X,\ \nabla\boldsymbol{\Gamma}/_X)$, where $\nabla\boldsymbol{\chi}$ and $\nabla\boldsymbol{\Gamma}$ stand for the linear mappings from τ_x into $\mathcal{V}$ and $\mathrm{Inv}\,(\mathfrak{S}, \mathcal{V})$, respectively, that approximate $\boldsymbol{\chi}$ and $\boldsymbol{\Gamma}$ around X.

For every placement $\mathfrak{p} = (\boldsymbol{\chi}, \boldsymbol{\Gamma})$, we can endow $\mathcal{B}_s$ with a metric structure by transporting the metrics of $\mathcal{E}$ onto $\mathcal{B}$ and, for each X, the inner product of $\mathcal{V}$ onto $\mathfrak{S}$ by means of the linear mapping $\boldsymbol{\Gamma}/_X$. More structure is added by thinking of configurations as elements of $\mathscr{P}$ modulo the isometries of $\mathfrak{D}$. This assumption is basic in the theory of the structured bodies. In fact, it implies that the images $\nabla\boldsymbol{\chi}/_X(\tau_x)$ and $\boldsymbol{\Gamma}/_X(\mathfrak{S})$ in $\mathcal{V}$ are solidal in all the

(2) By $\mathfrak{d} \circ \mathfrak{p}$ we mean the pair $(\mathbf{K}(\boldsymbol{\chi}(\cdot)), \mathbf{G}(\boldsymbol{\chi}(\cdot))\,\boldsymbol{\Gamma}(\cdot))$. An analogous definition holds for the composition product between elements of $\mathfrak{D}$ to which the notion of closure for $\mathfrak{D}$ makes reference.

placements corresponding to the same configuration and brings into the theory that *solidification* of $\mathcal{B}_s$ (see [1]$_2$) which is the more or less remote origin of the moment of momentum equation in the various approaches.

In the following we suppose that a reference placement $\mathfrak{p}_0 = (\boldsymbol{\chi}, \boldsymbol{\Gamma})$ is given and denote by $\mathbf{X}$ the position of X in $\mathfrak{p}_0$. Accordingly, any placement $\mathfrak{p}$ is described by a displacement field $\mathfrak{d} = (\mathbf{K}, \mathbf{G})$ and the local placements by the triplets $\gamma = (\mathbf{F}, \mathbf{G}, \nabla\mathbf{G})$, where $\mathbf{F} = \partial\mathbf{K}/\partial\mathbf{X}$ is the *macro-deformation gradient*, $\mathbf{G}$ is the *micro-deformation* and $\nabla\mathbf{G} = \partial\mathbf{G}/\partial\mathbf{X}$ is the *micro-deformation gradient*. We call the triplet a *site*.

We suppose that a class $\mathscr{P}$ of right - differentiable functions from the reals R into $\mathfrak{D}$ (*processes*) is given. That class is assumed to be closed with respect to suitable operations of composition and to time translations and its section at any time $t \in R$ is $\mathfrak{D}$. We do not dwell upon a more detailed definition of these properties because they can be easily obtained by adapting to this case the general frame of [**6**]. Confining ourselves to their mechanical meaning, we notice that they assure in particular that

(i) combining pieces of processes gives still a process,

(ii) the kinematical feature of the body is time-independent,

(iii) all the admissible placements may be reached at any time t.

We call *flux* a triplet $\lambda = (\dot{\mathbf{F}}\mathbf{F}^{-1}, \dot{\mathbf{G}}\mathbf{G}^{-1}, \mathrm{grad}\,(\dot{\mathbf{G}}\mathbf{G}^{-1})^T)$, where the time derivatives of $\mathbf{F}$ and $\mathbf{G}$ are evaluated for some process in $\mathscr{P}$ (3). Clearly

$$\gamma \in \mathrm{Lin}\,(\mathscr{V}_0, \mathscr{V}) \times \mathrm{Inv}\,(\mathfrak{V}) \times \mathrm{Lin}\,(\mathscr{V}_0, \mathrm{Lin}\,(\mathfrak{V}, \mathfrak{V}))\,,$$

$$\lambda \in \mathrm{Lin}\,(\mathscr{V}, \mathscr{V}) \times \mathrm{Lin}\,(\mathfrak{V}, \mathfrak{V}) \times \mathrm{Lin}\,(\mathscr{V}, \mathrm{Lin}\,(\mathfrak{V}, \mathfrak{V})) \equiv \Lambda\,,$$

where $\mathscr{V}$, $\mathscr{V}_0$ are the images of the tangent space at X in the present and reference placements, respectively. $\mathscr{P}$ is constrained if there is some site $\bar{\gamma}$ such that the *flux-cross section* $\Lambda_0(\bar{\gamma}) = \{\lambda = \dot{p}(0) \colon p \in \mathscr{P}, p(0) = \bar{\gamma}\}$ is different from Λ (4). Then, the internal agencies are specified by constitutive equations to within reactions $\bar{\mathbf{T}}, \bar{\mathbf{Z}}, \bar{\mathbf{H}}$ that must satisfy the condition

$$(\bar{\mathbf{T}}, -\bar{\mathbf{T}} - \bar{\mathbf{Z}}^T, \bar{\mathbf{H}}) \in \Lambda_0^{\perp}(\gamma(t)) \tag{2.1}$$

(3) Capriz-Podio Guidugli use the symbols $\mathbf{V} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ (*velocity gradient*) and $\mathbf{W} = \dot{\mathbf{G}}\mathbf{G}^{-1}$ (*wrenching*). So we do in the following.

