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Distributive unions in semilattices and in inverse symmetrizations (**)

Introduction

In this work distributive unions in semilattices are studied and used to characterize the exactness of short sequences and functors, in the context of distributive exact categories.

We recall that subquotients behave well in *distributive* exact categories: in this case, and only in this one, canonical isomorphisms are composable, i.e. form a transitive system **[5]**$_{5,7}$; any exact category $\mathscr{E}$ has an associated distributive exact category $\mathscr{E}^{\#}$, called the *distributive expansion* of $\mathscr{E}$ **[5]**$_6$.

Now, a distributive exact category $\mathscr{D}$ has two interesting symmetrizations: the well-known embedding $\mathscr{D} \to \mathscr{D}^{O}$ in the category of correspondences (or relations) of $\mathscr{D}$ **[4]**, **[3]**, **[5]**$_1$, and another embedding $\mathscr{D} \to \mathscr{D}^{\Theta} = \mathscr{D}^{O}/\mathbf{\Phi}$, where $\mathbf{\Phi}$ is a congruence of $\mathscr{D}^{O}$ consistent with the involution **[5]**$_5$.

If A is an object of $\mathscr{D}$ (hence of $\mathscr{D}^{O}$ and $\mathscr{D}^{\Theta}$), denote by $M(A)$, $M_{O}(A)$ and $M_{\Theta}(A)$ the ordered sets of subobjects of A with regard to $\mathscr{D}$, $\mathscr{D}^{O}$ and $\mathscr{D}^{\Theta}$ respectively: the elements of $M_{O}(A)$ are the subquotients of A, while those of $M_{\Theta}(A)$ are the classes of canonically isomorphic subquotients of A. $M_{\Theta}(A)$ is an (intersection) semilattice; its unions (not always existing) constitute a rather useless notion, because they are not preserved by functors obtained by Θ-symmetrization of exact ones. The « good » notion (and a useful one, as it will be shown in future works) appears to be *distributive* union (i.e. a union which is distributive with regard to binary intersection): the latter is preserved by exact functors, and yields characterizations of exactness (**6.1**, **6.3**).

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In No. **1** we recall the definition of distributive union in a semilattice, which goes back (at least) to Macneille [**6**]; a disjoint distributive union is called a *partition*. No. **2** introduces the category of *partition semilattices*, or *p-semilattices*: the objects are those semilattices with zero such that, if $a_0 \prec a$, then a_0 belongs to a finite partition of a; the morphisms are the mappings preserving zero, as well as finite intersections and finite partitions (or, equivalently, finite distributive unions).

In No. **3** we deal with a functor from distributive lattices to p-semilattices, denoted by $X \mapsto \hat{X}$, $f \mapsto \hat{f}$. No. **4** shows that, if $X \subset \mathfrak{P}S$ is a lattice of parts, $\hat{X}$ is canonically isomorphic to $\overline{X} = \{x - x' \mid x, x' \in X\} \subset \mathfrak{P}S$, while finite distributive unions in $\hat{X}$ correspond to (set-theoretical) unions in $\mathfrak{P}S$.

Numbers **5** and **6** are concerned with the study of inverse symmetrizations of distributive exact categories: a semilattice $M_\Theta(A)$ is isomorphic to $(M(A))^\wedge$, hence it is a p-semilattice; the connections between distributive unions in semilattices $M_\Theta(A)$ and exactness of short sequences or functors are considered in No. **6**.

Numbers **1-4** can probably be of some use in the general theory of semilattices, and are independent of symmetrization theory.

1 - Distributive unions in semilattices

We recall here the notion of distributive union in a semilattice (see [**6**], def. 3.10 and th. 6.3, 6.5).

1.1 Definition. Let E be a semilattice (i.e. a commutative idempotent semigroup, provided with the canonical order: $a \prec b$ iff $a = ab$); we say that $a \in E$ is the *distributive union* of the family $(a_i)_{i\in I}$ of elements of E if

(1) $a_i \prec a$ for any $i \in I$,
(2) if $b \in E$, ab is the union of $(a_i b)_{i\in I}$ in E.

In particular, a is the union of (a_i): take $b = a$ in (2); if E has 1, condition (1) is superfluous: take $b = 1$ in (2). Each of the following conditions is trivially equivalent to (2)

(2)′ if $b \prec a$ in E, then b is the union of $(a_i b)_{i\in I}$,
(2)″ if $a_i b \prec c$ in E $(i \in I)$, then $ab \prec c$.

If E is a 0-semilattice, we say that a family $(a_i)_{i\in I}$ in E is *disjoint* if

(3) $a_i a_j = 0$ for any $i, j \in I$, $i \neq j$.

When (1), (2), (3) hold we say that a is the *disjoint distributive union of* (a_i), or equivalently that (a_i) is a *partition of* a.

We also use the following (metatheorical) expressions: $a = \bigcup a_i$ *is a distributive union, or a disjoint distributive union* (*a partition*).

1.2 Examples. In a distributive lattice all finite unions are distributive; in a completely distributive lattice all unions are distributive.

Let $E \subset \mathfrak{P}\boldsymbol{Z}$ be the semilattice of intervals of integers; then in E, $\{0\} \cup \{2\} = \{0, 1, 2\}$ is a non-distributive union (intersect by $\{1\}$); it is obvious, and will be shown in a more general context in **4.2**, that distributive unions in E coincide with unions in $\mathfrak{P}\boldsymbol{Z}$ (belonging to E).

