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On partial boundedness of differential equations with time delay (**)

1 - Introduction

The Lyapunov function method has been used to investigate the boundedness properties of solutions of ordinary differential equations [3], [5]. Analogous results for partial boundedness, that is, boundedness with respect to a part of the variables were obtained for ordinary differential equations by Oziraner [4]. Recently Corduneanu [2] investigated some problems of partial stability related to linear differential equations with time delays:

$$(*) \qquad \dot{x}(t) = A(t, x_t) + B(t, y_t)\,, \qquad \dot{y}(t) = C(t, x_t) + D(t, y_t)\,,$$

where $x \in R^n$, $y \in R^m$, $t \in R^+ = [0, \infty)$ and the subscript t indicates the restriction of the corresponding function to the interval $[t-h, t]$, with $h > 0$ fixed: $x_t(\theta) = x(t+\theta)$, $-h \leqslant \theta \leqslant 0$. A, B, C, and D are defined in the following ways

$$A(t, \varphi) = \int_{-h}^{0} [d_s a(t, s)]\varphi(s)\,, \quad t \in R^+, \qquad B(t, \varphi) = \int_{-h}^{0} [d_s b(t, s)]\varphi(s)\,, \quad t \in R^+,$$

$$C(t, \varphi) = \int_{-h}^{0} [d_s c(t, s)]\varphi(s)\,, \quad t \in R^+, \qquad D(t, \varphi) = \int_{-h}^{0} [d_s d(t, s)]\varphi(s)\,, \quad t \in R^+,$$

where $a(t, s)$, $b(t, s)$, $c(t, s)$ and $d(t, s)$ are matrices.

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In Corduneanu's investigations only the x-component of the unknown solution of (*) was under consideration with respect to its behaviour. The problem of searching simultaneously for informations on both the x-component and the y-component of the solution is however interesting. In [**1**], we began these considerations in respect of stability, uniform stability, equi-asymptotic stability and generalized asymptotic stability of a more general system of equations than (*).

In this paper, we search for informations on the boundedness properties of the system considered in [**1**] with respect to the x-component and the y-component simultaneously. As in [**1**] we approach the problem by making use of two Lyapunov functionals and the theory of differential inequalities. We finally give an example to illustrate the usefulness of the results.

2 - Preliminaries

Let R^n denote the n-dimensional Euclidean space with convenient norm $\|\cdot\|$. Denote by R^+, the non-negative real numbers. For $h > 0$, let $l^n = C([-h, 0], R^n)$ denote the space of continuous functions with domain $[-h, 0]$ and range in R^n. For $\varphi \in l^n$, we define $\|\varphi\|_0 = \sup_{-h \leqslant s \leqslant 0} \|\varphi(s)\|$. Suppose that $x \in C([-h, 0], R^n)$ and for $t \in R^+$, x_t denotes a translation of the restriction of x to the interval $[t-h, t]$, then x_t is an element of l^n defined by $x_t(s) = x(t+s)$, $-h \leqslant s \leqslant 0$.

Consider the functional differential system with time delay

$$(1) \qquad \dot{x}(t) = f(t, x_t, y_t) , \qquad \dot{y}(t) = g(t, x_t, y_t) ,$$

where $t \in R^+$, $x \in R^n$, $y \in R^m$ and f, g are continuous functions from $R^+ \times C([-h, 0], R^n) \times C([-h, 0], R^m)$ into R^n and R^m respectively. Let (t_0, φ, ψ) belong to $R^+ \times l^n \times l^m$; we denote by $x = x(t; t_0, \varphi, \psi)$ and $y = y(t; t_0, \varphi, \psi)$ the solution of (1) such that $x_{t_0} = \varphi$ and $y_{t_0} = \psi$. For any $t \geqslant t_0$, we denote by $x_t(t_0; \varphi, \psi)$ and $y_t(t_0; \varphi, \psi)$ the corresponding elements of $C([-h, 0], R^n)$ and $C([-h, 0], R^m)$ respectively such that $x_{t_0}(t_0; \varphi, \psi) = \varphi$ and $y_{t_0}(t_0; \varphi, \psi) = \psi_0$. We assume that f and g of system (1) are continuous and satisfy the conditions for the uniqueness of the solution for $\|\varphi\|_0 + \|\psi\|_0 < \infty$, $t \geqslant 0$ and any solution $x_t(t_0; \varphi, \psi)$, $y_t(t_0; \varphi, \psi)$ is defined for all $t \geqslant 0$ for which $\|x_t(t_0; \varphi, \psi\| < \infty$ and $\|y_t(t_0; \varphi, \psi)\| < \infty$.

Definition 2.1. (i) The solutions of (1) are equi-bounded with respect to the x-component (y-component) if for any $\alpha > 0$ and $t_0 \geqslant 0$, there exists a

$\beta = \beta(t_0, \alpha) > 0$ which is continuous in t_0 for each α such that if $\|\varphi_0\|_0 + \|\psi_0\|_0 \leqslant \alpha$, then $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta$ $(\|y_t(t_0; \varphi_0, \psi_0)\| < \beta)$, for all $t \geqslant t_0$.