(4) Because of the assumptions on $\mathscr{P}$, the *flux cross section* does not depend on the time and we can choose the cross section at the time $t = 0$. Λ_0 may differ from Λ because of the restrictions either on $\mathfrak{D}$ (*holonomic constraints*) or directly on the class $\mathscr{P}$. The cases where the restrictions on $\mathscr{P}$ do not come from restrictions on $\mathfrak{D}$ correspond to the *anholonomic constraints*. Here we deal with holonomic constraints.

at any time t. Once one has the algebraic characterization of $\Lambda_0^{\perp}$, it is possible to eliminate the reactions from the balance equations and obtain a simpler theory where only the determined part of the internal agencies are present.

3 - Equations of balance

In addition to the kinematical feature of $\mathscr{B}_s$ certain inertial and mechanical quantities and their balances are to be specified in order to complete the description of the oriented body.

Borrowing symbols and equations from [**1**]$_1$, we assume that two measure of inertia-mass and microinertia are given whose densities, $\varrho \in R^+$ and $\varrho \mathbf{I} \in \operatorname{Sym}(\mathcal{V})$, evolve according to

$$(3.1) \qquad \dot{\varrho} = -\varrho \operatorname{div} \mathbf{x}\,, \qquad \dot{\mathbf{I}} = 2 \operatorname{sym} (\mathbf{I}\mathbf{W}^T)\,.$$

Furthermore, we assume that the mechanical balance is expressed by the field equations

$$(3.2) \qquad \varrho \ddot{\mathbf{x}} = \varrho \mathbf{b} + \operatorname{div} \mathbf{T}\,, \quad \varrho(\dot{S} - \mathbf{W}\mathbf{S}) = \varrho \mathbf{L} + \operatorname{div} \mathbf{H} + \mathbf{T}^T + \mathbf{Z}\,, \quad \mathbf{Z} = \mathbf{Z}^T\,,$$

where $\mathbf{S} \equiv \mathbf{I}\mathbf{W}^T$ and $\mathbf{b} \in \mathcal{V}$ and $\mathbf{L} \in \operatorname{Lin}(\mathcal{V})$ are the body forces and generalized couples, and by the boundary conditions

$$(3.3) \qquad \mathbf{T}\mathbf{n} = \hat{\mathbf{t}}\,, \qquad \mathbf{H}\mathbf{n} = \hat{\mathbf{m}}\,,$$

where $\mathbf{n}$ is the outer normal and $\hat{\mathbf{t}} \in \mathcal{V}$, $\hat{\mathbf{m}} \in \operatorname{Lin}(\mathcal{V})$ are the contact force and generalized couple per unit deformed area.

T, **H**, **Z** are the internal agencies of the director theories. For a more detailed explanation of these and of the balance equations the reader is sent to [**1**]$_1$. We only point out a difference. As we refer to n-dimensional bodies embedded in the Euclidean space, here **T** and **H** act upon unit vectors **n** belonging to $\mathscr{V} \subset \mathcal{V}$. Then, in $(3.2)_2$,the same symbol is used for the Cauchy stress $\mathbf{T} \in \operatorname{Lin}(\mathscr{V}, \mathcal{V})$ and its trivial extension to $\operatorname{Lin}(\mathcal{V})$ and, (in $(3.2)_{1,2}$, the operator «div» is defined so as the divergence theorem to hold, see [7].

With this understood, the balance equations are formally the same whatever the dimensions of $\mathscr{B}_s$ are. In particular, the power of the internal agencies in the kinetic energy theorem is defined by

$$(3.4) \qquad \varrho \dot{\varepsilon} \equiv \mathbf{T}\cdot\mathbf{V} - (\mathbf{T} + \mathbf{Z})\cdot\mathbf{W} + \mathbf{H}\cdot \operatorname{grad} \mathbf{W}^T \ (^5)\,,$$

that enlightens the orthogonality requirement (2.1).

(5) From now on dots stand for the inner products.

Remark 1. The formulation of new balance equations in theories of structured media is needed to account for the major kinematical structure of the model and is the most subtle point in principle. In [**1**]$_1$, $(3.2)_{2,3}$ are directly suggested by analogy with the study of an affine system of mass points. $(3.2)_2$ is shown to have the usual form of a momentum balance. As in other approaches to the subject, the most distinctive point of the theory concerns the moment of momentum balance. Indeed, $(3.2)_3$ generates a moment of momentum equation. Formally it is a constitutive assumption about the internal agencies, being similar, in this respect, to the constitutive axioms added to the momentum balance in theories for simpler models, i.e. the third law in Newton's Mechanics and the symmetry of the Cauchy's stress in Continuum Mechanics. From a comparison with Toupin's theory [**12**], where it is possible, $(3.2)_3$ turns out to be equivalent to his assumption of Galilean invariance of the strain energy. Then, in some sense, it introduces into the theory a particular notion of *observer* and *frame indifference* in a form that is consistent with the kinematical notion of configuration given in section **2**.

Remark 2. The equations (3.3) clear the meaning of the tensors $\mathbf{T}$ and $\mathbf{H}$: they describe the contact forces and generalized couples through any given surface in the body ([6]). When we consider constrained materials, however, a queer ambiguity may be left in the solution of the boundary value problem. In fact, associated to reactions, there might well exist fields $(\mathbf{T}_0, \mathbf{Z}_0, \mathbf{H}_0)$ which constitute a *null stress state*, i.e. such that

$$(3.5) \qquad \operatorname{div} \mathbf{T}_0 = \mathbf{0}\,, \qquad \mathbf{Z}_0 + \mathbf{T}_0^{\mathsf{T}} + \operatorname{div} \mathbf{H}_0 = \mathbf{0}\,, \qquad \mathbf{Z}_0 \in \mathrm{Sym}\,,$$

and satisfy homogeneous boundary conditions.