Distributive unions need not be preserved by homomorphisms of semilattices: if E' is the subsemilattice of E (see the preceding example) having elements: $\emptyset$, $a = \{0\}$, $b = \{1\}$, $c = \{0, 1, 2, 3\}$, then $c = a \cup b$ is a distributive union in E', non preserved by the embedding $E' \to E$.

1.3 Obvious properties for distributive unions in a semilattice E (resp. for partions in a 0-semilattice E):

(a) distributive unions (resp. partitions) are associative: if $a = \bigcup_{i \in I} a_i$ and $a_i = \bigcup_{j \in J_i} a_{ij}$ $(i \in I)$ are distributive unions (resp. partitions) then $a = \bigcup a_{ij}$ $(j \in J_i,\ i \in I)$ is a distributive union (resp. partition);

(b) remark, however, the following fact; if $a = b \cup c \cup d$ (distributive or simple union), one cannot conclude $a = (b \cup c) \cup d$ unless it is already known that b and c have union in E; consequently, binary distributive unions are not « sufficient to study » finite distributive unions in E (e.g. see the proof of **2.2**);

(c) the product of E is distributive with regard to distributive unions (resp. partitions): if $a = \bigcup a_i$ and $b = \bigcup b_j$ are distributive unions (resp. partitions) in E, then $ab = \bigcup a_i b = \bigcup a b_j = \bigcup a_j b_j$ are three distributive unions (resp. partitions);

(d) if $a = \bigcup a_i$ is a distributive union and $a_i \prec a'_i \prec a$ for any i, then $a = \bigcup a'_i$ is again a distributive union (use **1.1** $(2)''$);

(e) repeated elements in a distributive union can be erased;

(f) any intersection in E is distributive with regard to the product.

1.4 Lemma. *An embedding* (*i.e. one-to-one homomorphism*) *of semilattices* $f\colon E \to E'$ *reflects order, product and distributive unions.*

Proof. It is obvious that f reflects the product, hence also the order. Let $a, a_i \in E$ $(i \in I)$ and fa be the distributive union of (fa_i); then $a_i \prec a$ (f reflects the order) and, using **1.1** (2)$''$, if $a_i b \prec c$ in E $(i \in I)$ then $(fa_i)\cdot(fb) = f(a_i b) \prec fc$ for any i, therefore $(fa)(fb) \prec fc$ and $ab \prec c$.

2 - Partition semilattices

2.1 Definition. A *partition semilattice* (or *p-semilattice*) will be a 0-semilattice E satisfying

(1) if $a_0 \prec a$ in E, there exists a finite partition $a_0, a_1, \ldots, a_n$ of a.

If E has an identity 1, it is sufficient to require property (1) for $a = 1$. Any relatively complemented distributive 0-lattice is a p-semilattice $\big(a_0 \prec a$ gives the partition $a = a_0 \cup (a - a_0)\big)$; in particular this holds for any lattice $\mathfrak{P}S$ (parts of a set S). Here we are interested in p-semilattices arising from distributive lattices (No. **3**): in this case property (1) is always satisfied with $n = 2$ $\big($see **3.7** (2)$\big)$, but the theory here developped would not have any essential simplification by this assumption.

2.2 Proposition. *If $f: E \to E'$ is a homomorphism of 0-semilattices, the following conditions are equivalent:*

(a) *E is a p-semilattice, and f preserves finite distributive unions;*

(b) *E is a p-semilattice, and f preserves finite partitions;*

(c) *if $a_0 \prec a$ in E, there exists a finite partitions $a_0, a_1, \ldots, a_n$ of a which is preserved by f $\big($i.e. $(fa_i)_{i=0,1,\ldots,n}$ is a partition of fa in $E'\big)$.*

Proof. As (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is obvious, we need only to prove that (c) $\Rightarrow$ (a). It is not sufficient to consider binary distributive unions (**1.3**(b)), therefore we proceed by induction: unary distributive unions being trivially preserved, suppose that n-ary distributive unions are preserved and take

(1) $$a = a_0 \cup a_1 \cup \ldots \cup a_n \qquad \text{(distributive union)}\,.$$

As $a_0 \prec a$, we have by (c) a finite family $b_1, \ldots, b_m \in E$ such that

(2) $$a = a_0 \cup b_1 \cup \ldots \cup b_m \qquad \text{(partition preserved by } f)\,.$$

By (1) and (2)

(3) $$b_j = ab_j = \bigcup_{i=0}^{n} a_i b_j = \bigcup_{i=1}^{n} a_i b_j \qquad (j = 1, 2, \ldots, m)$$

is an n-ary distributive union (**1.3** (c)), hence preserved by f

$$(4) \qquad fb_j = \bigcup_{i=1}^{n} f(a_i b_j) \qquad \text{(distributive union)}.$$

Therefore, using (2), (4) and properties **1.3** (a), (d), (e), we have the following distributive unions

$$(5) \qquad fa = fa_0 \cup fb_1 \dots \cup fb_m = fa_0 \cup \Big(\bigcup_{i=1}^{n} f(a_i b_1)\Big) \cup \dots \cup \Big(\bigcup_{i=1}^{n} f(a_i b_m)\Big)$$

$$= fa_0 \cup \Big(\bigcup_{i=1}^{n} fa_i\Big) \cup \dots \cup \Big(\bigcup_{i=1}^{n} fa_i\Big) = \bigcup_{i=0}^{n} fa_i .$$

2.3 Let E and E' be p-semilattices; a *homomorphism of p-semilattices* (or *p-homomorphism*) $f\colon E \to E'$ will be a homomorphism of 0-semilattices preserving finite distributive unions (or, equivalently, finite partitions (**2.2**)).