(ii) The solutions of (1) are uniformly bounded with respect to the x-component (y-component) if β in (i) is independent of t_0.

(iii) The solutions of (1) are uniformly bounded in (φ_0, ψ_0) with respect to the x-component (y-component) if for any $t_0 \geqslant 0$ and a compact set S of the space $C([-h, 0], R^n) \times C([-h, 0], R^m)$, we can find $\beta(t_0, S) > 0$ such that if $(\varphi_0, \psi_0) \in S$, $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta$ $(\|y_t(t_0; \varphi_0, \psi_0)\| < \beta)$, for all $t \geqslant t_0$.

(iv) The solutions of (1) are uniformly bounded in $\{t_0, \varphi_0, \psi_0\}$ with respect to the x-component (y-component) if we can choose β in (iii) independent of t_0 for any compact set S.

(v) The solutions of (1) are quasi-equi-ultimately bounded with respect to the x-component (y-component) if for each $\alpha > 0$ and $t_0 \geqslant 0$, there exists positive numbers $\beta = \beta(t_0, \alpha)$ and $T = T(t_0, \alpha)$ such that $\|\varphi_0\|_0 + \|\psi_0\|_0 \leqslant \alpha$ implies $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta$ $(\|y_t(t_0; \varphi_0, \psi_0)\| < \beta)$, for all $t \geqslant t_0 + T$.

(vi) The solutions of (1) are quasi-uniform ultimately bounded with respect to the x-component (y-component) if T in (v) is independent of t_0.

(vii) The solutions of (1) are uniform-ultimately bounded with respect to the x-component (y-component) if (ii) and (vi) hold together.

Corresponding to Definition 2.1 we can formulate definitions of boundedness of solutions of (1) with respect to both the x-component and the y-component. We give one such definition since the others follow very easily.

Definition 2.2. The solutions of (1) are equi-bounded with respect to the x-component and the y-component if for any $\alpha > 0$ and $t_0 \geqslant 0$, there exist a $\beta_1 = \beta_1(t_0, \alpha) > 0$ and $\beta_2 = \beta_2(t_0, \alpha)$ which are continuous in t_0 for each α such that if $\|\varphi_0\|_0 + \|\psi_0\|_0 \leqslant \alpha$, then $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta_1$ and $\|y_t(t_0; \varphi_0, \psi_0)\| < \beta_2$, for all $t \geqslant t_0$.

We now apply Lyapunov's method to obtain necessary and sufficient conditions for the properties of boundedness of solutions of (1), enumerated in Definitions 2.1 and 2.2 to hold.

Theorem 2.3. *The solutions of the system* (1) *are said to be equibounded with respect to the x-component if and only if there exists a Lyapunov functional $V(t, \varphi, \psi)$ defined for $0 \leqslant t < \infty$, $\|\varphi\|_0 < \infty$, $\|\psi\|_0 < \infty$ and satisfying*

(i) $a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi)$, *where $a(r)$ is a continuous monotonically increasing function and $a(0) = 0$, $a(r) \to \infty$ as $r \to \infty$;*

(ii) $V(t, x_t(t_0; \varphi_0, \psi_0), y_t(t_0; \varphi_0, \psi_0))$ *does not grow for any solution* $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ *of* (1).

Proof. Let $K = V(t_0, \varphi_0, \psi_0)$ where $\|\varphi_0\|_0 + \|\psi_0\|_0 \leqslant \alpha$, then there exists $\beta(K) = \beta(t_0, \alpha) > 0$ such that provided $\|x_t(t_0; \varphi_0, \psi_0)\| > \beta$, then $a(\|x_t(t_0; \varphi_0, \psi_0)\|) > K$. By (i) and (ii)

$$a(\|x_t(t_0; \varphi_0, \psi_0)\|) \leqslant V(t, x_t(t_0; \varphi_0, \psi_0), y_t(t_0; \varphi_0, \psi_0)) \leqslant V(t_0, \varphi_0, \psi_0) .$$

We now claim that for any solution $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ of (1),

$$\|x_t(t_0; \varphi_0, \psi_0)\| < \beta \qquad \text{for all } t \geqslant t_0 .$$

Suppose not, then $\exists t_1 > t_0$ such that $\|x_{t_1}(t_0; \varphi_0, \psi_0)\| > \beta$ and hence

$$K < a(\|x_{t_1}(t_0, \varphi_0, \psi_0)\|) \leqslant V(t_1, \varphi, \psi) \leqslant V(t_0, \varphi_0, \psi_0) = K ,$$

which is a contradiction. Hence $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta$ for all $t \geqslant t_0$.