The equations (3.5) are equivalent to the conditions

$$(3.5)' \qquad \int_S \mathbf{T}_0 \mathbf{n} = \mathbf{0}\,, \qquad \int_S [\mathbf{x} \wedge \mathbf{T}_0 \mathbf{n} + \operatorname{skew}(\mathbf{H}_0 \mathbf{n})] = \mathbf{0}\,,$$

where S is the boundary of any given part of $\mathcal{B}$. As in the theory the term skew $(\mathbf{H}_0\mathbf{n})$ corresponds to the usual couple stresses, $(3.5)'$ means that the stresses and couple stresses associated with null stress fields are equipollent to zero. Then, for the same data the boundary value problem exhibits several solutions all of which correspond to equipollent distributions of couples and

---

([6]) The interpretation of $\mathbf{Z}$ is more concealed and requires us to regard the discrete model whence the continuum theory is derived.

forces through any surface inside the body [7]. This is not different, in essence, from what happens in the theory of constrained classical continua but for the peculiarity that here the notion of static equivalence involves stresses and couple stresses at the same time.

4 - Special theories

The special cases follow from the general theory by suitable assumptions on inertial measure, kinematics and constitutive equations of the body. The following results will be useful.

B1. (Embeddings). If $\mathcal{U}$ and $\mathcal{Z}$ are subspaces of inner product spaces $\mathcal{A}$ and $\mathcal{C}$ an embedding J_p: Lin $(\mathcal{U}, \mathcal{Z}) \to$ Lin $(\mathcal{A}, \mathcal{C})$ is defined by

$$J_p[\mathbf{A}] \equiv \mathbf{EAP}\,, \tag{4.1}$$

for $\mathbf{A} \in \mathrm{Lin}\,(\mathcal{U}, \mathcal{Z})$, where $\mathbf{E}: \mathcal{Z} \to \mathcal{C}$ is the immersion and $\mathbf{P}$ is some projection from $\mathcal{A}$ onto $\mathcal{U}$. Clearly, (4.1) is equivalent to require that $[\mathbf{A}]$ coincide with $\mathbf{A}$ on $\mathcal{U}$ and vanish on some complementary subspace $\dot{\mathcal{U}}$: $\mathcal{U} \oplus \dot{\mathcal{U}} = \mathcal{A}$. We will denote by the same symbol $\mathbf{A}$ and $J_p[\mathbf{A}]$ when $\mathbf{P}$ is the orthogonal projection onto $\mathcal{U}$. We remark that, in this case, the following properties hold $J_p[\mathbf{A}] \cdot J_p[\mathbf{B}] = \mathbf{A} \cdot \mathbf{B}$, $J_p[\mathbf{A}]^T = J_p[\mathbf{A}^T]$.

B2. (Splittings). Let $\mathcal{A}$ and $\mathcal{C}$ be given as under B1. On dealing with equations into Lin $(\mathcal{A}, \mathcal{C})$, it is sometimes expedient in the following to split Lin $(\mathcal{A}, \mathcal{C})$ into the direct sum of subspaces. This may be performed in an obvious way for any choice of complementary subspaces $\mathcal{U}$, $\dot{\mathcal{U}}$ and $\mathcal{Z}$, $\dot{\mathcal{Z}}$ in $\mathcal{A}$ and $\mathcal{C}$. In particular

$$\mathrm{Lin}\,(\mathcal{A}, \mathcal{C}) = \mathrm{Lin}\,(\mathcal{U}, \mathcal{Z}) \oplus \mathrm{Lin}\,(\mathcal{U}^{\perp}, \mathcal{Z}) \oplus \mathrm{Lin}\,(\mathcal{U}, \mathcal{Z}^{\perp}) \oplus \mathrm{Lin}\,(\mathcal{U}^{\perp}, \mathcal{Z}^{\perp}) \tag{4.2}$$

and the factors are mutually orthogonal.

B3. (Material microstructures). In what follows, many of the cases concern the material behaviour of the microstruture, which implies the respect of relations of the form

$$\mathbf{V} = V(\mathbf{W})\,. \tag{4.3}$$

---

(7) This property was already noticed by Toupin and Grioli for their structured model and find a convenient interpretation when their theories are viewed as constrained (see $[\mathbf{1}]_2$).

As we consider constraints in finite regions of $\mathcal{B}$, we regard (4.3) both as a restriction on the local values of $\mathbf{V}$ and $\mathbf{W}$ and on the fields $\mathbf{V}$ and $\mathbf{W}$ and include restrictions on grad $\mathbf{W}$, coming from the symmetry of grad $\mathbf{V}$, in the algebraic characterization of the constraint. This point of view is absent in the abstract scheme proposed in $[\mathbf{I}]_2$.

In particular, in 4c, 4d and 4e $\mathbf{F}$ and $\mathbf{G}$ coincide along material surfaces that spread all over the body. Accordingly, it is not difficult to show that the wrenching gradient must satisfy

$$[(\mathrm{grad}\,\mathbf{W}_{|\mathbf{x}})\,\mathbf{u}]\,\mathbf{v} = [(\mathrm{grad}\,\mathbf{W}|_{\mathbf{x}})\,\mathbf{v}]\,\mathbf{u} \qquad \text{for } \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}, \tag{4.4}$$

where $\mathscr{V}$ is the tangent plane to the constraint surface at $\mathbf{x}$.

