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Summability factors for certain sequence spaces (**)

1 - Introduction

In this paper we study summability factors for the following subspaces of the space s of complex sequences.

m: bounded sequences, C: Cesàro summable sequences, c: convergent sequences, ac: absolutely convergent sequences, qc: quasiconvex convergent sequences.

We note that $\mathrm{ac} = \{x \in s: \sum |\Delta x_p| < \infty\}$ and $\mathrm{qc} = \{x \in \mathrm{c}: \sum p\,|\Delta^2 x_p| < \infty\}$. Let $E = \{\mathrm{m, C, c, ac, qc}\}$, and if $u, v \in E$, let (u, v) denote the set of all $f \in s$ such that the sequence of partial sums of $\sum f_p a_p$ is in v whenever the sequence of partial sums of $\sum a_p$ is in u. Our problem is then to characterize the elements of $E \times E$.

Hadamard [2] characterized (c, c) in 1903, and Kojima [4] proved a theorem in 1917 from which characterizations of (C, C), (C, c), and (c, C) can be obtained. It is well known (or at least clear) that (ac, m) = (ac, c) = (ac, ac) = m. Although there are twenty-five elements of $E \times E$, each of which is used to name a sequence space, there are only twelve distinct spaces so named, as we shall see in the summary at the end of the paper.

2 - Notation and background

Throughout, we will use $\alpha = \{\alpha_p\}$ and $\beta = \{\beta_p\}$ to denote the partial sums

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of $\sum a_p$ and $\sum f_p a_p$, respectively. The matrix $F = (f_{pq})$, defined as follows

$$f_{pq} = \begin{matrix} 0 & \text{if } p < q \\ f_q & \text{if } p = q \\ \Delta f_q & \text{if } p > q\,, \end{matrix}$$

will be useful, since $F\alpha = \beta$.

Let $l = \{x \in s\colon \sum |x_p| < \infty\}$, $c_0 = \{x \in s\colon \lim_n x_n = 0\}$, $ac_0 = ac \cap c_0$, $c_0' = \{x \in s\colon \{nx_n\} \in m\}$, $qc_0' = qc \cap c_0'$, and let M denote the set of all complex sequences having bounded Cesàro means.

If A is a complex matrix, let ΣA denote the matrix whose p, q-entry is $\sum_{n=1}^{q} a_{pn}$, let $\Sigma^2 A = \Sigma(\Sigma A)$, and let ΔA denote the matrix whose p, q-entry is $a_{pq} - a_{p,q+1}$. If $x \in s$, let $\Delta x = \{\Delta x_p\}$, $\Delta^2 x = \Delta(\Delta x)$, $\sigma x_p = x_1 + \ldots + x_p$, and $\sigma x = \{\sigma x_p\}$.

We state the theorem of Kojima previously mentioned.

Theorem K. *In order for f to have the property that β is Cesàro summable of order t whenever α is Cesàro summable of order r, it is necessary and sufficient that the following properties hold*

$$\sup_n n^{r-t} |f_n| < \infty\,, \qquad \sup_n \frac{1}{A_n^{(t)}} \sum_{i=1}^{n-r-1} A_i^{(r)} \Big| \sum_{q=0}^{r+1} \binom{r+1}{q} A_{n-i+1}^{(t-q)} \Delta^{r-(q-1)} f_{i+q} \Big| < \infty\,,$$

where $A_n^{(q)} = (n+q-1)!/(q!(n-1)!)$, $A_n^{(-q)} = 0$, $n, q = 1, 2, 3, \ldots$, *and each of r and t is a nonnegative integer.*

3 - Characterizations

Hadamard [**2**] showed in 1903 that $(c, c) = ac$. The Silverman-Toeplitz theorem, which gives three conditions necessary and sufficient for a matrix to be conservative ($Ax \in c$ whenever $x \in c$), appeared some eight years later, and, when applied to the matrix F, affords a simple proof of Hadamard's result, as well as characterizations of (c, m) and (m, m) as follows. If $f \in ac$, then $f \in (m, m)$ and $f \in (c, c) \subseteq (c, m)$. Now suppose $f \in (c, m)$ but $f \notin ac$. We see that f satisfies two of the $S-T$ conditions but $\sup_p \sum_{q=1}^{\infty} |f_{pq}| = \infty$. It is classical under these conditions to construct a null sequence which the matrix transforms into an unbounded sequence, and this would contradict our assumption that $f \in (c, m)$. Thus $f \in (c, m)$ implies $f \in ac$. So $(c, m) = ac \subseteq (m, m) \subseteq (c, m)$. Hence $(c, m) = (m, m) = ac$.