Conversely, define a Lyapunov functional as follows

$$V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|x_{t+\sigma}(t_0; \varphi_0, \psi_0)\|\} .$$

Since the system (1) is equi-bounded, V is defined. Now if $\sigma = 0$, then $\|\varphi\|_0 \leqslant V(t, \varphi, \psi)$ and if we set $a(\|\varphi\|_0) = \|\varphi\|_0$, we have (i). Suppose $t_1 < t_2$, then

$$V(t_1, x_{t_1}(t_0; \varphi_0, \psi_0), y_{t_1}(t_0; \varphi_0, \psi_0)) = \sup_{\sigma \geqslant 0} \{\|x_{t_1+\sigma}(t_0; \varphi_0, \psi_0)\|\}$$

$$\geqslant \sup_{\sigma \geqslant 0} \{\|x_{t_2+\sigma}(t_0; \varphi_0, \psi_0)\|\} = V(t_2, x_{t_2}(t_0; \varphi_0, \psi_0), y_{t_2}(t_0; \varphi_0, \psi_0)) .$$

Hence $V(t, \varphi, \psi)$ does not grow for any solution $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ of (1).

Theorem 2.4. *The solutions of the system* (1) *are said to be equi-bounded with respect to the y-component if and only if there exists a Lyapunov functional* $W(t, \varphi, \psi)$ *defined for* $0 \leqslant t < \infty$, $\|\varphi\|_0 < \infty$, $\|\psi\|_0 < \infty$ *and satisfying*

(i) $a(\|\psi\|_0) \leqslant W(t, \varphi, \psi)$, *where* $a(r)$ *is a continuous monotonically increasing function and* $a(0) = 0$ *with* $a(r) \to \infty$ *as* $r \to \infty$;

(ii) $W(t, \varphi, \psi)$ *does not grow for any solution* $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ *of system* (1).

Proof. Using arguments similar to the last theorem with obvious modifications the result follows.

Theorem 2.5. *Assume that there exists a Lyapunov functional* $V(t, \varphi, \psi)$ *defined on* $0 \leqslant t < \infty$, $\|\psi\|_0 \geqslant K > 0$ *which satisfies*

(i) $a(\|\psi\|_0) \leqslant V(t, \varphi, \psi)$, *where* $a(r) \to \infty$ *as* $r \to \infty$ *and* $a(r)$ *is continuous and increasing*;

(ii) $V(t, \varphi, \psi)$ *does not grow for any solution* $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ *of the system* (1).

Corresponding to each L, *we can find another Lyapunov functional* $W(t, \varphi, \psi)$ *defined on* $0 \leqslant t < \infty$, $\|\varphi\|_0 \geqslant K_1(L)$, $\|\psi\|_0 \leqslant L$ *and satisfying*

(iii) $a_1(\|\varphi\|_0) \leqslant W(t, \varphi, \psi)$, *where* $a_1(r) \to \infty$ *as* $r \to \infty$, $a_1(r)$ *is continuous and increasing*;

(iv) $W(t, \varphi, \psi)$ *does not grow for any solution* $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ *of the system* (1).

Then the solutions of the system (1) *are equi-bounded with respect to the x-component and the y-component simultaneously.*

Proof. Let $\alpha > K$ and $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ be a solution of (1) where $t_0 \geqslant 0$, $\|\varphi_0\|_0 \leqslant \alpha$ and $\|\psi_0\|_0 \leqslant \alpha$. Let $L = V(t_0; \varphi_0, \psi_0)$, then there exists $\beta(L) = \beta(t_0, \alpha) \geqslant K$ such that $a(\|y_t(t_0; \varphi_0, \psi_0)\|) > L$ provided $\|y_t(t_0; \varphi_0, \psi_0)\| > \beta$. We claim that $\|y_t(t_0; \varphi_0, \psi_0)\| < \beta$ for all $t \geqslant t_0$, provided $\|\varphi_0\|_0 \leqslant \alpha$ and $\|\psi_0\|_0 \leqslant \alpha$. Suppose not, then there exists $t_1 > t_0$ such that $\|y_{t_1}(t_0; \varphi_0, \psi_0)\| > \beta$ and so by (i) and (ii)

$$L < a(\|y_{t_1}(t_0; \varphi_0, \psi_0)\|) \leqslant V(t_1, x_{t_1}(t_0; \varphi_0, \psi_0), y_{t_1}(t_0; \varphi_0, \psi_0) \leqslant V(t_0, \varphi_0, \psi_0) = L\,,$$

which is a contradiction. Hence for $t \geqslant t_0$, $\|y_t(t_0; \varphi_0, \psi_0)\| < \beta$.