4a. *Materials of second grade.* We suppose that $\mathcal{B}_s$ is a three-dimensional body where

$$\mathbf{I} = \mathbf{0}, \qquad \mathfrak{D} = \{\mathfrak{d}: \mathbf{K} \text{ is smooth, } \mathbf{G} = \nabla K\}.$$

Then the *site cross-section* Γ and the *flux cross-section* Λ_0 are

$$\Gamma = \{\gamma = (\mathbf{F}, \mathbf{G}, \nabla\mathbf{G});\ \mathbf{F} = \mathbf{G}, \quad [\nabla\mathbf{G}\,\mathbf{v}]\mathbf{w} = [\nabla\mathbf{G}\,\mathbf{w}]\mathbf{v},\ \forall \mathbf{v}, \mathbf{w} \in \mathcal{V}\},$$

$$\Lambda_0 = \{\lambda = (\mathbf{V}, \mathbf{W}, \mathrm{grad}\,\mathbf{W}^T), \qquad \mathbf{V} = \mathbf{W} \in \mathrm{Lin}\,(\mathcal{V}, \mathcal{V}),$$

$$[(\mathrm{grad}\,\mathbf{W}^T)\,\mathbf{v}]^T\mathbf{w} = [(\mathrm{grad}\,\mathbf{W}^T)\,\mathbf{w}]^T\mathbf{v}, \text{ for } \forall \mathbf{v}, \mathbf{w} \in \mathcal{V}\}.$$

From (2.1) it follows

$$\overline{\mathbf{Z}} = \mathbf{0}, \qquad (\overline{\mathbf{H}}\mathbf{v})^T\mathbf{w} = -(\overline{\mathbf{H}}\mathbf{w})^T\mathbf{v} \qquad \text{for } \forall \mathbf{v}, \mathbf{w} \in \mathcal{V},$$

which yield, introducing the extra-stresses,

$$\mathbf{T} = \overline{\mathbf{T}}, \qquad \mathbf{Z} = \mathbf{Z}_E, \qquad \mathbf{H} = \mathbf{H}_E + \overline{\mathbf{H}}, \tag{4.5}$$

where for any fixed coordinate system we have

$$\mathbf{H}_E^{ijk} = \mathbf{H}_E^{kji}, \qquad \overline{\mathbf{H}}^{ijk} = -\overline{\mathbf{H}}^{kji}. \tag{4.6}$$

If, for simplicity, we omit the generalized body couples from $(3.2)_2$, we obtain

$$\overline{\mathbf{T}} = -(\operatorname{div}\mathbf{H}_E)^T - \mathbf{Z}_E - (\operatorname{div}\overline{\mathbf{H}})^T \tag{4.7}$$

and, from $(3.2)_1$, the pure balance equation

$$\varrho\ddot{\mathbf{x}} - \varrho\mathbf{b} = \operatorname{div}\left(-\mathbf{Z}_E - (\operatorname{div}\mathbf{H}_E)^T\right). \tag{4.8}$$

Thus, the constrained theory is determined to within an arbitrary null stress state defined by $\mathbf{T}_0 = -(\operatorname{div}\mathbf{H}_0)^T$ with $\mathbf{H}_0$ satisfying $(4.6)_2$ and $\mathbf{Z}_0 = 0$, but when it is formulated in terms of

$$\mathbf{T}_E \equiv -\mathbf{Z}_E - (\operatorname{div}\mathbf{H}_E)^T, \tag{4.9}$$

it can be put into correspondence with Toupin's theory of the hyperelastic materials of second grade [**12**]. In particular, the comparison between the definition (4.9) and the constitutive equation for the Cauchy stress in [**12**] emphasizes that the symmetry of $\mathbf{Z}$ is equivalent to the Euclidean invariance of the strain energy function assumed by Toupin. This agrees with the interpretation of the symmetry of $\mathbf{Z}$ given in section **3**.

4b. *Classical continuum.* Classical continuum, theory trivially follows from the previous one by assuming $\mathbf{H}_E = \mathbf{0}$ and identifying $\mathbf{T}_E$ and $-\mathbf{Z}_E$ in (4.9).

4c. *Ericksen's liquid crystals.* Let $\mathcal{B}_s$ be three-dimensional. We call $\mathcal{B}_s$ a *liquid crystal* of Ericksen's type if the following statements apply.

(i) In the reference placement $\mathfrak{p}_0$ the microinertia has rank 1. We denote by $\mathscr{W}$ the non-trivial invariant subspace of I_0 at X.

(ii) There exists a family of disjoint smooth surfaces, spreading over the body, such that $\mathfrak{D} = \{\mathfrak{d} : G = \nabla\mathbf{K} \text{ on } \mathscr{V}_0\}$, where $\mathscr{V}_0$ is the tangent plane at the surface at X in $\mathfrak{p}_0$ (8).

(iii) At any X we have

$$\mathfrak{V} = \mathscr{W}_0 \oplus \mathscr{V}_0\,. \tag{4.10$_1$}$$

(8) In the scheme by Toupin, this assumption is equivalent to admit two material directors.