Partition semilattices and their homomorphisms yield a subcategory of the category of 0-semilattices; we write the former as [*p-semilattices*].

2.4 Remark. Proposition **2.2** suggests that in a p-semilattice E it should be sufficient to consider finite partitions instead of finite distributive unions; actually, if

$$(1) \qquad a = a_1 \cup a_2 \cup \dots \cup a_m \qquad \text{(distributive union)}$$

it is possible (see later on) to «rewrite» a as

$$(2) \qquad a = b_1 \cup b_2 \cup \dots \cup b_n \qquad \text{(partition)}$$

so that, for any $i = 1, \dots, m$ and any $j = 1, \dots, n$

$$(3) \qquad \text{either } a_i b_j = 0 \qquad \text{or} \qquad a_i b_j = b_j$$

hence any a_i is the disjoint distributive union of those b_j such that $b_j \prec a_i$ (and any b_j precedes some a_i); a partition (2), satisfying condition (3), can be said *subordered* to the distributive union (1). It can be obtained as follows: as $a_i \prec a$, there are m finite partitions of a (which we can always suppose to have the same index set, possibly by introducing null terms)

$$(4) \qquad a = \bigcup_{k=1}^{p} a_{ik}, \qquad a_{i1} = a_i \qquad (i = 1, 2, \dots, m)$$

and, the product being distributive with regard to partitions (**1.3** (c)), we have the partition

$$(5) \qquad a = a^m = \Big(\bigcup_{k=1}^{p} a_{1k}\Big)\cdot\Big(\bigcup_{k=1}^{p} a_{2k}\Big) \ldots \Big(\bigcup_{k=1}^{p} a_{mk}\Big) = \bigcup a_{1k_1} a_{2k_2} \ldots a_{mk_m} \qquad (k_i = 1, 2, \ldots, p)$$

which solves our problem, as $a_i(a_{1k_1} \cdot a_{2k_2} \ldots a_{mk_m})$ is zero if $k_i > 1$ and $a_{1k_1} \ldots a_{mk_m}$ otherwise.

3 - From distributive lattices to partition semilattices

We define here the canonical functor

$$\hat{\ }: [\textit{distributive lattices}] \to [\textit{p-semilattices}] .$$

3.1 If X is a distributive lattice, the set X_2 of decreasing pairs of elements of X can be provided with the following product

$$(1) \qquad (x, x') \,\square\, (y, y') = \big(x \cap (y \cup x'),\ x' \cup (y' \cap x)\big) = \big((x \cap y) \cup x',\ (x' \cup y') \cap x\big)$$

which is associative ([**5**]$_7$, n. 3.14). Moreover the semigroup $(X_2, \square)$ is idempotent and left inverse [1]; as an idempotent semigroup, it has a canonical preorder $(x, x') \,\mathbf{\alpha}\, (y, y')$ ([**5**]$_2$, n. 1.7), characterized by the following equivalent properties

$$(2) \qquad (x, x') = (x, x') \,\square\, (y, y') ,$$

$$(3) \qquad x \leqslant x' \cup y , \qquad x \cap y' \leqslant x' ,$$

$$(4) \qquad x \cup y' \leqslant x' \cup y , \qquad x \cap y' \leqslant x' \cap y ,$$

while the associated congruence $(x, x')\, \mathbf{\Phi}\, (y, y')$ is characterized by the equivalent conditions

$$(5) \qquad (x, x') \,\mathbf{\alpha}\, (y, y') \,\mathbf{\alpha}\, (x, x') ,$$

$$(6) \qquad x \cup y' = x' \cup y , \qquad x \cap y' = x' \cap y ,$$

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(1) For an idempotent semigroup this means the following identity: $\alpha \square \beta \square \alpha = \alpha \square \beta$ [**7**]; left inverse idempotent semigroups are more usually called *left regular bands*.

and satisfies

(7) $$(\alpha \square \beta)\, \Phi\, (\beta \square \alpha) \qquad (\alpha, \beta \in X_2)\,.$$

3.2 The quotient $\hat{X} = X_2/\Phi$ is therefore a semilattice, whose canonical order ($<$) coincides with the one induced by the preorder of X_2.

If $(x, x') \in X_2$, we write $x|x'$ its image in $\hat{X}$ under the canonical projection. $\hat{X}$ is a 0-semilattice, with

(1) $$0 = x|x \qquad (\text{for any } x \in X)\ ({}^2)$$

and it will be shown (**3.7**) that it is a p-semilattice.

Remark also that, if $(x, x') \in X_2$

(2) $$x|x' = 0 \qquad \text{implies} \quad x = x'$$

for $(x, x')\, \Phi\, (x, x)$ gives $x = x \cap x = x' \cap x = x'$ $\big($**3.1** (6)$\big)$.