Choose $\alpha_1(\alpha) = \max\{\alpha, K_1(\beta)\}$ and consider the Lyapunov functional $W(t, \varphi, \psi)$ defined on $0 \leqslant r < \infty$, $\|\varphi\|_0 \geqslant K_1(\beta)$, $\|\psi\|_0 \leqslant \beta$. Now let $M = W(t_0, \varphi_0, \psi_0)$, then there exists $\beta_1(M) = \beta_1(t_0, \alpha) \geqslant K_1 > 0$ such that $a_1(\|x_t(t_0; \varphi_0, \psi_0)\|) > M$ provided $\|x_t(t_0; \varphi_0, \psi_0)\| > \beta_1$. We claim that $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta_1$ for all $t \geqslant t_0$. In fact, suppose there exists $t_1 > t_0$ such that $\|x_{t_1}(t_0; \varphi_0, \psi_0)\| > \beta_1$, then

$$M < a_1(\|x_{t_1}(t_0; \varphi_0, \psi_0)\|) \leqslant W(t_1; \varphi_0, \psi_0), x_{t_1}(t_0; \varphi_0, \psi_0), y_{t_1}(t_0; \varphi_0, \psi_0))$$

$$\leqslant W(t_0; \varphi_0, \psi_0) = M\,,$$

which is a contradition, hence the result.

Theorem 2.6. *Assume that there exists a Lyapunov functional* $V(t, \varphi, \psi)$ *defined on* $0 \leqslant t < \infty$, $\|\varphi\|_0 \geqslant K$ $\|\psi\|_0 < \infty$ *which satisfies, for any compact subset* S *of* $l^n \times l^m$,

(i) $a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \leqslant \psi_S(t)$, $(\varphi, \psi) \in S$, *where* $a(r) \to \infty$ *as* $r \to \infty$, $a(r)$ *is continuous and increasing;*

(ii) $V(t, \varphi, \psi)$ *does not grow for any solution* $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ *of system* (1).

Then the solution of (1) *are uniformly bounded in* (φ_0, ψ_0) *with respect to the x-component.*

Proof. For $t_0 \geqslant 0$ and any compact set $S \subset l^n \times l^m$, there exists $\beta(t_0, S) > 0$ such that $a(\|x_t(t_0; \varphi_0, \psi_0)\|) > \varphi_S(t_0)$ provided $\|x_t(t_0; \varphi_0, \psi_0)\| > \beta$. Let $(\varphi_0, \psi_0) \in S$. and $t \geqslant t_0$, then by (i) and (ii)

$$a(\|x_t(t_0; \varphi_0, \psi_0)\|) \leqslant V(t, x_t(t_0; \varphi_0, \psi_0), y_t(t_0; \varphi_0, \psi_0)) \leqslant V(t_0, \varphi_0, \psi_0) \leqslant \varphi_s(t_0) .$$

We claim that for $(\varphi_0, \psi_0) \in S$, $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta$ for all $t \geqslant t_0$. Suppose not, then $\exists t_1 > t_0$ such that $\|x_{t_1}(t_0, \varphi_0, \psi_0)\| > \beta$ and so

$$\varphi_S(t_0) < a(\|x_{t_1}(t_0; \varphi_0, \psi_0)\|) \leqslant V(t_0, \varphi_0, \psi_0) \leqslant \varphi_S(t_0) ,$$

which is impossible. Hence $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta$ for all $t \geqslant t_0$.

Theorem 2.7. *If the solutions of* (1) *are uniformly bounded in* (φ_0, ψ_0) *with respect to the x-component. Then there exists a Lyapunov functional such that* (i) *and* (ii) *of Theorem* 2.6 *hold.*

Proof. Define $V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|x_{t+\sigma}(t_0; \varphi_0, \psi_0)\|\}$, then the result follows.

Remark. In view of Theorems 2.6 and 2.7 necessary and sufficient conditions for the uniform boundedness in (φ_0, ψ_0) with respect to the y-component can be obtained for the system (1).

We now state results concerning the uniform boundedness of (1) with respect to both the x-component and the y-component.

Theorem 2.8. *Assume that there exists a Lyapunov functional* $V(t, \varphi, \psi)$ *defined on* $0 \leqslant t < \infty$, $\|\varphi\|_0 < \infty$, $\|\psi\|_0 \geqslant K > 0$ *which satisfies, for any compact set* $S \subset l^n \times l^m$,

(i) $a(\|\psi\|_0) \leqslant V(t, \varphi, \psi) \leqslant \varphi_S(t)$, *for* $(\varphi, \psi) \in S$, *where* $a(r) \to \infty$ *as* $r \to \infty$, $a(r)$ *is continuous and increasing;*

(ii) *$V(t, \varphi, \psi)$ does not grow for any solution $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ of the system* (1).

Corresponding to any N, we can find another Lyapunov functional $W(t, \varphi, \psi)$ defined on $0 \leqslant t < \infty$, $\|\varphi\| \geqslant K_1(N)$, $\|\psi\| \leqslant N$ and satisfying

(iii) *$a_1(\|\varphi\|_0) \leqslant W(t, \varphi, \psi) \leqslant \Phi_S(t)$, $(\varphi, \psi) \in S$, where $a_1(r) \to \infty$ as $r \to \infty$, $a(r)$ is continuous and increasing;*

(iv) *$W(t, \varphi, \psi)$ does not grow for any solution, $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ of the system* (1).

Then the solution of (1) *are uniformly bounded in (φ_0, ψ_0) with respect to the x-component and the y-component.*

Proof. Using arguments parallel to Theorem 2.6 modified along the lines of Theorem 2.5, the result follows.