Note that the above definition implies that

$$(4.10)_{2,3} \qquad \mathcal{U} = \mathcal{W}_F \oplus \mathcal{V} = \mathcal{W}_G \oplus \mathcal{V} ,$$

where $\mathcal{V} = \mathbf{F}(\mathcal{V}_0) = \mathbf{G}(\mathcal{V}_0)$, $\mathcal{W}_F = \mathbf{F}(\mathcal{W}_0)$, $\mathcal{W}_G = \mathbf{G}(\mathcal{W}_0)$.

Because of the constraint, the fluxes are completely free but for the conditions

$$(4.11)_1 \qquad (\mathbf{V} - \mathbf{W})\,\mathbf{v} = \mathbf{0} , \qquad \mathbf{v} \in \mathcal{V}$$

and by B3

$$(4.11)_2 \qquad [(\operatorname{grad} \mathbf{W}^T)\,\mathbf{u}]^T \mathbf{v} = [(\operatorname{grad} \mathbf{W}^T)\,\mathbf{v}]^T \mathbf{u} \qquad \text{for } \mathbf{u}, \mathbf{v} \in \mathcal{V} .$$

By (4.2) and the orthogonality conditions (2.1), from $(4.11)_{1,2}$ it follows

$$(4.12) \qquad \begin{aligned} &\overline{\mathbf{T}} \in \operatorname{Lin}(\mathcal{V}, \mathcal{U}) , && \overline{\mathbf{Z}} = \mathbf{0} , \\ &\overline{\mathbf{H}} \in \operatorname{Lin}\big(\mathcal{V}, \operatorname{Lin}(\mathcal{U}, \mathcal{V})\big) , && (\overline{\mathbf{H}}\mathbf{u})^T \mathbf{v} = -(\overline{\mathbf{H}}\mathbf{v})^T \mathbf{u} \quad \text{for } \mathbf{u}, \mathbf{v} \in \mathcal{V} . \end{aligned}$$

The conditions $(4.12)_3$ are equivalent to

$$(4.13) \qquad \overline{\mathbf{H}}\mathbf{u} = \mathbf{u}\overline{\mathbf{H}} = \mathbf{0} \ \text{ for } u \in \mathcal{V}^{\perp} , \qquad (\overline{\mathbf{H}}\mathbf{u})^T \mathbf{v} = -(\overline{\mathbf{H}}\mathbf{v})^T \mathbf{u}, \ \text{for } \mathbf{u}, \mathbf{v} \in \mathcal{U} ,$$

from which we have

$$(4.14) \qquad \operatorname{div} \overline{\mathbf{H}} \in \operatorname{Lin}(\mathcal{U}, \mathcal{V}) , \qquad \operatorname{div}\,[(\operatorname{div} \overline{\mathbf{H}})^T] = \mathbf{0} \ ({}^9) .$$

From $(4.14)_{1,2}$ $\overline{\mathbf{H}}$ and the part of the reaction in the Cauchy stress, $\overline{\overline{\mathbf{T}}} = -(\operatorname{div} \overline{\mathbf{H}})^T$, are a *null stress* that remains completely arbitrary in the constrained theory. Apart from this, however, no ambiguity is left in the theory.

Ericksen's theory of the liquid cristals agrees with the present theory, if one ignores $\overline{\overline{\mathbf{T}}}$ and $\overline{\mathbf{H}}$ and assumes

$$(4.15) \qquad \varrho \mathbf{L} = \varrho \mathbf{n} \otimes \mathbf{d} , \qquad \mathbf{H}_E = \mathbf{n} \otimes \mathbf{w}_E ,$$

---

(9) $(4.14)_1$ follows, by $(4.13)_{1,2}$, from the chain

$$\overline{\mathbf{H}}^{ijk}{}_{,k}\, t_i = (\overline{\mathbf{H}}^{ijk} t_i)_{,k} - \overline{\mathbf{H}}^{ijk} t_{i,k} = \overline{\mathbf{H}}^{ijk} (-B_{ik} + t_{i,r} t^r t_k) = 0$$

where t_i is the unit normal to the constraint surfaces and B_i are the components of their curvature tensor with respect to some coordinate system.

where $\mathbf{n} \in \mathscr{W}$, $\mathbf{d} \in \mathcal{V}$ and $\mathbf{w}_E \in \text{Lin}(\mathcal{V}, \mathcal{V})$ stand, respectively, for the director and the generalized body- and surface-forces in [**2**].

As a matter of fact, the evolution law of micro-inertia $(3.1)_2$ is equivalent to assume $\mathbf{I} = \mathbf{G}\mathbf{I}_0\mathbf{G}^T$, and requires that

$$\varrho\mathbf{I} = \varrho\nu\mathbf{n}\otimes\mathbf{n}\,, \qquad \varrho(\dot{\mathbf{S}} - \mathbf{W}\mathbf{S}) = \varrho\mathbf{I}(\mathbf{W}^2 + \dot{\mathbf{W}})^T = \varrho\nu\mathbf{n}\otimes\ddot{\mathbf{n}}\,,$$

where $\dot{\nu} = 0$ and $\mathbf{n} = G\mathbf{n}_0$ for some given $n_0 \in \mathscr{W}_0$.