3.3 If X has 0, there is a canonical embedding of 0-semilattices

(1) $$i_X\colon X \to \hat{X}\,, \qquad x \mapsto x|0$$

and

(2) $$\big(i_X(x) = y|y'\big) \quad \text{iff} \quad (x \cup y' = y \text{ and } x \cap y' = 0)$$

the right hand condition saying that x is the relative complement of y' with regard to y (notation: $y - y'$). Therefore $i_X\colon X \to \hat{X}$ is an isomorphism of semilattices iff X is relatively complemented; in such a case the reciprocal isomorphism is

(3) $$i_X^{-1}: \hat{X} \to X\,, \qquad x|x' \mapsto x - x'\,.$$

Instead, if X has 1, there is an embedding of semilattices

(4) $$j_X\colon X^* \to \hat{X}\,, \qquad x \mapsto 1|x\,,$$

where X^* is the opposite lattice of X. If X has both 0 and 1, $\hat{X}$ has an identity: $1|0$.

(²) This would fail for $X = \emptyset$; therefore we agree that, in this case, $\hat{X} = \{0\}$.

3.4 Now, if $f\colon X \to Y$ is a homomorphism of distributive lattices, define $f_2\colon X_2 \to Y_2$ as

(1) $$f_2(x, x') = (fx, fx') \qquad (x, x') \in X_2\,,$$

f_2 respects $\square$-product, hence $\mathfrak{a}$ and Φ, therefore it induces a homomorphism of 0-semilattices

(2) $$\hat{f}\colon \hat{X} \to \hat{Y}\,, \qquad x|x' \mapsto fx|fx'$$

and we have a functor from distributive lattices to 0-semilattices.

3.5 Lemma. *Let X be a distributive lattice, $a, b \in \hat{X}$, $b = y|y'$. Then*

(1) $$(a\,\mathfrak{a}\,b) \quad \textit{iff} \quad (\exists x, x' \in X\colon\ a = x|x' \ \textit{and}\ y' \prec x' \prec x \prec y)\,.$$

Proof. If $a\,\mathfrak{a}\,b$, and $a = z|z'$, take

(2) $$(x, x') = (y, y')\,\square\,(z, z') = \big(y \cap (z \cup y'),\, y' \cup (z' \cap y)\big)$$

so that $x|x' = (y|y')(z|z') = z|z' = a$ and $y' \prec x' \prec x \prec y$; conversely if such a pair (x, x') exists, $a\,\mathfrak{a}\,b$ because

(3) $$(x, x')\,\square\,(y, y') = \big(x \cap (y \cup x'),\, x' \cup (y' \cap x)\big) = (x, x')\,.$$

3.6 Lemma. *If X is a distributive lattice and*

(1) $$x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n$$

is a finite chain of elements of X, then in $\hat{X}$ the element $x_n|x_0$ is given by the following disjoint distributive union

(2) $$x_n|x_0 = (x_n|x_{n-1}) \cup \ldots \cup (x_2|x_1) \cup (x_1|x_0)\,.$$

Proof. It is sufficient to prove the fact for $n = 2$.
Then

(3) $$(x_2|x_1)\cdot(x_1|x_0) = \big(x_2 \cap (x_1 \cup x_1)\big)\,|\,\big(x_1 \cup (x_0 \cap x_2)\big) = x_1|x_1 = 0$$

moreover $x_1|x_0$, $x_2|x_1 \prec x_2|x_0$ by **3.5**, and if $x_1|x_0$, $x_2|x_1 \prec y|y'$ then (**3.1**)

(4) $$x_1 \leqslant y \cup x_0\,; \qquad x_1 \cap y' \leqslant x_0\,; \qquad x_2 \leqslant y \cup x_1\,; \qquad x_2 \cap y' \leqslant x_1$$

hence

(5) $$x_2 \leqslant y \cup x_1 \leqslant y \cup x_0\,, \qquad x_2 \cap y' \leqslant x_1 \cap y' \leqslant x_0$$

that is $x_2|x_0 \prec y|y'$; this proves that $x_2|x_0$ is the disjoint union of $x_2|x_1$ and $x_1|x_0$; as regards the distributivity, take $y|y' \prec x_2|x_0$ and assume that $x_0 \prec y' \prec y \prec x_2$ (it is possible by **3.5**); then

(6) $$(y|y')(x_2|x_1) = \big(y \cap (x_2 \cup y')\big)\,|\,\big(y' \cup (x_1 \cap y)\big) = y\,|\,\big(y' \cup (x_1 \cap y)\big)\,,$$

(7) $$(y|y')(x_1|x_0) = \big(y \cap (x_1 \cup y')\big)\,|\,\big(y' \cup (x_0 \cap y)\big) = \big(y' \cup (x_1 \cap y)\big)\,|\,y'\,.$$

By the preceding argument, $y|y'$ is the union of the elements (6) and (7).

3.7 Theorem. *If X is a distributive lattice, $\hat{X}$ is a p-semilattice; if $f\colon X \to Y$ is a homomorphism of distributive lattices, $\hat{f}\colon \hat{X} \to \hat{Y}$ is a p-homomorphism. In other words, the functor* $\hat{}$ *defined in* **3.1, 3.4** *can be regarded as a functor*

(1) $$\hat{}\,\colon [\textit{distributive lattices}] \to [\textit{p-semilattices}]\,.$$

Moreover if f is one-to-one (resp. onto), so is $\hat{f}$.

