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On transformations of sets of positive linear measure (**)

1 - Introduction and definitions

If A is a subset of the set of real numbers then the set $D(A) = \{|x-y|: x \in A, y \in A\}$, is called *the distance set of* A. If A is of positive Lebesgue measure, Steinhaus [7] proved that $D(A)$ contains an interval with the origin as an end point. This result has been generalised in the n-dimensional Euclidean space with Lebesgue measure and in topological groups with Haar measure in various ways in the papers ([4], $[5]_{1,2,3}$).

If (E, ϱ) is a metric space with a measure on E and A is a subset of E, then the distance set of A is defined to be $D(A) = (\varrho(x, y): x \in A, y \in A)$.

If A is a measurable subset of E with positive measure and if $D(A)$ contains an interval with the origin as an end point, then this property will be referred to as the Steinhaus property of distance sets. For $A \subset E$, let

$$\Lambda^*(A) = \sup_{\delta>0} \left[\inf\{ \sum_{i=1}^{\infty} d(A_i): A_i \subset E,\ d(A_i) < \delta\},\ A \subset \bigcup_{i=1}^{\infty} A_i\right],$$

where $d(A_i)$ stands for the diameter of A_i. Then Λ^* is a metric outer measure and the restriction of Λ^* to the measurable sets is known as the linear measure Λ. With respect to the outer measure Λ^*, all Borel sets are measurable.

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If f is a continuous function from $[0, 1]$ to E, then $C = f[0, 1] \subset E$ is called *a curve in* E. The curve is called *simple* if f is injective. It is called *rectifiable* if $\Lambda(C) < \infty$.

It is proved in [**1**] that any simple rectifiable curve in the plane have the Steinhaus property for distance sets, while in [**2**] it is shown that there exists a simple rectifiable curve in a general metric space which has not the Steinhaus property. However, if a certain smoothness condition is satisfied by C, then Boardman [**3**] proved that C has the Steinhaus property. In **2** we obtain a generalisation of the theorem of Boardman along with certain other results. In **3** also we extend the result of Boardman in a normed linear space but to a somewhat different directions.

If $C \subset E$ is a simple rectifiable curve determined by the map $f\colon [0, 1] \to E$, then f induces a linear ordering on C defined as follows: if $x, y \in C$ then $x < y$ if and only if $f^{-1}(x) < f^{-1}(y)$.

If $a, b \in C$ and $a < b$ then the subarc $\langle a, b \rangle$ of C is defined by $\langle a, b \rangle = (c \in C\colon a \leqslant c \leqslant b)$.

It may be verified that Λ is continuous in the sense that if $b \in C$, $a_n \in C$, $a_n < a_{n+1} < b$ and $\lim \varrho(a_n, b) = 0$, then

$$\lim_{n \to \infty} \Lambda(\langle a_n, b \rangle) = 0 . \tag{1}$$

Also, if the simple curve C is determined by $C = f[0, 1]$, then because $[0, 1]$ is compact and (C, ϱ) is Hausdorff, the surjective restriction of $f\colon [0, 1] \to C$ is a homeomorphism.

Definition A [**3**]. Suppose that $r > 0$ and B is a subset of C. Then $B(r) = \{z \in C\colon \exists u \in B$ such that $u < z$ and $\varrho(u, z) = r\}$ and $B(-r) = \{z \in C\colon \exists u \in B$ such that $z < u$ and $\varrho(u, z) = r\}$.

Definition 1. The simple rectifiable curve C is said to satisfy the condition (A), if there exists $c > 0$ and $d_0 > 0$ such that for each subset $B \subset C$, $0 < r < d_0$ implies $d(B) \geqslant c[d(B(-r))]$.

Definition 2. A compact subset K of C is said to satisfy the condition (B) if $\operatorname{dist}[K, f(1)] > 0$, where $\operatorname{dist}[K, f(1)]$ is the distance of the point $f(1)$ from the set K.