Then, on decomposing $(3.2)_2$ according to $\text{Lin}(\mathcal{V}, \mathscr{W}_G) \oplus \text{Lin}(\mathcal{V}, \mathscr{V})$, we have

$$\begin{aligned} &\overline{\mathbf{T}}^T + \mathbf{P}_{\mathscr{V}}[\mathbf{T}_E^T + \mathbf{Z}_E + (\text{grad}\,\mathbf{n})\,\mathbf{w}_E^T] = \mathbf{0}\,, \\ &(\mathbf{1} - \mathbf{P}_{\mathscr{V}})[\mathbf{T}_E^T + \mathbf{Z}_E + (\text{grad}\,\mathbf{n})\,\mathbf{w}_E^T] + \mathbf{n}\otimes\text{div}\,\mathbf{w}_E + \varrho\mathbf{n}\otimes\mathbf{d} = \varrho\nu\mathbf{n}\otimes\ddot{\mathbf{n}}, \end{aligned} \tag{4.16}$$

where $\mathbf{P}_{\mathscr{V}}$ is the projection onto $\mathscr{V}$ which performs the decomposition. The first equation furnishes the reaction $\mathbf{T}$ and the second one corresponds to the *micromomentum balance* (see [**2**], eq. (34)) if we put

$$\mathbf{n}\otimes\mathbf{r} \equiv (\mathbf{1} - \mathbf{P}_{\mathscr{V}})[\mathbf{Z}_E + \mathbf{T}_E^T + (\text{grad}\,\mathbf{n})\,\mathbf{w}_E^T]\,. \tag{4.17}$$

Finally, from (4.17), $(4.16)_1$ and $(3.2)_3$ we have

$$-\mathbf{Z}_E = (\overline{\mathbf{T}} + \mathbf{T}_E)^T + (\text{grad}\,\mathbf{n})\,\mathbf{w}_E^T - \mathbf{n}\otimes\mathbf{r} \in \text{Sym}\,, \tag{4.18}$$

that is just the *moment of momentum balance* in Ericksen's paper ([**2**], eq. (42)).

4d. *Cosserat surfaces.* $\mathscr{B}$ is a surface and $\mathscr{V}_0$ is a tangent plane at X in the reference placement. The same formulae and conclusions of the above case hold but for few differences due to the dimensions of $\mathscr{B}$. Because of the dimensions of $\mathscr{B}$, here we have $\mathbf{T} \in \text{Lin}(\mathscr{V}, \mathcal{V})$ and $\mathbf{H} \in \text{Lin}(\mathscr{V}, \text{Lin}(\mathcal{V}, \mathcal{V}))$ that imply from (4.12): $\mathbf{T}_E = \mathbf{0}$, $\mathbf{w}_E \in \text{Lin}(\mathscr{V}, \mathcal{V})$ and the conditions (4.14) for $\mathbf{H}$. However, the eqs. $(4.16)_2$ and (4.18) are unchanged and can be compared with the eqs. (3.20) and (3.24) of the Cosserat surface theory [**10**], with some obvious correspondences of notations.

4e. *Classical theory of shells.* The Kirchhoff-Love theory of shells is obtained by adding to the inertial and kinematical assumptions which characterize Cosserat surfaces, the following prescriptions

(i) $\mathscr{W}_0 \equiv \mathscr{V}_0^{\perp}$,

(ii) $\mathfrak{D} = \{\mathfrak{d}\colon\ \mathbf{G} = \nabla\mathbf{K} \text{ on } \mathscr{V}_0,\ \mathbf{G}^T\mathbf{G} = \mathbf{1} \text{ on } \mathscr{V}_0^{\perp}\}$.

The definition of $\mathfrak{D}$ assures not only that $\mathbf{G}(\mathscr{V}_0) = \mathbf{F}(\mathscr{V}_0) = \mathscr{V}$, as in the case 4d., but further that $\mathbf{G}(\mathscr{V}_0^{\perp}) = \mathscr{V}^{\perp}$.

Moreover, it is equivalent to the following conditions on $\mathbf{V}$ and $\mathbf{W}$

$$\mathbf{V} - \mathbf{W} = \mathbf{0} \quad \text{on } \mathscr{V}, \qquad \mathbf{W}^T + \mathbf{W} = \mathbf{0} \quad \text{on } \mathscr{V}^{\perp}. \tag{4.19}$$

$(4.19)_1$ implies the respect of the symmetry conditions (4.4). By differentiation, from $(4.19)_2$ we have also

$$\mathbf{n}\,\text{grad}\,\mathbf{W}^T \in \text{Lin}\,(\mathscr{V}, \mathscr{V})\,, \quad \mathbf{n}\,\text{grad}\,\mathbf{W}^T = -\,\mathbf{n}\,\text{grad}\,\mathbf{W} + (\mathbf{W}^T + \mathbf{W})\mathbf{B}\,, \tag{4.20}$$

where $\mathbf{n}$ is the unit normal and $\mathbf{B}$ is the curvature tensor of the surface. Then, if we split the reaction hyperstress according to the orthogonal decomposition

$$\text{Lin}\,(\mathscr{V}, \text{Lin}\,(\mathcal{V}, \mathscr{V})) \oplus \text{Lin}\,(\mathscr{V}, \text{Lin}\,(\mathscr{V}^{\perp}, \mathscr{V}^{\perp}) \oplus \text{Lin}\,(\mathscr{V}, \text{Lin}\,(\mathscr{V}, \mathscr{V}^{\perp}))$$

and denote its components, respectively, by $\bar{\bar{\mathbf{H}}}$, $\mathbf{n}\otimes\bar{\mathbf{M}}_{\perp}$ and $\mathbf{n}\otimes\bar{\mathbf{M}}_{\tau}$, from (2.1) we have

$$\bar{\mathbf{T}}\cdot(\mathbf{V} - \mathbf{W}) - \bar{\mathbf{Z}}^T\cdot\mathbf{W} + \bar{\bar{\mathbf{H}}}\cdot\text{grad}\,\mathbf{W}^T - \bar{\mathbf{M}}_{\tau}\cdot(\mathbf{n}\,\text{grad}\,\mathbf{W}) + \bar{\mathbf{M}}_{\tau}\cdot(\mathbf{W}^T + \mathbf{W})\mathbf{B} = 0 \tag{4.21}$$

for every $(\mathbf{V}, \mathbf{W}, \text{grad}\,\mathbf{W}^T)$ satisfying (4.4), (4.19) and (4.20).