Kojima's theorem for $r=0$ and $t=1$ states that $f \in (\mathrm{c}, \mathrm{C})$ iff

$$\sup_n |f_n|/n < \infty , \qquad \sup_n (1/n) \sum_{i=1}^{n-1} |(n-i+1)\Delta f_i + f_{i+1}| < \infty .$$

For i fixed we have $((n-i+1)/n)\Delta f_i + f_{i+1}/n \to \Delta f_i$ as $n \to \infty$. Hence the second condition implies $\sum_{i=1}^{\infty} |\Delta f_i| < \infty$, so that $f \in \mathrm{ac}$. Clearly $f \in \mathrm{ac}$ implies both of Kojima's conditions. Hence $(\mathrm{c}, \mathrm{C}) = \mathrm{ac}$. Note that Kojima's first condition for this case is superfluous, since his second condition implies $f \in \mathrm{c}$.

Next we investigate (m, c) and (m, C). Suppose $f \in (\mathrm{m}, \mathrm{c})$. Then $f \in \mathrm{ac}$ since $(\mathrm{m}, \mathrm{c}) \subseteq (\mathrm{c}, \mathrm{c}) = \mathrm{ac}$. If $\alpha \in \mathrm{m}$, then

$$(*) \qquad \beta_n = \sum_{p=1}^{n} \alpha_p \Delta f_p + f_{n+1}\alpha_n ,$$

which means that $\{f_{n+1}\alpha_n\}$ converges since $\beta \in \mathrm{c}$ and $\sum \alpha_p \Delta f_p$ converges. But convergence of $\{f_{n+1}\alpha_n\}$ for every $\alpha \in \mathrm{m}$ implies that $f \in \mathrm{c}_0$. Hence $f \in \mathrm{ac}_0$. Conversely, if $f \in \mathrm{ac}_0$, it follows from $(*)$ that $\beta \in \mathrm{c}$ whenever $\alpha \in \mathrm{m}$. Thus $(\mathrm{m}, \mathrm{c}) = \mathrm{ac}_0$. We also find that $(\mathrm{m}, \mathrm{C}) = \mathrm{ac}_0$ as follows. If $f \in \mathrm{ac}_0$, then $f \in (\mathrm{m}, \mathrm{c}) \subseteq (\mathrm{m}, \mathrm{C})$. Conversely, if $f \in (\mathrm{m}, \mathrm{C})$, then $f \in (\mathrm{c}, \mathrm{C}) = \mathrm{ac}$, and consequently from $(*)$, $\{f_{n+1}\alpha_n\}$ is Cesàro summable for every $\alpha \in \mathrm{m}$ since for $\alpha \in \mathrm{m}$, β is Cesàro summable and $\sum \alpha_p \Delta f_p$ is convergent. This of course implies that $f \in \mathrm{c}_0$. Thus $f \in \mathrm{ac}_0$. Hence $(\mathrm{m}, \mathrm{C}) = \mathrm{ac}_0$.

We could use Kojima's theorem to characterize (C, c), but instead will use the matrix ΔF and obtain a characterization of (C, m) as well. We note that $(\Delta F)(\sigma\alpha) = \beta$ for all α. Form a matrix G from ΔF by multiplying each element in the q-th column of ΔF by q, $q = 1, 2, 3, \ldots$. Then

$$g_{pq} = \begin{array}{ll} 0 & \text{if } p < q \\ q f_q & \text{if } p = q \\ q(f_q - 2f_{q+1}) & \text{if } p = q+1 \\ q\Delta^2 f_q & \text{if } p > q+1 . \end{array}$$