Proof. Let X be a distributive lattice, and $x|x' \prec y|y'$ in $\hat{X}$; then we may suppose that $y' \prec x' \prec x \prec y$ (**3.5**), so that by **3.6**

(2) $$y|y' = (y|x) \cup (x|x') \cup (x'|y')\,, \qquad \text{partition in } \hat{X}\,.$$

Now, if $f\colon X \to Y$ is a homomorphism of distributive lattices, it is sufficient to use criterium **2.2** (c) to verify that $\hat{f}\colon \hat{X} \to \hat{Y}$ is a p-homorphism: if $x|x' \prec \prec y|y'$ in $\hat{X}$, the partition (2) is preserved by f (use again **3.6**)

(3) $$fy|fy' = (fy|fx) \cup (fx|fx') \cup (fx'|fy')\,, \qquad \text{partition in } \hat{Y}\,.$$

As regards the last assertion, suppose that f is one-to-one, and $\hat{f}(a) = \hat{f}(b)$, with $a, b \in \hat{X}$; take $c = ab \prec a$, $c = x|x'$, $a = y|y'$ with $y' \prec x' \prec x \prec y$ (**3.5**); hence (3) holds, and $fy|fy' = \hat{f}a = \hat{f}c = fx|fx'$: it follows that $fx'|fy' = 0$ and $fy|fx = 0$, that is $fx' = fy'$ and $fy = fx$ (**3.2** (2)); hence $x' = y'$, $x = y$ and $c = a$; in the same way $c = b$.

Last, suppose that f is onto, and take $b = y|y' \in \hat{Y}$: then there exist $x, x' \in X$ such that $fx = y$, $fx' = y'$ and $\hat{f}(x|x \cap x') = fx|fx' = b$.

4 - Lattices of parts; consecutive partitions

Here S is a set, and $\mathfrak{P}S$ the (completely distributive) lattice of its parts; unions in $\mathfrak{P}S$ (i.e. set-theoretical unions) are always distributive, while partitions are « usual partitions ».

4.1 Proposition. *Let X be a sublattice of $\mathfrak{P}S$; then there is an embedding of p-semilattices*

$$(1) \qquad \iota_X\colon \hat{X} \to \mathfrak{P}S\,, \qquad x|x' \mapsto x - x'$$

so that $\hat{X}$ is isomorphic to $\bar{X} = \{x - x' \,|\, x, x' \in X\}$, a sub-$p$-semilattice of $\mathfrak{P}S$.

Proof. One can prove by direct (and tedious) computation that ι_X is well defined, one-to-one and preserves product and finite distributive unions; the following argument is far quicker.

Take $f\colon X \to \mathfrak{P}S$ the natural embedding; then $\hat{f}\colon \hat{X} \to (\mathfrak{P}S)\hat{}$ is an embedding of p-semilattices (**3.7**) and $(\mathfrak{P}S)\hat{}$ is isomorphic to $\mathfrak{P}S$ (**3.3**) under

$$(2) \qquad i\colon \quad \mathfrak{P}S \to (\mathfrak{P}S)\hat{}\,, \qquad x \mapsto x|\emptyset\,,$$

$$(3) \qquad i^{-1}\colon (\mathfrak{P}S)\hat{} \to \mathfrak{P}S\,, \qquad x|x' \mapsto x - x'\,.$$

By composing $\hat{f}$ and (3) one gets just the map (1), which is therefore an embedding of p-semilattices.

4.2 Corollary. *In the same hypothesis, a is the distributive union of a finite family (a_i) in $\hat{X}$ iff $\iota_X(a_i)$ is the (set-theoretical) union of $(\iota_X(a_i))$ in $\mathfrak{P}S$. If X is stable for unrestricted unions in $\mathfrak{P}S$ or contains all subsets $\{p\}$, for $p \in S$, the same holds for infinite families; in the contrary, counter-examples (for infinite families) can be given.*

Proof. By **4.1**, ι_X preserves finite distributive unions, by **1.4** it reflects all distributive unions.

As regards preservation in the infinite case, let a be the distributive union of a family (a_i) in $\bar{X}$ (isomorphic to $\hat{X}$), and a' the union of the same family in $\mathfrak{P}S$: obviously $a' \subset a$. Now, if $\bar{X}$ is stable for unions in $\mathfrak{P}S$, $a' \in \bar{X}$ hence $a' \supset a$ (for $a' \supset a_i$ for any i) and $a' = a$; suppose instead that $\bar{X}$ contains all point-like subsets of S: if $a' \neq a$, take $p \in a - a'$ and $b = \{p\} \in \bar{X}$, so that $b \cap a_i \subset (a - a') \cap a_i = \emptyset$ for any i, which is absurd because $b = b \cap a$ is the union of all $b \cap a_i$.

Last, we give the following counter-example for preservation of infinite distributive unions, in a case in which none of two preceding assumptions holds. X is the sublattice of $\mathfrak{P}\boldsymbol{R}$ having for elements the closed left-unbounded intervals $]-\infty, \alpha]$; $\bar{X}$ the subsemilattice of $\mathfrak{P}\boldsymbol{R}$ consisting of bounded left-open intervals $]\alpha, \beta]$, $\alpha \leqslant \beta$ in $\boldsymbol{R}$. Then the family $]0, \beta]$ $(0 < \beta < 1)$ has distributive union $]0, 1]$ in $\bar{X}$, while it has union $]0, 1[$ in $\mathfrak{P}\boldsymbol{R}$.

4.3 As any distributive lattice X is isomorphic to a sublattice of a suitable $\mathfrak{P}S$ [2], any p-semilattice associated to a distributive lattice has representations of kind **4.1** (1).