Theorem 2.9. *If the solution of* (1) *are uniformly bounded in (φ_0, ψ_0) with respect to the x-component and the y-component, then there exist two Lyapunov functionals satisfying hypothesis* (i), (ii), (iii) and (iv) *of Theorem* 2.8.

Proof. Define

$$V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|y_{t+\sigma}(t_0; \varphi_0, \psi_0)\|\}, \qquad W(t, \varphi, \psi) = \sup_{\tau \geqslant 0} \{\|x_{t+\tau}(t_0; \varphi_0, \psi_0)\|\}.$$

The functionals V and W are defined and satisfy for $(\varphi, \psi) \in S$ where S is any compact set in $l^n \times l^m$ the inequalities,

$$V(t, \varphi, \psi) \leqslant \beta(t, S) = \varphi_S(t), \qquad W(t, \varphi, \psi) \leqslant \beta_1(t, S) = \Phi_S(t).$$

We now give in the next three theorems sufficient conditions for the uniform boundedness and uniform ultimate boundedness properties of the system (1) with respect to the x-component and the y-component.

Theorem 2.10. *Suppose $V(t, \varphi, \psi)$ is a Lyapunov functional defined on $H_1 = \{(t, \varphi, \psi): t \in R^+, \|\varphi\|_0 + \|\psi\|_0 \geqslant K, K \text{ large}\}$ which satisfies the following conditions:*

(i) *$V(t, \varphi, \psi) \to \infty$ uniformly for (t, φ) as $\|\psi\|_0 \to \infty$;*

(ii) *$V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0, \|\psi\|_0)$, where $b(r, s)$ is continuous;*

(iii) $$D^+V(t, \varphi, \psi) = \limsup_{\delta \to 0^+} \frac{1}{\delta} \left[V(t+\delta, x_{t+\delta}(t_0; \varphi_0, \psi_0), y_t(t_0; \varphi_0, \psi_0)) - V(t, x_t(t_0; \varphi_0, \psi_0), y_t(t_0, \varphi_0, \psi_0))\right] \leqslant 0.$$

Suppose corresponding to $M > 0$, *there exists a Lyapunov functional* $W(t, \varphi, \psi)$ *defined on* $H_2 = (t, \varphi, \psi): t \in R^+, \|\varphi\|_0 \geqslant K_1(M), \|\psi\|_0 \leqslant M, K_1$ *large enough}, with the following properties:*

(iv) $W(t, \varphi, \psi) \to \infty$ *uniformly for* (t, ψ) *as* $\|\varphi\|_0 \to \infty$;

(v) $W(t, \varphi, \psi) \leqslant C(\|\varphi\|_0)$ *where* $C(r)$ *is continuous;*

(vi) $D^+ W(t, \varphi, \psi) \leqslant 0$.

Then the solutions of (1) *are uniformly bounded with respect to the x-component and y-component simultaneously.*

Proof. Let $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ be a solution of (1) such that $\|\varphi_0\|_0 + \|\psi_0\|_0 \leqslant \alpha$ where $\alpha > K$. Choose $\beta(\alpha) > 0$ so large that

$$\sup_{\|\varphi\|_0 + \|\psi\|_0 \leqslant \alpha} V(t, \varphi, \psi) < \inf_{\|\psi\| = \beta} V(t, \varphi, \psi) .$$

This choice is possible by (i) and (ii) and we claim that for any solution which exists, $\|y_t(t_0; \varphi_0, \psi_0)\| < \beta(\alpha)$ for any $t \geqslant t_0 \geqslant 0$. Suppose not then there exists $t_1 > t_0$ such that $\|x_{t_1}(t_0; \varphi_0, \psi_0)\| = \beta$. If

$$J_\alpha = \sup \{V(t, \varphi, \psi): t \geqslant 0 \text{ and } \|\varphi\|_0 + \|\psi\|_0 \leqslant \alpha\}$$

then

$$\sup_{\|\varphi\|_0 + \|\psi\|_0 \leqslant 0} V(t_1, \varphi, \psi) < \inf_{\|y_{t_1}(t_0; \varphi, \psi)\| = \beta} V(t_1, \varphi, \psi) \leqslant V(t_1, \varphi, \psi)$$

$$\leqslant V(t_0, \varphi_0, \psi_0) \leqslant J_\alpha ,$$

which is a contradiction.