Definition 3. Let G be the family of all linearly measurable subsets of C and $A_r \in G$, $r = 1, 2, \ldots$. If there exists a set $A \in G$ such that $\Lambda[A_r \vartriangle A] \to 0$ as $r \to \infty$, then the sequence of sets $\{A_r\}$ is said to converge to the set A in G, where $X \vartriangle Y$ denotes the symmetric difference of the sets X and Y.

The smoothness condition required by Boardman is that $d(B)$ is not too small compared with $d[B(-r)]$. His theorem may be stated as follows.

Theorem A [**3**]. Let C be a simple rectifiable curve in a metric space (E, ϱ) satisfying the condition (A). If S is a linearly measurable subset of C, with $\Lambda(S) > 0$, then $D(S)$ contains an interval with the origin as an end point, that is C has the Steinhaus property for distance sets.

2 - Extension of Theorem A

Theorem 1. *Let C be a simple rectifiable curve in a metric space (E, ϱ). Suppose that there exist constants $c > 0$ and $d_0 > 0$ such that for any finite number of sets $B_1, B_2, \dots, B_q \subset C$*

$$d(B_1 \cap B_2 \cap \dots \cap B_q) \geqslant c \qquad \big(d[B_1(-r_1) \cap B_2(-r_2) \cap \dots \cap B_q(-r_q)]\big)$$

whenever $0 < r_i < d_0$, $i = 1, 2, \dots, q$. Then if S is a linear measurable subset of C with $\Lambda(S) > 0$ and p be any positive integer, there exists $\eta > 0$ such that if $r_1, r_2, \dots, r_p$ are chosen any p numbers from $(0, \eta)$, then the set of points $x \in S$ such that there exists $x_i \in S$ with $\varrho(x, x_i) = r_i$, $i = 1, 2, \dots, p$ is a set of positive linear measure.

Proof. Suppose that C is determined by $f\colon [0, 1] \to C$. Then because Λ is continuous by (1), we may assume, without any loss of generality that $\operatorname{dist}[f(1), S] > 0$.

Since C is rectifiable, i.e., $\Lambda(C) < \infty$, there exists a compact set K and an open (in C) set G such that

$$K \subset S \subset G \subset C\,, \tag{2}$$

$$f(1) \in C - G\,, \qquad \Lambda(K) > \frac{1}{C}\Lambda(G - K)\,. \tag{3}$$

The relations (2) and (3) may be assumed by following the same technique as adopted in the proof of theorem 13.5 of [**6**].

Since $K \subset S$, it is sufficient to show that K has the property as stated in the theorem.

Let $\delta > 0$ be such that $\Lambda(K) - \delta > (1/c)\Lambda(G - K)$.

From the definition of $\Lambda(K)$ it follows that there exists $\varepsilon_0 > 0$ such that for all ε' with $0 < \varepsilon' \leqslant \varepsilon_0$, all covers $\{A_i\}_{i=1}^{\infty}$ of K with $d(A_i) < \varepsilon'$ has the property

$$\sum_{i=1}^{\infty} d(A_i) \geqslant \Lambda(K) - \delta \,. \tag{4}$$

If $d_1 = \operatorname{dist}(K, C - G)$, then $d_1 > 0$. Let $\eta = \min(d_0, d_1, \varepsilon_0/3)$. Then if $r_1, r_2, \ldots, r_p$ be any p numbers such that $0 < r_i < \eta$, $i = 1, 2, \ldots, p$, then it is clear that

$$K(r_i) \subset G\,, \qquad i = 1, 2, \ldots, p, \,. \tag{5}$$

Let ε be any number satisfying $0 < \varepsilon < \varepsilon_0/4$ and $B_j^i \subset C$ be such that $d(B_j^i) < \varepsilon$ and $K(r_i) \subset \bigcup_{j=1}^{\infty} B_j^i$, $i = 1, 2, \ldots, p$.