It follows that the reaction stresses are such that

(4.22) $\bar{\bar{\mathbf{H}}}$ belongs to the subspace defined by $(4.12)_3$,

$$\bar{\mathbf{M}}_{\perp} \in \text{Lin}\,(\mathscr{V}, \mathscr{V}^{\perp})\,, \tag{4.23}$$

$$\bar{\mathbf{M}}_{\tau} \in \text{Skew}\,(\mathscr{V})\,, \tag{4.24}$$

$$\bar{\mathbf{T}} \in \text{Lin}\,(\mathscr{V}, \mathcal{V})\,. \tag{4.25}$$

It is interesting to notice that besides the ambiguity left by the reactions $\bar{\bar{\mathbf{H}}}$, the theory is undetermined as to the term $\mathbf{n}\otimes\bar{\mathbf{M}}_\tau$. The indetermination (4.24) in $\bar{\mathbf{M}}_\tau$ was noticed in [4] where it was simply suggested to put the skewsymmetric part of the tangential component of the couple stresses equal to zero.

Here we properly interpret this indeterminacy as a reaction. Associated with $\bar{\mathbf{M}}_\tau$, there is a null stress given by

$$
\text{(4.26)}\qquad
\begin{aligned}
&\mathbf{T}_0 = \operatorname{div}(\mathbf{n}\oplus\bar{\mathbf{M}}_\tau)\,, \qquad \mathbf{H}_0 = \mathbf{n}\otimes\bar{\mathbf{M}}_\tau\,,\\
&\mathbf{Z}_0 = -\operatorname{div}(\mathbf{n}\otimes\bar{\mathbf{M}}_\tau) - [\operatorname{div}(\mathbf{n}\otimes\bar{\mathbf{M}}_\tau)]^T\,.
\end{aligned}
$$

In fact, these fields satisfy the balance equations (3.5). $(3.5)_{2,3}$ trivially hold and $(3.5)_1$ follows from

$$
\operatorname{div}(\mathbf{n}\otimes\bar{M}_\tau) = \bar{M}_\tau{}^{\alpha\beta}{}_{|\beta}\,\mathbf{n}\otimes\mathbf{a}_\alpha + \bar{M}_\tau{}^{\alpha\beta}B^\gamma_\beta\,\mathbf{a}_\gamma\otimes\mathbf{a}_\alpha\,,
$$

and

$$
\begin{aligned}
\operatorname{div}(\operatorname{div}(\mathbf{n}\otimes\bar{\mathbf{M}}_\tau)) = \bar{M}_\tau{}^{\alpha\beta}{}_{|\beta\alpha}\,\mathbf{n} + (\bar{M}_\tau{}^{\alpha\beta}{}_{|\beta}B^\gamma{}_\alpha + \bar{M}_\tau^{\alpha\beta}{}_{|\alpha}B^\gamma_\beta)\,\mathbf{a}_\gamma\\
+ \bar{M}_\tau{}^{\alpha\beta}B_\beta{}^\gamma{}_{|\alpha}\,\mathbf{a}_\gamma + \bar{M}_\tau{}^{\alpha\beta}B^\gamma{}_\beta B_{\gamma\alpha}\,\mathbf{n}\,,
\end{aligned}
$$

where the right hand side term vanishes because of the skewsymmetry of $\bar{M}_\tau$ and taking account of (A.2.39) of [10] and Codazzi equations.

Here, as is customary in shell theory [10], we have used an intrinsic reference $\{\mathbf{a}_\alpha,\ \alpha=1,2$ and $\mathbf{n}\perp\mathbf{a}_\alpha\}$ and denoted by «$|$» the surface covariant derivative. $B^\gamma{}_\alpha$ is for the curvature tensor.

The indeterminacy on $\bar{\mathbf{M}}_\tau$ reflects upon all the components of the internal agencies. If one cancels $\bar{\mathbf{M}}_\tau$ in the present theory and accordingly denotes the extra hyperstress by $\mathbf{n}\otimes\mathbf{M}_E$, with $\mathbf{M}_E\in\operatorname{Sym}(\mathscr{V})$, the classical shell theory is obtained. As $\mathbf{Z}$ is symmetric, it follows from (4.21) that

$$
\text{(4.27)}\qquad -\bar{\mathbf{Z}} = \mathbf{n}\otimes\bar{\mathbf{m}} + \bar{\mathbf{m}}\otimes\mathbf{n}\,, \qquad \mathbf{Z}_E\in\operatorname{Sym}(\mathscr{V})\,,
$$

where the reaction $\bar{\mathbf{m}}$ stands for the intrinsic director couple in the theory by Green-Naghdi-Wainwright. It follows further from $(3.2)_2$ that

$$
\text{(4.28)}\quad \varrho\nu\ddot{n} = \varrho\mathbf{d} + \operatorname{div}\mathbf{M}_E + \operatorname{div}\mathbf{M}_\perp - \bar{\mathbf{m}}\,,\ \ \bar{\mathbf{T}}^T + \mathbf{P}_{\mathscr{V}}\bar{\mathbf{Z}} + \mathbf{Z}_E - \mathbf{B}\bar{\mathbf{M}}_\perp - \mathbf{B}\mathbf{M}_E^T = \mathbf{0}\,,
$$

where $\mathbf{P}_{\mathscr{V}}$ is now the orthogonal projection onto the tangent plane $\mathscr{V}$.