Clearly G transforms the Cesàro means of α into β, and so $\beta \in \mathrm{c}$ whenever α is Cesàro summable iff G is conservative. Since G has convergent columns and the sequence of row sums of G is convergent (in fact constant), then by the $S-T$ theorem G is conservative iff

$$(**) \quad \sup_p \sum_{q=1}^{\infty} |g_{pq}| = \sup_p \Big(\sum_{q=1}^{p-2} q|\Delta^2 f_q| + (p-1)|f_{p-1} - 2f_p| + p|f_p| \Big) < \infty ,$$

and this condition holds iff $\sum_{q=1}^{\infty} q|\Delta^2 f_q| < \infty$ and $\{qf_q\} \in \mathrm{m}$. Hence

$(C, c) = qc \cap c_0' = qc_0'$. If (**) does not hold, then there is a null sequence x such that $Gx \notin m$. There exists $\alpha \in C$ such that the nth Cesàro mean of α is x_n, $n = 1, 2, 3, \ldots$, so that $Gx = \beta \notin m$. Hence if $f \in (C, m)$, then (**) holds, and so $f \in qc_0'$. Conversely, if $f \in qc_0'$, then $f \in (C, c) \subseteq (C, m)$. Thus $(C, m) = qc_0'$.

Next we show that $(C, C) = qc$ by utilizing Kojima's theorem, which states that for $r = t = 1$, $f \in (C, C)$ iff

$$\sup_n |f_n| < \infty, \qquad \sup_n \sum_{i=1}^{n-2} (i/n)\,|(n-i+1)\Delta^2 f_i + 2\Delta f_{i+1}| < \infty .$$

For i fixed, we have

$$(i/n)(n-i+1)\Delta^2 f_i + (2i/n)\Delta f_{i+1} \to i\Delta^2 f_i \qquad \text{as } n \to \infty ,$$

so that Kojima's second condition implies that $\sum_{i=1}^{\infty} i|\Delta^2 f_i| < \infty$, and this together with his first condition, implies that $f \in qc$. Conversely, if $f \in qc$, then Kojima's two conditions hold since $f \in c$, $\sum i|\Delta^2 f_i| < \infty$, and $\{i\Delta f_i\}$ is a null sequence. Thus $(C, C) = qc$.

We can show that

$$\begin{aligned}
(c, ac) &= (m, ac) = l ,\\
(c, qc) &= (m, qc) = (C, ac) = \{f\colon \textstyle\sum p\,|f_p| < \infty\} ,\\
(ac, qc) &= c_0',\\
(C, qc) &= \{f\colon \textstyle\sum p^2 |f_p| < \infty\} ,
\end{aligned}$$

but will include a proof of the last statement only since the other proofs are similar to the one we will give and much simpler.

Suppose $\sum p^2|f_p| < \infty$ and $\alpha \in C$. We have

$$\Delta\beta_p = f_{p+1}a_{p+1} , \qquad p\,\Delta^2\beta_p = p(f_{p+1}a_{p+1} - f_{p+2}a_{p+2}) .$$

Now since $\alpha \in C$, then by a well known property of Cesàro summable series $p^{-1}a_p \to 0$ as $p \to \infty$ ([**3**], p. 484). Thus

$$\sum |\Delta\beta_p| \leqq \sum (p+1)\,|f_{p+1}|\,\frac{|a_{p+1}|}{p+1} < \infty ,$$

$$\sum p\,|\Delta^2\beta_p| \leqq \sum (p+1)^2 |f_{p+1}|\,\frac{|a_{p+1}|}{p+1} + \sum (p+2)^2 |f_{p+2}|\,\frac{|a_{p+2}|}{p+2} < \infty ,$$

and so $\beta \in qc$. Conversely, suppose $f \in (C, qc)$ but $\sum p^2|f_p| = \infty$. Then either

$\sum (2p)^2 |f_{2p}| = \infty$ or $\sum (2p+1)^2 |f_{2p+1}| = \infty$. Without loss of generality, assume the latter. There exists a nonincreasing null sequence $\{r_{2p-1}\}$ such that $\sum p^2 |f_{2p+1}| r_{2p+1} = \infty$. Let

$$a_1 = r_1 , \qquad a_{2n} = 0 , \qquad n = 1, 2, 3, \dots ;$$
$$a_{4n-1} = (2-4n) r_{2n-1} , \qquad n = 1, 2, 3, \dots ;$$
$$a_{4n+1} = (2n-1) r_{2n-1} + (2n+1) r_{2n+1} , \qquad n = 1, 2, 3, \dots .$$