Now r_i is less than d_1 and $f(1) \in C - G$, so if $u \in K$, then $\varrho[f(1), u] > r_i$ and since $\langle u, f(1)\rangle$ is connected, there exists $z \in C$ with $u < z$ such that $\varrho(u, z) = r_i$. So, $z \in K(r_i)$ and so there exists m such that $z \in B_m^i$ and therefore $u \in B_m^i(-r_i)$ and so $K \subset \bigcup_{j=1}^{\infty} B_j^i(-r_i)$ for $i = 1, 2, \ldots, p$.

Let $A = K(r_1) \cap K(r_2) \cap \ldots \cap K(r_p)$, then

$$A \subset \Big(\bigcup_{j=1}^{\infty} B_j^1\Big) \cap \Big(\bigcup_{i=1}^{\infty} B_j^2\Big) \cap \ldots \cap \Big(\bigcup_{j=1}^{\infty} B_j^p\Big) \subset \bigcup_{j,k,\ldots,m=1}^{\infty} [B_j^1 \cap B_k^2 \cap \ldots \cap B_m^p]\,. \tag{6}$$

and

$$K \subset \Big[\bigcup_{j=1}^{\infty} B_j^1(-r_1)\Big] \cap \Big[\bigcup_{j=1}^{\infty} B_j^2(-r_2)\Big] \cap \ldots \cap \Big[\bigcup_{j=1}^{\infty} B_j^p(-r_p)\Big]$$

$$\subset \bigcup_{j,k,\ldots,m=1}^{\infty} [B_j^1(-r_1) \cap B_k^2(-r_2) \cap \ldots \cap B_m^p(-r_p)]\,.$$

Now, $z_1, z_2 \in B_q^i(-r_i)$ imply that there exist $u_1, u_2 \in B_q^i$ such that $\varrho(u_j, z_j) = r_i$ for $j = 1, 2$. So, $\varrho(z_1, z_2) \leqslant 2r_i + d(B_q^i) < (11/12)\,\varepsilon_0$ because $r_i < \eta \leqslant \varepsilon_0/3$ and $d(B_q^i) < \varepsilon < \varepsilon_0/4$.

So, $d[B_q^i(-r_i)] < \varepsilon_0$ and therefore $d[B_j^1(-r_1) \cap B_k^2(-r_2) \cap \ldots \cap B_m^p(-r_p)] < \varepsilon_0$. So, by (4) $\sum_{j,k,\ldots,m=1}^{\infty} d[B_j^1(-r_1) \cap B_k^2(-r_2) \cap \ldots \cap B_m^p(-r_p)] \geqslant \Lambda(K) - \delta > (1/c)\,\Lambda(G - K)$.

Now since each r_i is less than d_0, we have by the condition of the theorem

$$\sum_{j,k,\ldots,m=1}^{\infty} d[B_j^1 \cap B_k^2 \cap \ldots \cap B_m^p] \geqslant c$$

$$\sum_{j,k,\ldots,m=1}^{\infty} d[B_j^1(-r_1) \cap B_k^2(-r_2) \cap \ldots \cap B_m^p(-r_p)] \geqslant c[\Lambda(K)-\delta] > \Lambda(G-K)\,.$$

Since $d[B_j^1 \cap B_K^2 \cap \ldots \cap B_m^p] < \varepsilon$, it follows from (6) that

$$\Lambda(A) \geqslant c[\Lambda(K)-\delta] > \Lambda(G-K)\,. \tag{7}$$

By (5), $A \subset G$ and so

$$[K \cap K(r_1) \cap \ldots \cap K(r_p)] = \Lambda(K \cap A) \geqslant \Lambda(G) - \Lambda(G-K) - \Lambda(G-A)$$
$$= \Lambda(A) - \Lambda(G-K) > 0\,,$$

by (7). Hence $K \cap K(r_1) \cap \ldots \cap K(r_p)$ is a set of positive linear measure. Let $x \in K \cap K(r_1) \cap \ldots \cap K(r_p)$, then $x \in K$ and because $x \in K(r_i)_1$ $i = 1, 2, \ldots, p$, there exist $x_i \in K$ such that $\varrho(x, x_i) = r_i$, $i = 1, 2, \ldots, p$. This proves the theorem.