4.4 Let X be a distributive lattice; lemma **3.6** has brought to evidence the existence of a peculiar type of partitions (**3.6** (2)); so we say that a finite *ordered* family $a_1, a_2, \ldots, a_n$ is a *consecutive partition* of $a \in \hat{X}$ if there exists a chain $x_0 \prec x_1 \prec \ldots \prec x_n$ in X such that

(1) $$a = x_n | x_0 , \qquad a_i = x_i | x_{i-1} \qquad (i = 1, 2, \ldots, n) .$$

It can be shown that consecutive partitions are associative, and distributive with regard to the product.

4.5 Remark however that consecutive partitions are confined to p-semilattices *associated to (given) distributive lattices*. Moreover there exist (in some semilattice $\hat{X}$) partitions which cannot be ordered into consecutive ones as shown by the following example. Take $S = \boldsymbol{Z}^3$, the set of triples of integers with the product order

(1) $$(m, n, k) \leqslant (m', n', k') \qquad \text{iff} \qquad m \leqslant m',\ n \leqslant n',\ k \leqslant k'$$

and $X \subset \mathfrak{P}S$ the (complete) sublattice of closed subsets in the order topology

(2) $$x \in X \qquad \text{iff} \qquad (\alpha \in x,\ \beta \in S,\ \beta \leqslant \alpha \cdot\Rightarrow\cdot \beta \in x) .$$

Then $\bar{X} = \{x - x' \mid x, x' \in X\} \subset \mathfrak{P}S$ is a p-semilattice (isomorphic to $\hat{X}$: **4.1**) characterized by

(3) $$z \in \bar{X} \qquad \text{iff} \qquad (\alpha, \gamma \in z,\ \beta \in S,\ \alpha \leqslant \beta \leqslant \gamma \cdot\Rightarrow\cdot \beta \in z) .$$

Now $a = \{(0, 0, 1), (1, 0, 1)\}$ and $b = \{(1, 0, 0), (0, 1, 1)\}$ belong to $\bar{X}$ (by (3)), and so does their union (in $\mathfrak{P}S$) c: hence $c = a \cup b$ is a partition in $\bar{X}$, but there are no closed subsets $x_0, x_1, x_2 \in X$ such that either (4) or (5) holds

$$(4) \qquad c = x_2 - x_0 , \qquad a = x_1 - x_0 , \qquad b = x_2 - x_1 ,$$

$$(5) \qquad c = x_2 - x_0 , \qquad b = x_1 - x_0 , \qquad a = x_2 - x_1 ,$$

actually in the first case $(1, 0, 0) \leqslant (1, 0, 1) \in a \subset x_1$ hence $(1, 0, 0) \in x_1$, which is absurd for $(1, 0, 0) \in b = x_2 - x_1$; in the second case $(0, 0, 1) \leqslant (0, 1, 1) \in b \subset x_1$ hence $(0, 0, 1) \in x_1$, while it is in $a = x_2 - x_1$.

5 - Distributive unions of subquotients in exact distributive categories

The two last numbers (**5** and **6**) concern the study of inverse symmetrizations of distributive exact categories: partitions are shown to be (**6.1**, **6.3**) a good « surrogate » for exactness in these symmetrizations; some familiarity with preceding works $[\mathbf{5}]_{5,6,7}$ is supposed.

5.1 Let $\mathscr{K}$ be an inverse category (provided with its canonical involution) and A an object of $\mathscr{K}$: the set $\mathscr{K}_0(A) = \mathscr{K}_1(A)$ of projections of A is a semilattice with regard to composition, and its domination $\mathfrak{a}$ coincides with the canonical order of semilattices. For any $\alpha \in \mathscr{K}(A, B)$ the transfer mappings $([\mathbf{5}]_3, \ \S\, 2.17)$

$$(1) \qquad \alpha_\square \colon \mathscr{K}_0(A) \to \mathscr{K}_0(B) , \qquad \alpha_\square(\varepsilon) = \alpha\varepsilon\tilde{\alpha} ,$$

$$(2) \qquad \alpha^\square \colon \mathscr{K}_0(B) \to \mathscr{K}_0(A) , \qquad \alpha^\square(\eta) = \tilde{\alpha}\eta\alpha = \tilde{\alpha}_\square(\eta) ,$$

respect product $([\mathbf{5}]_4, \ \S\, 4.7)$ and order; we remark also that

$$(3) \qquad \alpha^\square \alpha_\square(\varepsilon) = \tilde{\alpha}(\alpha\, \varepsilon\, \tilde{\alpha})\,\alpha = (\tilde{\alpha}\alpha)\; \varepsilon\; (\tilde{\alpha}\alpha) = \varepsilon \cdot \alpha^\square(1) .$$

5.2 Proposition. *If $\mathscr{K}$ is an inverse category and $\alpha \in \mathscr{K}(A, B)$, the transfer mapping* **5.1** (1) *preserves (unrestricted) distributive unions; if α is monic,* **5.1** (1) *reflects them as well.*