Let $W(t, \varphi, \psi)$ be defined on $0 \leqslant t < \infty$, $\|\varphi\|_0 \geqslant K_1(\beta)$ and $\|y_t(t_0; \varphi, \psi)\| \leqslant \beta$. Choose $N(\alpha) = \max \{\alpha, K_1(\beta)\}$ where N depends only on α and choose $\gamma(\alpha)$ large enough so that

$$\sup \{W(t, \varphi, \psi): t \geqslant 0: \|\varphi\| = N, \|\psi\| \leqslant \beta(\alpha)\}$$

$$< \inf \{W(t, \varphi, \psi): t \geqslant 0, \|\varphi\| = \gamma, \|\psi\| \leqslant \beta(\alpha)\} .$$

Then by (vi) and the same type of arguments as in the first part it follows that for any solution $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ of (1), $\|x_t(t_0; \varphi_0, \psi_0)\| < \gamma(\alpha)$

for $t \geqslant t_0$. Thus for any solution (1) existing for $t \geqslant t_0$, $\|x_t(t_0; \varphi_0, \psi_0)\| < \gamma(\alpha)$ and $\|y_t(t_0; \varphi_0, \psi_0)\| < \beta(\alpha)$ for all $t \geqslant t_0$: The solutions are thus uniformly bounded with respect to the x-component and the y-component.

Theorem 2.11. *Assume that there exists a Lyapunov functional $V(t, \varphi, \psi)$ defined on $0 \leqslant t < \infty$, $\|\varphi\|_0 < \infty$, $\|\psi\| \geqslant K > 0$ which satisfies the following properties:*

(i) *$a(\|\psi\|_0) \leqslant V(t, \varphi, \psi) \leqslant b(\|\psi\|_0)$, where $a(r)$, $b(r)$ are continuous, increasing and $a(r) \to \infty$, as $r \to \infty$;*

(ii) *$D^+V(t, \varphi, \psi) \leqslant - C(\|\psi\|_0)$, where $C(r) > 0$ is continuous.*

Corresponding to each N, we can find another Lyapunov functional $W(t, \varphi, \psi)$ defined on $0 \leqslant t < \infty$, $\|\varphi\|_0 \geqslant K_1(N)$, $\|\psi\|_0 \leqslant N$ and satisfying the following conditions:

(iii) *$a_1(\|\varphi\|_0) \leqslant W(t, \varphi, \psi) \leqslant b_1(\|\varphi\|_0)$, where $a_1(r)$, $b_1(r)$ are continuous increasing and $a_1(r) \to \infty$ as $r \to \infty$;*

(iv) *$D^+W(t, \varphi, \psi) \leqslant 0$.*

Then the solutions of (1) *are uniformly bounded with respect to the x-component and y-component simultaneously.*

Proof. Let α be such that $\alpha > K$ and $x_t(t_0, \varphi_0, \psi_0)$, $y_t(t_0; \varphi_0, \psi_0)$ be a solution of (1) where $t_0 \geqslant 0$, $\|\varphi_0\|_0 \leqslant \alpha$ and $\|\psi_0\|_0 \leqslant \alpha$. Choose $\beta(\alpha) > 0$ large enough so that $b(\alpha) < a(\beta)$. If the solution x_t, y_t exists, then $\|y_t(t_0, \varphi_0, \psi_0)\| < \beta(\alpha)$ for all $t \geqslant t_0$. Suppose not then $\exists t_1 > t_0$ such that $\|y_{t_1}(t_0; \varphi_0, \psi_0)\| = \beta(\alpha)$. In that case also $\exists t_2, t_3$ with $t_0 \leqslant t_2 < t_3 \leqslant t_1$ such that $\|y_{t_2}(t_0; \varphi_0, \psi_0)\| = \alpha$, $\|y_{t_3}(t_0; \varphi_0, \psi_0)\| = \beta$ and $\alpha < \|y_t(t_0; \varphi_0, \psi_0)\| < \beta$ for $t_2 < t < t_3$. Now consider $V(t, x_t(t_0; \varphi_0, \psi_0), y_t(t_0; \varphi_0, \psi_0))$ on $t_2 < t < t_3$. By (i) and (ii),

$$a(\beta) \leqslant V(t_3, x_{t_3}(t_0; \varphi_0, \psi_0), y_{t_3}(t_0; \varphi_0, \psi_0)) \leqslant V(t_2, x_{t_2}(t_0; \varphi_0, \psi_0), y_{t_2}(t_0; \varphi_0, \psi_0)) \leqslant b(\alpha),$$

which is a contradiction. Hence for all $t \geqslant t_0$, $\|y_t(t_0; \varphi_0, \psi_0)\| < \beta(\alpha)$. Choose $\alpha_1(\alpha) = \max\{\alpha, K_1(\beta(\alpha))\}$ and consider the Lyapunov functional $W(t, \varphi, \psi)$ defined on $0 \leqslant t < \infty$, $\|\varphi\|_0 \geqslant K_1(\beta)$, $\|\psi\| \leqslant \beta$. Choose $\beta_1(\alpha)$ so large that $b_1(\alpha_1) < a_1(\beta_1)$. We claim that as long as the solution exists, $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta_1(\alpha)$ for all $t \geqslant t_0$, using the same type of arguments for the y-component. Hence the uniform boundedness with respect to the x-component and the y-component follow.