Then

$$\begin{aligned}
\alpha_{4n+1} = \alpha_{4n+2} &= a_1 + \sum_{p=1}^{n} a_{4p-1} + \sum_{p=1}^{n} a_{4p+1} \\
&= r_1 + \sum_{p=1}^{n} (2-4p) r_{2p-1} + \sum_{p=1}^{n} [(2p-1) r_{2p-1} + (2p+1) r_{2p+1}] \\
&= r_1 - 2r_1 + \sum_{p=2}^{n} (2-4p) r_{2p-1} + r_1 + \sum_{p=2}^{n} (2p-1) r_{2p-1} \\
&\qquad + \sum_{p=1}^{n-1} (2p+1) r_{2p+1} + (2n+1) r_{2n+1} \\
&= \sum_{p=2}^{n} (2-4p) r_{2p-1} + \sum_{p=2}^{n} (2p-1) r_{2p-1} + \sum_{q=2}^{n} (2q-1) r_{2q-1} + (2n+1) r_{2n+1} \\
&= (2n+1) r_{2n+1} , \qquad n = 0, 1, 2, \dots ,
\end{aligned}$$

and so

$$\begin{aligned}
\alpha_{4n+3} = \alpha_{4n+4} &= (a_1 + \dots + a_{4n+2}) + a_{4n+3} = (2n+1) r_{2n+1} + a_{4(n+1)-1} \\
&= (2n+1) r_{2n+1} + (-2-4n) r_{2n+1} = -(2n+1) r_{2n+1} , \qquad n = 0, 1, 2, \dots .
\end{aligned}$$

Thus $\sum_{p=1}^{4n} \alpha_p = 0$, so that

$$\sum_{p=1}^{4n+1} \alpha_p = \alpha_{4n+1} = (2n+1) r_{2n+1} ,$$

$$\sum_{p=1}^{4n+2} \alpha_p = (2n+1) r_{2n+1} + \alpha_{4n+2} = 2(2n+1) r_{2n+1} , \qquad \text{and}$$

$$\begin{aligned}
\sum_{p=1}^{4n+3} \alpha_p &= 2(2n+1) r_{2n+1} + \alpha_{4n+3} = 2(2n+1) r_{2n+1} - (2n+1) r_{2n+1} \\
&= (2n+1) r_{2n+1} .
\end{aligned}$$

Hence for $n = 0, 1, 2, \ldots,$ we have

$$0 \leqq \frac{\alpha_1 + \alpha_2 + \ldots + \alpha_{4n+q}}{4n+q} \leqq r_{2n+1}, \qquad q = 1, 2, 3, 4 .$$

So $\alpha \in \mathrm{C}$ since $(\sigma\alpha_p)/p \to 0$ as $p \to \infty$. But $\beta \notin \mathrm{qc}$ since

$$\begin{aligned}\sum_{n=1}^{\infty} 2n\,|\Delta^2\beta_{2n}| &= \sum_{n=1}^{\infty} 2n\,|f_{2n+1}a_{2n+1} - f_{2n+2}\,a_{2n+2}| = \sum_{n=1}^{\infty} 2n\,|f_{2n+1}a_{2n+1}| \\ &= \sum_{n=1}^{\infty} 4n\,|f_{4n+1}a_{4n+1}| + \sum_{n=1}^{\infty}(4n-2)\,|f_{4n-1}a_{4n-1}| \\ &\geqq \sum_{n=1}^{\infty} 4n\,|f_{4n+1}|(2n+1)\,r_{2n+1} + \sum_{n=1}^{\infty}(4n-2)\,|f_{4n-1}|(4n-2)\,r_{2n-1} \\ &\geqq \sum_{n=1}^{\infty}(2n)^2\,|f_{4n+1}|r_{4n+1} + \sum_{n=1}^{\infty}(2n-1)^2\,|f_{4n-1}|r_{4n-1} \\ &= \sum_{n=1}^{\infty} n^2\,|f_{2n+1}|r_{2n+1} = \infty .\end{aligned}$$

This contradicts our assumption that $f \in (\mathrm{C}, \mathrm{qc})$. Thus if $f \in (\mathrm{C}, \mathrm{qc})$, then $\sum p^2|f_p| < \infty$. Consequently $(\mathrm{C}, \mathrm{qc}) = \{f\colon \sum p^2|f_p| < \infty\}$.