Proof. Let ε be the distributive union of (ε_i) in $\mathscr{K}_0(A)$; we verify condition **1.1** (2)″ on $\alpha_\square(\varepsilon)$ and $(\alpha_\square(\varepsilon_i))$: let $\eta, \zeta \in \mathscr{K}_0(B)$ be such that $\alpha_\square(\varepsilon_i) \cdot \eta \prec \zeta$ for any i; then

$$(1) \qquad \alpha^\square(\zeta) \succ \alpha^\square\big((\alpha_\square(\varepsilon_i)\cdot\eta\big) = (\alpha^\square \alpha_\square \varepsilon_i)(\alpha^\square \eta) = \varepsilon_i \cdot (\alpha^\square 1) \cdot \alpha^\square \eta = \varepsilon_i(\alpha^\square \eta)$$

for any i; hence $\alpha^{\square}(\zeta) > \varepsilon \cdot (\alpha^{\square}\eta)$, and

$$(2) \qquad \zeta > \zeta(\alpha_{\square}1) = \alpha_{\square}(\alpha^{\square}\zeta) > \alpha_{\square}\big(\varepsilon(\alpha^{\square}\eta)\big)$$
$$= (\alpha_{\square}\varepsilon)\cdot(\alpha_{\square}\alpha^{\square}\eta) = (\alpha_{\square}\varepsilon)\eta(\alpha_{\square}1) = (\alpha_{\square}\varepsilon)\eta \, .$$

If α is monic, $\alpha^{\square}1 = 1$ and $\alpha^{\square}\alpha_{\square}\varepsilon = \varepsilon$ for any $\varepsilon \in \mathscr{K}_0(A)$, therefore our thesis follows from $\alpha^{\square} = (\tilde{\alpha})_{\square}$ preserving distributive unions.

5.3 Let $\mathscr{D}$ be a distributive exact category, A a $\mathscr{D}$-object and $X = M(A)$ the distributive lattice of subobjects of A with regard to $\mathscr{D}$; for brevity we write $M_O(A)$ and $M_\Theta(A)$ the sets of subobjects of A with regard to the symmetrized categories $\mathscr{H} = \mathscr{D}^O$ and $\mathscr{K} = \mathscr{D}^\Theta$; the last one is inverse.

By [**5**]$_7$, § 3.13, there are isomorphisms of sets with operation

$$(1) \qquad (X_2, \square) \to \big(M_O(A), \square_M\big) \to \big(\mathscr{H}_0(A), \square\big) \, ,$$

$$(2) \qquad (m, n) \;\mapsto\; m/n = \mu \;\mapsto \mu\tilde{\mu} \, ,$$

hence the three are left-inverse idempotent semigroups, and by quotientation with regard to $\boldsymbol{\Phi}$ $\big(\boldsymbol{\Phi}_M$ in $M_O(A)\big)$ one gets two canonical isomorphisms of 0-semilattices (actually p-semilattices by **3.7**)

$$(3) \qquad \hat{X} \to M_\Theta(A) \to \mathscr{K}_0(A) \, .$$

We shall often identify $\big(M(A)\big)^{\wedge}$ and $M_\Theta(A)$ via the first isomorphism of (3); consequently we write $m|n = (m/n)^-$ the $\boldsymbol{\Phi}_M$-class of $m/n \in M_O(A)$.

5.4 If $\alpha \in \mathscr{K}(A, B)$, it is easy to verify that the canonical isomorphism in **5.3** (3) « translates » the mapping

$$(1) \qquad \alpha_M \colon M_\Theta(A) \to M_\Theta(B) \, , \qquad \alpha_M(\mu) = \operatorname{im} \alpha\mu$$

into $\alpha_{\square} \colon \mathscr{K}_0(A) \to \mathscr{K}_0(B)$; thus, by **5.2**, α_M preserves (also infinite) distributive unions, and reflects them if α is monic.

5.5 Corollary. *If $\alpha \in \mathscr{K}(A, B)$ is monic, and $\mu, \mu_i \in M_\Theta(A)$ $(i \in I)$ then μ is the distributive union of (μ_i) in $M_\Theta(A)$ iff $\alpha_M(\mu) = \operatorname{im}(\alpha\mu)$ is the distributive union of $\big(\alpha_M(\mu_i)\big)$ in $M_\Theta(B)$.*

6 - Partitions and exactness

6.1 Theorem. *In the distributive exact category* $\mathscr{D}$ *the short sequence*

(1) $$\cdot \overset{m}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} \cdot$$

is of order two (*i.e.* $pm = 0$) *iff*

(2) $$m \cap \tilde{p} = 0 \qquad \text{in } M_\Theta(A)$$

while it is exact ($\operatorname{im} m = \ker p$) *iff*

(3) $$1_A = m \cup \tilde{p}\,, \qquad \text{partition in } M_\Theta(A)\,.$$

Proof. We may suppose that m is a subobject of A, and p a quotient; the condition (2) is equivalent to $(\tilde{p}p)(m\tilde{m}) = 0$ in $\mathscr{D}^\Theta$ (use the second isomorphism of **5.3** (3)), hence to $pm = 0$ in $\mathscr{D}^\Theta$ (or in $\mathscr{D}$).

As regards the exactness, suppose that $pm = 0$. If $p = \operatorname{coker} m = (1/m)^\sim$ then $\tilde{p} = 1/m$ and (3) holds by **3.6**. Conversely, let (3) be satisfied; as $pm = 0$, $m <_M \ker p = m'$ and we consider $\mu = m' | m \in M_\Theta(A)$

(4) $$m \cap \mu = (m \,\square_M\, (m'/m))^- = (m/m)^- = 0\,,$$

(5) $$\tilde{p} \cap \mu = ((1/m') \,\square_M\, (m'/m))^- = (m'/m')^- = 0\,,$$

(6) $$\mu = 1_A \cap \mu = (m \cap \mu) \cup (\tilde{p} \cap \mu) = 0\,,$$

hence $m = m' = \ker p$.