Theorem 2.12. *Assume that the hypothesis* (i), (ii), (iii) and (iv) *of Theorem* 2.11 *hold. Assume further that B is chosen such that $b(K) < a(B)$ and there exists a Lyapunov functional $G(t, \varphi, \psi)$ defined on $T_0 \leqslant t < \infty$, $\|\varphi\|_0 \geqslant K_2 > 0$, $\|\psi\| \leqslant B$ and satisfying the following conditions:*

(i) $a_2(\|\varphi\|_0) \leqslant G(t, \varphi, \psi) \leqslant b_2(\|\varphi\|_0)$, *where $a_2(r)$, $b_2(r)$ are continuous and increasing with $a_2(r) \to \infty$ as $r \to \infty$;*

(ii) $D^+G(t, \varphi, \psi) \leqslant - C_2(\|\varphi\|_0)$ *where $C_2(r) > 0$ is continuous.*

Then the solution of (1) *are uniformly ultimately bounded with respect to the x-component and y-component simultaneously.*

Proof. By Theorem 2.11, the uniform boundedness with respect to the x-component and y-component are guaranted. Let $\alpha > K$ and $x_t(t_0; \varphi_0, \psi_0)$, $y_t(t_0, \varphi_0, \psi_0)$ be a solution of (1) such that $\|\varphi_0\| \leqslant \alpha$ and $\|\psi_0\| \leqslant \alpha$, then there exist $\beta(\alpha)$ and $\beta_1(\alpha)$ such that $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta_1(\alpha)$, $\|y_t(t_0; \varphi_0, \psi_0)\| < \beta(\alpha)$. Suppose $\|y_t(t_0; \varphi_0, \psi_0)\| \geqslant K$ for all $t \geqslant t_0$, then we can find $\lambda(\alpha) > 0$ such that for $\|\varphi_0\|_0 < \infty$ and $K \leqslant \|y_t(t_0; \varphi_0, \psi_0)\| < \beta(\alpha)$, $D^+V(t, \varphi, \psi) \leqslant - \lambda(\alpha)$. Hence

$$V(t, \varphi, \psi) - V(t_0, \varphi_0, \psi_0) \leqslant - \lambda(\alpha)(t - t_0)\,.$$

Choosing

$$T_1(\alpha) = \frac{b(\alpha) - a(K)}{\lambda(\alpha)}\,,$$

then if $t > t_0 + T_1(\alpha)$, we have

$$a(K) \leqslant V(t, \varphi, \psi) \leqslant V(t_0, \varphi_0, \psi_0) - \lambda(\alpha)(t - t_0) < b(\alpha) - \lambda(\alpha)\Big(\frac{b(\alpha) - a(K)}{\lambda(\alpha)}\Big) = a(K)\,,$$

which is a contradiction. Hence $\exists t_1$ such that $t_0 \leqslant t_1 \leqslant t_0 + T_1(\alpha)$ with $\|y_{t_1}(t_0; \varphi_0, \psi_0)\| < K$. Now $a(K) < b(K) < a(B)$, hence $\|y_t(t_0; \varphi_0, \psi_0)\| < B$ for all $t \geqslant t_0 + T_1(\alpha)$.

Choose $M = \max\{B, K_2\}$, then if $\|\varphi_0'\|_0 \leqslant M$ and $\|\psi_0'\|_0 \leqslant M$, $\|x_t(t_0; \varphi_0', \psi_0')\| < \beta_1(M)$ for all $t \geqslant t_0$: $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta_1(\alpha)$ for all $t \geqslant t_0$ and $\|y_t(t_0; \varphi_0, \psi_0)\| < B$ for all $t \geqslant t_0 + T_1(\alpha)$, so that $\|y_t(t_0; \varphi_0, \psi_0)\| < B$ for all $t \geqslant t_0 + T_1(\alpha) + T_0$. We now claim that $\|x_t(t_0; t_0; \varphi_0, \psi_0)\| < M$ for all $t \geqslant t_0 + T_1(\alpha) + T_0$. Suppose not, then $\|x_t(t_0, \varphi_0, \psi_0)\| \geqslant M$ for all $t \geqslant t_0 + T_1(\alpha) + T_0$. However, for $M \leqslant \|x_t(t_0; \varphi_0, \psi_0)\| \leqslant \beta_1(\alpha)$, $\|y_t(t_0; \varphi_0, \psi_0)\| < B$ and $t \geqslant T_0$; there exists $\delta(\alpha) > 0$

such that $D^+G(t, \varphi, \psi) \leqslant -\delta(\alpha)$. Choose

$$T_2(\alpha) = \frac{b_2(\beta_1(\alpha)) - a_2(M)}{\delta(\alpha)},$$

then

$$G(t, \varphi, \psi) \leqslant G\big(t_0 + T_1(\alpha) + T_0,\ x_{t_0+T_1(\alpha)+T_0}(t_0; \varphi_0, \psi_0),\ y_{t_0+T_1(\alpha)+T_0}(t_0, \varphi_0, \psi_0)\big)$$
$$- \delta(\alpha)\big(t - t_0 - T_1(\alpha) - T_0\big) .$$

Hence for $t > t_0 + T_1(\alpha) + T_0 + T_2(\alpha)$,

$$a_2(M) \leqslant G(t, \varphi, \psi) \leqslant G\big(t_0 + T_1(\alpha) + T_0,\ x_{t_0+T_1(\alpha)+T_0}(t_0; \varphi_0, \psi_0), y_{t_0+T_1(\alpha)+T_0}(t_0; \varphi_0, \psi_0)\big)$$
$$- \delta(\alpha)\Big(\frac{b_2(\beta_1(\alpha)) - a_2(M)}{\delta(\alpha)}\Big) < b_2(\beta_1(\alpha)) - b_2(\beta_1(\alpha)) + a_2(M) = a_2(M) ,$$

which is a contradiction.