The equations $(4.28)_{1,2}$ can be given the form

$$
\begin{aligned}
&\overline{T}^{3\alpha} = \overline{m}^{\alpha} + \overline{M}^{3\gamma} B_{\gamma}{}^{\alpha}\,, \qquad \overline{T}^{\alpha\beta} = Z_{E}^{\beta\alpha} + M_{E}^{\alpha\gamma} B \\
(4.29)\qquad &\overline{m}^{3} = -\varrho\nu\ddot{n}^{3} + \varrho d^{3} + B_{\alpha\beta} M_{E}{}^{\alpha\beta} + \overline{M}^{3\alpha}{}_{|\alpha}\,, \\
&\overline{m}^{\alpha} = -\varrho\nu\ddot{n}^{\alpha} + \varrho d^{\alpha} + M_{E}{}^{\alpha\beta}{}_{|\beta} - \overline{M}^{3\gamma} B_{\gamma\alpha}\,.
\end{aligned}
$$

$(4.29)_{1,3,4}$ allows us to deduce the null stress field as a function of $\overline{\mathbf{M}}_{\perp}$. Equations (4.29) coincide with eqs. (7.9), (7.10), (7.16) and (7.17) of [**4**]; when (4.29) are inserted in (3.2), we arrive at

$$
(4.30)\qquad
\begin{aligned}
&\overline{T}^{\alpha\beta}{}_{|\beta} - M_{E}{}^{\beta\gamma}{}_{|\gamma} B_{\beta}{}^{\alpha} - \varrho(d^{\beta} - \ddot{n}^{\beta}) B_{\beta}{}^{\alpha} + \varrho b^{\alpha} = \varrho\ddot{x}^{\alpha}\,, \\
&M_{E}{}^{\alpha\beta}{}_{|\beta\alpha} + B_{\alpha\beta}\overline{T}^{\alpha\beta} + \varrho(d^{\alpha} - \ddot{n})_{|\alpha} + \varrho b^{3} = \varrho\ddot{x}^{3}\,.
\end{aligned}
$$

These balance equations are the same as equations (7.13) and (7.14) of [**4**]. It is important to notice that the reactions $\overline{T}^{\alpha\beta}$ in (4.30) can be eliminated by means of $(4.29)_2$.

A c k n o w l e d g e m e n d. The present paper has been completed while the author was a fellow at the Johns Hopkins University, supported by N.S.F. under the grants No. MCS 78-00396 and ENG 76-14765A01.

The author wishes to thank the members of the Group for Natural Phylosophy of that University for their warm hospitality.

References

[**1**] G. CAPRIZ and P. PODIO GUIDUGLI. $[\bullet]_1$ *Discrete and continuous bodies with affine structure*, Ann. Mat. Pura Appl. (4) **111** (1976), 195-217; $[\bullet]_2$ *Formal structure and classification of theories of oriented materials*, Ann. Mat. Pura Appl. (4) **115** (1977), 17-39.

[**2**] J. L. ERICKSEN, *Conservative laws for liquid crystals*, Trans Soc. Rheol **5** (1961), 23-34.

[**3**] A. C. ERINGEN, *Mechanics of micromorphic continua*, « Proc. IUTAM Symposium », Freudenstadt-Stuttgart 1967, 18-35, Kröner ed. Springer, Berlin 1968.

[**4**] A. E. GREEN, P. M. NAGHDI and W. L. WAINWRIGHT, *A general theory of a Cosserat surface*, Arch. Rational Mech. Anal. **20** (1965), 287-308.

[**5**] A. E. GREEN and W. ZERNA, *Theoretical elasticity*, II edition, Clarendon Press, Oxford 1968.

[6] M. E. GURTIN and P. PODIO GUIDUGLI, *The thermodynamics of constrained materials*, Arch. Rational Mech. Anal. **56** (1973), 192-208.

[7] M. E. GURTIN and A. I. MURDOCH, *A continuum theory of elastic material surfaces*, Arch. Rat. Mech. Anal. **57** (1975), 291-323.

[8] E. KRÖNER, *Mechanics of generalized continua*, Proc. IUTAM Symposium, Freudenstadt-Stuttgart 1967, Springer, Berlin 1968.

[9] R. D. MINDLIN, *Micro-structure in linear elasticity*, Arch. Rational Mech. Anal. **16** (1964), 51-78.

[10] P. M. NAGHDI, *The theory of shells and plates*, Handbuch der physik VI a/2, Springer, Berlin-Göttingen-Heidelberg 1972.

[11] W. NOLL, *Lectures on the foundation of continuum mechanics and thermodynamics*, Arch. Rational Mech. Anal. **52** (1973), 62-92.

[12] R. A. TOUPIN, *Theory of elasticity with couple stresses*, Arch. Rational Mech. Anal. **17** (1964), 85-112.

Sommario

Si deducono alcune teorie speciali per materiali con struttura riguardando i modelli considerati come vincolati. Si discutono anche le indeterminazioni nello stato di tensione dovute ai vincoli.

* * *