6.2 Let $\mathscr{D}$, $\mathscr{E}$ be distributive exact categories and $f\colon \mathscr{D} \to \mathscr{E}$ a zero-preserving functor; we recall that f is an $\boldsymbol{O}$-functor (i.e. has an involution-preserving extension $f^{\boldsymbol{O}}\colon \mathscr{D}^{\boldsymbol{O}} \to \mathscr{E}^{\boldsymbol{O}}$) iff it is exact ([**5**]$_1$, § 6.15), while it is a $\boldsymbol{\Theta}$-functor iff it preserves monics, epics, pullbacks of monics and pushout of epics ([**1**], th. 2.4); any $\boldsymbol{O}$-functor is also a $\boldsymbol{\Theta}$-functor (this results also, in a more direct way, from [**5**]$_5$, § 1.6). The « gap » between these two notions is characterized by the following statement.

6.3 Theorem. *Let* $f\colon \mathscr{D} \to \mathscr{E}$ *be a* $\boldsymbol{\Theta}$*-functor between distributive exact categories; for any* A *object of* $\mathscr{D}$, *write*

(1) $$f_A\colon M(A) \to M(fA)\,, \qquad m \mapsto \operatorname{im}(fm)\ (^3)\,,$$

(2) $$f_A^\Theta\colon M_\Theta(A) \to M_\Theta(fA)\,, \qquad \mu \mapsto \operatorname{im}(f^\Theta \mu)\ (^3)\,,$$

(2′) $$\mathscr{K}_0(A) \to \mathscr{K}'_0(fA)\,, \qquad \varepsilon \mapsto f^\Theta(\varepsilon) \qquad (\mathscr{K} = \mathscr{D}^\Theta,\ \mathscr{K}' = \mathscr{E}^\Theta)\,.$$

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(3) f and f^Θ preserve *monics*, and not *subobjects*, generally.

The following conditions are equivalent:

(a) *f is exact;*

(b) *for any A in $\mathscr{D}$*

(3) $$f_A^{\ominus}(m\,|\,n) = (f_A m)\,|\,(f_A n);$$

(c) *for any A in $\mathscr{D}$, the mapping* (2) *is a p-homomorphism;*

(c)′ *the same for* (2)′.

When these conditions are satisfied, all f_A are lattice homomorphisms and $f_A^{\ominus} = (f_A)^{\wedge}$, according to (3).

Proof. The conditions (c) and (c)′ are trivially equivalent, by the second isomorphism in **5.3** (3).

(a) $\Rightarrow$ (b). By [**5**]$_6$, § 2.11.

(b) $\Rightarrow$ (c). It is sufficient to prove that each mapping f_A is a lattice homomorphism, so that $f_A^{\ominus} = (f_A)^{\wedge}$ and the conclusion follows from **3.7**. We write here

(4) $$f_A'\colon P(A) \to P(fA)\,, \qquad f_A'(p) = \operatorname{coim}(fp)$$

the analogous mapping with regard to quotients in $\mathscr{D}$ and $\mathscr{E}$. By **6.2**, both f_A and f_A' preserve intersections (pullbacks of monics and pushouts of epics); moreover, if $m, n \in M(A)$ and $p = \operatorname{cok} m = (1/m)^{\sim}$, $q = \operatorname{cok} n = (1/n)^{\sim}$ are the corresponding quotients of A, by using the property (3)

$$1/f_A(m \cup n) = f_A^{\ominus}(1/(m \cup n)) = f_A^{\ominus}((1/m) \cap (1/n)) = f_A^{\ominus}(\tilde{p} \cap \tilde{q})$$

$$= f_A^{\ominus}((p \cap q)^{\sim}) = (f_A'(p \cap q))^{\sim} = (f_A'(p) \cap f_A'(q))^{\sim}$$

$$= f_A^{\ominus}(\tilde{p}) \cap f_A^{\ominus}(\tilde{q}) = f_A^{\ominus}(1/m) \cap f_A^{\ominus}(1/n) = (1/f_A m) \cap (1/f_A n) = 1/((f_A m) \cup (f_A n))\,.$$

(c) $\Rightarrow$ (a). Short exact sequences must be preserved by **6.1**.

6.4 Non-distributive unions in semilattices $M_{\ominus}(A)$ are probably useless. Actually, in situation **6.1** (1), the condition

(1) $$1_A = m \cup \tilde{p} \qquad (\text{union in } M_{\ominus}(A))$$

can be strictly weaker than **6.1** (3), and in some case trivial (satisfied for *all* $m \neq 0$ and $p \neq 0$); for example, take $\mathscr{G}$ the category of abelian groups, $\mathscr{D} = \mathscr{G}^{\#}$, and $A = (G, X)$ where G is an abelian group and X a chain of subobjects of G containing 0 and 1_G.

It is not difficult to show that exact functors between distributive exact categories (more exactly: their Θ-symmetrized functors) generally do not preserve *simple* unions of Θ-subquotients.

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Sunto

Si studiano le unioni distributive (rispetto all'intersezione) nei semireticoli. Nell'ambito delle categorie esatte distributive, tali unioni vengono utilizzate per caratterizzare l'esattezza delle sequenze corte e dei funtori.

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