Next we study (qc, m), (qc, c), and (qc, C), using the matrix Σ^2F. We note that

$$p, q\text{-term of } \Sigma^2F = \begin{cases} qf_1 & \text{if } p = 1 \\ qf_1 - f_2 - \ldots - f_p & \text{if } 1 < p \leqq q \\ qf_1 - f_2 - \ldots - f_{q+1} & \text{if } p > q ,\end{cases}$$

and $(\Sigma^2F)(\Delta^2\alpha) = \beta - k$ if $\alpha \in \mathrm{qc}$, where k is the constant term sequence each term of which equals $f_1 \cdot \sum_{p=1}^{\infty} a_p$. Divide each element of the qth column of Σ^2F by q, $q = 1, 2, 3, \ldots$. Call the new matrix H. Then H transforms $\{p\,\Delta^2\alpha_p\}$ into $\beta - k$ if $\alpha \in \mathrm{qc}$, and

$$h_{pq} = \begin{cases} f_1 & \text{if } p = 1 \\ f_1 - (f_2 + \ldots + f_p)/q & \text{if } 1 < p \leqq q \\ f_1 - (f_2 + \ldots + f_{q+1})/q & \text{if } p > q .\end{cases}$$

Since H has convergent columns, H is an l to c map iff $\sup_{p,q}|h_{pq}| < \infty$, a condition which alone is necessary and sufficient for H to be an l to m map. It is clear that $\sup_{p,q}|h_{pq}| < \infty$ implies $\{(\sigma f_q)/q\} \in \mathrm{m}$, and the converse can be shown. Since each $x \in l$ gives rise to an $\alpha \in \mathrm{qc}$ such that $x = \{p\Delta^2\alpha_p\}$

([1], lemma 4), then $f \in (\mathrm{qc}, \mathrm{m})$ implies $\{(\sigma f_q)/q\} \in \mathrm{m}$. Conversely, $\{(\sigma f_q)/q\} \in \mathrm{m}$ implies $f \in (\mathrm{qc}, \mathrm{c}) \subseteq (\mathrm{qc}, \mathrm{m})$. Thus $(\mathrm{qc}, \mathrm{c}) = (\mathrm{qc}, \mathrm{m}) = M$. Next we show that $(\mathrm{qc}, \mathrm{C}) = (\mathrm{qc}, \mathrm{c})$. Suppose $f \in (\mathrm{qc}, \mathrm{C})$ and again consider the matrix H. Replace each column of H with the sequence of Cesàro means of that column. Call the resulting matrix K and note that K transforms $\{p\Delta^2\alpha_p\}$ into $(-k)$ plus the sequence of Cesàro means of β if $\alpha \in \mathrm{qc}$. Furthermore we see that

$$\lim_p k_{pq} = f_1 - (f_2 + \dots + f_{q+1})/q \, .$$

Since $f \in (\mathrm{qc}, \mathrm{C})$, then K is an l to c map so that $\sup_{p,q} |k_{pq}| < \infty$, which implies that

$$\sup_q \left| f_1 - \frac{f_2 + \dots + f_{q+1}}{q} \right| < \infty ,$$

and this by the proof above implies that $f \in (\mathrm{qc}, \mathrm{c})$. So $(\mathrm{qc}, \mathrm{C}) \subseteq (\mathrm{qc}, \mathrm{c}) \subseteq (\mathrm{qc}, \mathrm{C})$, and consequently $(\mathrm{qc}, \mathrm{C}) = (\mathrm{qc}, \mathrm{c})$.