Remark. If $p = 1$, we obtain that there exists a positive number η such that if r_1 be any number with $0 < r_1 < \eta$, then the set of points $x \in K$ for which there exists $y \in K$ with $\varrho(x, y) = r_1$, is a set of positive linear measure. This result itself is more general than Theorem A which assured only the existence of a pair of points x and y of K such that $\varrho(x, y) = r_1$.

Theorem 2. *Suppose that the curve C satisfies the condition* (A) *for $c > 1$ and K is a compact subset of C with $\Lambda(K) > 0$ that satisfies the condition* (B). *If $\{r_m\}$ is a sequence of positive numbers converging to zero, then $K(r_m) \to K$ in G.*

Proof. Under the supposition of the theorem it follows from the proof of Theorem A, because $c > 1$, that there exists a positive number η such that if $0 < r < \eta$, then $\Lambda[K(r)] \geqslant \Lambda(K)$.

Equivalently, since $r_n \to 0$, there exists a positive integer N_1 such that

$$\Lambda[K(r_n)] \geqslant \Lambda(K) \quad \text{whenever } n \geqslant N_1\,. \tag{8}$$

Since K is a compact subset of C, for $\varepsilon > 0$ arbitrary, there exists an open (in C) set G such that

$$K \subset G \subset C \quad \text{and} \quad \Lambda(G-K) < \varepsilon/3\,. \tag{9}$$

Let $d = \text{dist}(K, C - G)$, then $d > 0$. There exists a positive integer N_2 such that $r_n < d$ for $n \geqslant N_2$ and so $K(r_n) \subset G$ for $n \geqslant N_2$.

Let $X_n = K \cap K(r_n)$, then $X_n \subset G$ for $n \geqslant N_2$ and so $X_n = G - (G - K) - [G - K(r_n)]$.

So for $n \geqslant N_2$,

$$\Lambda(X_n) \geqslant \Lambda(G) - \Lambda(G - K) - \Lambda[G - K(r_n)] > \Lambda[K(r_n)] - \varepsilon/3$$

from (9). If $N = \max(N_1, N_2)$, then from (8) for $n \geqslant N$, $\Lambda(X_n) > \Lambda(K) - \varepsilon/3 > \Lambda(G) - 2\varepsilon/3 > \Lambda(G) - \varepsilon$. Consequently, $\Lambda[K(r_n) \vartriangle K] < \varepsilon$ for $n \geqslant N$. So, $K(r_n) \to K$ in G.

Corollary. *Under the hypotheses of the above theorem, if A is any measurable subset of C, then $K(r_n) \cap A \to K \cap A$ in G.*

Proof. We have

$$\Lambda[\{K(r_n) \cap A\} \vartriangle \{K \cap A\}] \leqslant \Lambda[K(r_n) \vartriangle K] \to 0 \qquad \text{as } n \to \infty .$$

Definition 4. For $r > 0$, let $\varphi(r) = \Lambda[K \cap K(r)]$, where K is a compact subset of C with $\Lambda(K) > 0$ that satisfies the condition (B). For $r = 0$ let $\varphi(0) = \Lambda(K)$.

Theorem 3. *The function $\varphi(r)$ is right continuous at the origin provided the curve C satisfies the condition* (A) *for $c > 1$.*

Proof. Let $\{r_n\}$ be a sequence of positive numbers converging to zero. Then by the corollary and Theorem 2, $K(r_n) \cap K \to K$ in G. Clearly then $\Lambda[K(r_n) \cap K] \to \Lambda(K)$, i.e., $\varphi(r_n) \to \varphi(0)$. This proves the theorem.