Hence we can find $t_1 \in [t_0 + T_1(\alpha) + T_0, t_0 + T_1(\alpha) + T_0 + T_2(\alpha)]$ such that $\|x_{t_1}(t_0; \varphi_0, \psi_0)\| < M$. Hence $\|x_t(t_0; \varphi_0', \psi_0')\| < \beta_1(M)$ for all $t \geqslant t_0$ implies $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta_1(M)$ for all $t \geqslant t_1$ in view of the fact that $\|y_{t_1}(t_0; \varphi_0, \psi_0)\| < B \leqslant M$. Therefore for all $t \geqslant t_0 + T_1(\alpha) + T_0 + T_2(\alpha)$, $\|x_t(t_0; \varphi_0, \psi_0)\| < \beta_1(M)$ and for all $t \geqslant t_0 + T_1(\alpha)$, $\|y_t(t_0; \varphi_0, \psi_0)\| < B$. We conclude the proof by setting $T(\alpha) = T_1(\alpha) + T_0 + T_2(\alpha)$ and $\gamma = \beta_1(M)$, then $B < \gamma$ and if $\|\varphi_0\|_0 \leqslant \alpha$, $\|\psi_0\| \leqslant \alpha$ we have for all $t \geqslant t_0 \geqslant 0$, $\|x_t(t_0; \varphi_0, \psi_0)\| < \gamma$ and $\|y_t(t_0; \varphi_0, \psi_0)\| < \gamma$.

3 - Application

We now give an example to illustrate the theory in the case of uniform boundedness. We shall construct two Lyapunov functionals and make use of one of our results.

Consider the time delay differential equation

$$\ddot{x}(t) + f(x(t))\,\dot{x}(t) + g(x(t - r)) = 0 , \tag{2}$$

where $r > 0$, $f(x)$ is continuous, $g(x)$ has continuous first derivative $F(x) = \int_0^x f(s)\,ds \to \pm\infty$ as $|x| \to \infty$ and $\exists \beta > 0$ such that $F(x) > 0$ and is monotone increasing for $x > \beta$. Furthermore, $g'(x) > 0$ and $xg(x) > 0$ for $x \neq 0$, with $g(x)\,F(x) > 0$.

The equation (2) is equivalent to the system

$$\dot{x}(t) = y(t) - F\big(x(t)\big) , \qquad \dot{y}(t) = - g\big(x(t-r)\big) ,$$

which is a particular example of system (1).

Let a, b be two positive numbers chosen suitably and sufficiently large and define two Lyapunov functionals $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$ as follows

$$V(t, \varphi, \psi) = \begin{cases} G\big(\varphi(0)\big) + \frac{1}{2}\psi^2(0) & \text{if } \varphi(0) \geqslant a,\ |\psi(0)| < \infty , \\ G\big(\varphi(0)\big) + \frac{1}{2}\psi^2(0) - \varphi(0) + a & \text{if } |\varphi(0)| \leqslant a,\ \psi(0) \geqslant b , \\ G\big(\varphi(0)\big) + \frac{1}{2}\psi^2(0) + 2a & \text{if } \varphi(0) \leqslant -a,\ \psi(0) \geqslant b , \\ G\big(\varphi(0)\big) + \frac{1}{2}\psi^2(0) + \frac{2a}{b}\psi(0) & \text{if } \varphi(0) \leqslant -a,\ |\psi(0)| \leqslant b , \\ G\big(\varphi(0)\big) + \frac{1}{2}\psi^2(0) - 2a & \text{if } \varphi(0) \leqslant -a,\ \psi(0) \leqslant -b , \\ G\big(\varphi(0)\big) + \frac{1}{2}\psi^2(0) + \varphi(0) - a & \text{if } |\varphi(0)| \leqslant a,\ \psi(0) \leqslant -b , \end{cases}$$

where $G(x) = \int\limits_0^x g(s)\,\mathrm{d}s$;

$$W(t, \varphi, \psi) = |\varphi(0)| , \qquad \text{if } |\varphi(0)| > K_1(M),\ |\psi(0)| \leqslant M .$$

It can be easily checked that V and W satisfy the hypothesis of Theorem 2.10 and hence the solutions of system (2) are uniformly bounded with respect to the x-component and the y-component.

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Riassunto

In questo lavoro studiamo la parziale limitatezza delle soluzioni delle equazioni differenziali con ritardo.

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