Similarly, by using the matrix ΣF, we can show that $(\mathrm{ac}, \mathrm{C}) = \mathrm{m}$, but will omit the proof.

The author's theorem in $[\mathbf{1}]_1$ for $k = j = 2$ can be stated as follows.

In order for $Ax \in \mathrm{qc}$ whenever $x \in \mathrm{qc}$, it is necessary and sufficient that the following conditions hold:

(1) A has convergent columns,

(2) $$\{\sum_{q=1}^{\infty} a_{pq}\}_{p=1}^{\infty} \in \mathrm{qc} \, ,$$

(3) $$\sup_{n,p} (1/n) \left| \sum_{r=1}^{n} \sum_{q=1}^{r} a_{pq} \right| < \infty \, ,$$

(4) $$\sup_n (1/n) \sum_{p=1}^{\infty} p \left| \sum_{r=1}^{n} \sum_{q=1}^{r} \Delta_1^2 a_{pq} \right| < \infty \, .$$

The spaces denoted by BV_1^* and BV_2^* in $[\mathbf{1}]_1$ are denoted by ac and qc, respectively, in the present paper. We will apply this theorem for $A = F$ and $x = \alpha$ in order to characterize (qc, qc). Note that (1) and (2) hold for F. Simple calculations show that

$$\sum_{r=1}^{n} \sum_{q=1}^{r} f_{pq} = \begin{cases} nf_1 & \text{if } p = 1 \\ nf_1 - f_2 - \dots - f_p & \text{if } 1 < p \leqq n \\ nf_1 - f_2 - \dots - f_{n+1} & \text{if } p > n \, , \end{cases}$$

and

$$\sum_{r=1}^{n}\sum_{q=1}^{r}\Delta_1^2 f_{pq}=\Delta_1^2\sum_{r=1}^{n}\sum_{q=1}^{r} f_{pq}=\begin{matrix} f_{p+1}-f_{p+2} & \text{if } p<n \\ f_{p+1} & \text{if } p=n \\ 0 & \text{if } p>n\,. \end{matrix}$$

Thus (3) holds for F iff $\{(\sigma f_q)/q\}\in \mathrm{m}$, and (4) holds for F iff

$$\left\{\frac{1\cdot|\Delta f_2|+2\cdot|\Delta f_3|+\ldots+(n-1)\cdot|\Delta f_n|}{n}\right\}\in \mathrm{m} \qquad \text{and} \qquad f\in \mathrm{m}\,.$$

This last condition is equivalent to

$$\left\{\frac{1\cdot|\Delta f_1|+2\cdot|\Delta f_2|+\ldots+n\cdot|\Delta f_n|}{n}\right\}\in \mathrm{m} \qquad \text{and} \qquad f\in \mathrm{m}\,.$$

Clearly $f\in \mathrm{m}$ implies (3) for F. Hence (qc, qc) $=\{f\in \mathrm{m}\colon \{|p\Delta f_p|\}\in M\}$.

Similarly, by using the author's theorem in [1]$_1$ for $k=2$ and $j=1$, we can show that (qc, ac) $=\{f\colon \{|f_p|\}\in M\}$, but will omit the proof.

4 - Summary

Since the preceding section was written in narrative style, we include the following summary of our results for the convenience of the reader.

1. (c, c) = (c, m) = (m, m) = (c, C) = ac .
2. (m, c) = (m, C) = ac$_0$.
3. (C, c) = (C, m) = qc$'_0$.
4. (C, C) = qc .
5. (c, ac) = (m, ac) = l .
6. (c, qc) = (m, qc) = (C, ac) = $\{f\colon \sum p|f_p|<\infty\}$.
7. (C, qc) = $\{f\colon \sum p^2|f_p|<\infty\}$.
8. (ac, m) = (ac, C) = (ac, c) = (ac, ac) = m .
9. (ac, qc) = c$'_0$.
10. (qc, m) = (qc, C) = (qc, c) = M .
11. (qc, ac) = $\{f\colon \{|f_p|\}\in M\}$.
12. (qc, qc) = $\{f\in \mathrm{m}\colon \{|p\Delta f_p|\}\in M\}$.

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Summary

In this paper we study summability factors with respect to five sequence spaces.

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