3 - Extension of Theorem A in normed linear space

In this section, we suppose that E is a real normed linear space. Let f be a continuous injective map of $[0, 1]$ into E and $C = f[0, 1]$, where $\Lambda(C) < \infty$. The definition of curve, linear measure etc. will have the same meaning where $\varrho(x, y)$ is to be replaced by $\|x - y\|$.

Definition 5. We say that f is homogeneous in a neighbourhood of 1 if there exists $\delta' > 0$ such that for any real number α, $1 \leqslant \alpha \leqslant 1 + \delta'$, $f(\alpha x) = \alpha f(x)$, for all $x, \alpha x \in [0, 1]$. In this case we say that $f \in H(1)$.

Definition 6. Let $B \subset C$ and $r > 0$. For real number $\boldsymbol{a}$, let $B(r, a) = \{z \in C\colon \exists u \in B,\ u < z \text{ and } \|au - z\| = r\}$, $B(-r, a) = \{z \in C\colon \exists u \in B,\ z < u \text{ and } \|au - z\| = r\}$.

It may be noted that $B(r, 1) = B(r)$ and $B(-r, 1) = B(-r)$ in the normed linear space E.

Theorem 4. *Let C be a simple rectifiable curve in a normed linear space E determined by $f \in H(1)$. Suppose that C has the following property; there exists $c > 0$, $\delta_0 > 0$ and $d_0 > 0$ such that, for each subset B of C, $0 < r < d_0$ and $1 \leqslant a \leqslant 1 + \delta_0$ imply $d(B) \geqslant c[d\{B(-r/a, 1/a)\}]$. If S is a linearly measurable subset of C with $\Lambda(S) > 0$, then there exist $\eta > 0$ and $\delta > 0$ such that if r and a be any numbers with $0 < r < \eta$ and $1 \leqslant a \leqslant 1+\delta$, then the set of points $x \in S$ for which there exists $u \in S$ with $\|x - au\| = r$ is a set of positive linear measure (δ depends on r).*

Proof. We can assume by (1) that $\operatorname{dist}[f(1), S] > 0$. Since S is measurable and $\Lambda(C) < \infty$, there exists a compact set K and an open (in C) set G such that $K \subset S \subset G \subset C$ and moreover $f(1) \in C - G$ and $\Lambda(K) > (1/c)\Lambda(G - K)$. Let $\delta_1 > 0$ be such that $\Lambda(K) - \delta_1 > (1/c)\Lambda(G - K)$. There exists then $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_0$, all covers $\{A_i\}_{i=1}^{\infty}$ of K with $d(A_i) < \varepsilon$ satisfy

$$\sum_{i=1}^{\infty} d(A_i) \geqslant \Lambda(K) - \delta_1 > \frac{1}{c}\Lambda(G - K)\,. \tag{10}$$

Let $d_1 = \operatorname{dist}(K, C - G)$, then $d_1 > 0$ and $M > 0$ be a number such that $\|x\| \leqslant M$ for all x in K. Let $\delta_2 > 0$ be chosen small enough to ensure $M\delta_2 < d_1$. Suppose that $\eta = \min[d_0, (1/3)\varepsilon_0, d_1 - M\delta_2]$. So $\eta > 0$. Let $0 < r < \eta/2$ and $1 \leqslant a \leqslant 1+\delta_2$, then it may be verified easily that

$$K(r, a) \subset G\,. \tag{11}$$

Let ε be any number with $0 < \varepsilon < \varepsilon_0/3$ and $B_i \subset C$ be such that $d(B_i) < \varepsilon$ and

$$K(r, a) \subset \bigcup_{i=1}^{\infty} B_i\,. \tag{12}$$

Suppose that $u \in K$, then $\|f(1) - u\| > 2r$. It is then clear that there exists $\delta_3 > 0$ independent of $u \in K$ such that $\|f(1) - au\| > r$ for all α with $1 \leqslant a \leqslant 1+\delta_3$. Since $f \in H(1)$ and f is continuous, $|f^{-1}(u) - 1| > 0$ for all $u \in K$. So, there exists δ_4, $0 < \delta_4 < \min(\delta_2, \delta_3, \delta')$ which is independent of $u \in K$ such that $au \in C$ for all α with $1 \leqslant a \leqslant 1+\delta_4$ whenever $u \in K$.

As $\langle au, f(1)\rangle$ is connected, there exists $z \in C$ with $z > au$ and therefore $> u$ because $f \in H$ (1) such that $\|au - z\| = r$. So, $z \in K(r, a)$ for $0 < r < \eta/2$ and $1 \leqslant a \leqslant 1+\delta_4$.

This means that there exists i such that $z \in B_i$ and so $u \in B_i(-r/a, 1/a)$. Therefore

$$K \subset \bigcup_{i=1}^{\infty} B_i\left(-\frac{r}{a}, \frac{1}{a}\right). \tag{13}$$

If now $z_1, z_2 \in B_i(-r/a, 1/a)$ then there are $u_1, u_2 \in B_i$ such that

$$\left\|\frac{u_1}{a} - z_1\right\| = \frac{r}{a} \quad \text{and} \quad \left\|\frac{u_2}{a} - z_2\right\| = \frac{r}{a}.$$

So,

$$\|z_1 - z_2\| = \left\|z_1 - \frac{u_1}{a} + \frac{u_1}{a} - \frac{u_2}{a} + \frac{u_2}{a} - z_2\right\| \leqslant \frac{r}{a} + \frac{r}{a} + \frac{1}{a} d(B_i)$$

$$= \frac{2r}{a} + \frac{1}{a} d(B_i) \leqslant 2r + d(B_i) < \eta + d(B_i)$$

$$\leqslant \frac{\varepsilon_0}{3} + d(B_i) < \frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{3} = \frac{2}{3}\varepsilon_0,$$

because $d(B_i) < \varepsilon < \varepsilon_0/3$.

So, $d[B_i(-r/a, 1/a)] < \varepsilon_0$. By (10)

$$\sum_{i=1}^{\infty} d\left[B_i\left(-\frac{r}{a}, \frac{1}{a}\right)\right] \geqslant \Lambda(K) - \delta_1 > \frac{1}{c}\Lambda(G - K).$$

We have $0 < r < \eta/2 < d_0$. Let $0 < \delta < \min(\delta_0, \delta_1, \delta_4, r/M)$ and suppose that $1 \leqslant a \leqslant 1+\delta$. Then $d(B_i) \geqslant c d[B_i(-r/a, 1/a)]$. Therefore $\sum_{i=1}^{\infty} d(B_i) \geqslant \Lambda(G - K)$ and hence from (12)

$$\Lambda[K(r, a)] > \Lambda(G - K). \tag{14}$$

Since by (11), $K(r, a) \subset G$, we have $[K(r, a) \cap K] = G - [G - K(r, a)] - (G - K)$ and so $\Lambda[K(r, a) \cap K] \geqslant \Lambda(G) - \Lambda[G - K(r, a)] - \Lambda(G - K) = \Lambda[K(r, a)] - \Lambda(G - K) > 0$ from (14).

So, the set $K(r, a) \cap K$ is of positive linear measure. Let $x \in K(r, a) \cap K$. Then $x \in K$ and $x \in K(r, a)$ implies the existence of $u \in K$ with $\|au - x\| = r$. Since $K \subset S$, this proves the theorem.

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Abstract

Boardman [3] *obtained an extension of Steinhaus theorem* [7] *for distance sets for sets of real numbers to the sets which are subsets of a simple rectifiable curve in a metris space. In this paper we obtain, with certain additional results, generalisations of Boardman's theorem in a metric space and in a normed linear space.*

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