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A generalization of Ménage and Lommel polynomials (**)

1 - Introduction

The Lommel polynomials $R_{n,\nu}(x)$ are polynomials in $1/x$ which occur as the coefficients in the expansion of the Bessel function of order $\nu+n$ in terms of the Bessel functions of order ν and $\nu-1$ [4]

$$J_{\nu+n}(x)=R_{n,\nu}(x)\,J_\nu(x)-R_{n-1,\nu+1}(x)\,J_{\nu-1}(x)\,.$$

The Ménage polynomials $U_n(t)$ enumerate the permutations on n elements where element i may not be in position i or $i+1$, for $i=1$ to $n-1$, and element n may not be in position n or 1. The Ménage polynomials satisfy the recurrence [3]

$$(1.1)\qquad \begin{aligned}&(n-2)\,U_n(t)=n(n-2)\,U_{n-1}(t)+n(t-1)^2\,U_{n-2}(t)-4(t-1)^n,\\ &U_1(t)=2(t-1)+1\,,\qquad U_2(t)=2(t-1)^2+4(t-1)+2\,.\end{aligned}$$

The main result of this paper is that there is a surprisingly close connection between these two sets of polynomials. In the following sections modified forms of the Lommel and Ménage polynomials are introduced and generalized, and formulas involving the generalized polynomials are obtained.

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2 - Modified Lommel polynomials

We define modified Lommel polynomials $f_{n,\nu}(x) = \exp(-n\pi i/2) R_{n,\nu}(-2i/x)$, which satisfy the recurrence

$$f_{n+1,\nu}(x) = (n+\nu)\, x f_{n,\nu}(x) + f_{n-1,\nu}(x) , \tag{2.1}$$

together with the initial conditions $f_{-1,\nu}(x) = 0$, $f_{0,\nu}(x) = 1$.

The explicit formula is

$$f_{n,\nu}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} \frac{(\nu)_{n-k}}{(\nu)_k} x^{n-2k},$$

where $(a)_k = (a)(a+1)(a+2) \dots (a+k-1)$, and if ν is a negative integer or zero, $(\nu)_{n-k}/(\nu)_k$ is to be replaced by $(\nu+k)_{n-2k}$. The following formulas for the modified Lommel polynomials follow immediately from similar formulas for the Lommel polynomials [4]

$$f_{n+1,\nu-1}(x) = (\nu-1)\, x f_{n,\nu}(x) + f_{n-1,\nu+1}(x) , \tag{2.2}$$

$$f_{n,\nu}(x) = (-1)^n f_{n,-\nu-n+1}(x) , \tag{2.3}$$

$$x^2 f'_{n,\nu}(x) = -x f_{n,\nu}(x) + f_{n+1,\nu}(x) + f_{n-1,\nu}(x) - f_{n-1,\nu+1}(x) - f_{n+1,\nu-1}(x) , \tag{2.4}$$

where the prime denotes a derivative with respect to x.

3 - Modified Ménage polynomials

We define modified Ménage polynomials by

$$w_n(x) = \frac{x^n U_{n+1}(x^{-1}+1) - 2x^{-1}}{n+1},$$

and substitute into (1.1) to obtain the recurrence

$$w_{n+1}(x) = (n+1)\, x w_n(x) + w_{n-1}(x) + 2 , \tag{3.1}$$

with the initial conditions $w_0(x) = 1$, $w_1(x) = x + 2$.

Explicitly, the modified Ménage polynomials are given by

$$w_n(x) = \sum_{k=0}^{n} \binom{2n-k+1}{k} (n-k)!\, x^{n-k}.$$

4 - The generalized polynomials

Recurrences (2.1) and (3.1) can be generalized by defining the polynomials $g_{n,\nu}^{(\lambda)}(x)$ by the recurrence

$$g_{n+1,\nu}^{(\lambda)}(x) = (n+\nu)\, x g_{n,\nu}^{(\lambda)}(x) + g_{n-1,\nu}^{(\lambda)}(x) + \lambda\,, \tag{4.1}$$

together with the initial conditions $g_{-1,\nu}^{(\lambda)}(x) = 0$, $g_{0,\nu}^{(\lambda)}(x) = 1$.

Thus we have $g_{n,\nu}^{(0)}(x) = f_{n,\nu}(x)$ and $g_{n,1}^{(2)}(x) = w_n(x)$. The first few polynomials are explicitly

$$g_{1,\nu}^{(\lambda)}(x) = \nu x + \lambda\,,$$

$$g_{2,\nu}^{(\lambda)}(x) = (\nu)(\nu+1)\, x^2 + \lambda(\nu+1)\, x + (\lambda+1)\,,$$

$$g_{3,\nu}^{(\lambda)}(x) = (\nu)_3 x^3 + \lambda(\nu+1)_2 x^2 + \big(\lambda(\nu+2) + 2(\nu+1)\big)\, x + 2\lambda\,,$$

$$g_{4,\nu}^{(\lambda)}(x) = (\nu)_4 x^4 + \lambda(\nu+1)_3 x^3 + (\nu+2)\big(\lambda(\nu+3) + 3(\nu+1)\big)\, x^2$$

$$+ \lambda(3\nu+7)\, x + (2\lambda+1)\,.$$

Theorem 4.1. *For* $n \geqslant 1$,

$$g_{n,\nu}^{(\lambda)}(x) = f_{n,\nu}(x) + \lambda \sum_{j=1}^{n} f_{n-j,\nu+j}(x)\,.$$

Proof. For $n = 1$ and 2 direct substitution verifies the result. We assume the theorem holds for all $n < k$ and use recurrence (4.1) together with the induction hypothesis to write

$$g_{k,\nu}^{(\lambda)}(x) = f_{k,\nu}(x) + \lambda(\nu+k-1)\, x \sum_{j=1}^{k-1} f_{k-1-j,\nu+j}(x) + \lambda \sum_{j=1}^{k-2} f_{k-2-j,\nu+j}(x) + \lambda\,. \tag{4.2}$$

In recurrence (2.1), replace n by $k-1-j$, ν by $\nu+j$ and substitute into

(4.2) to get

$$g_{k,\nu}^{(\lambda)}(x) = f_{k,\nu}(x) + \lambda \sum_{j=1}^{k-2} f_{k-j,\nu+j}(x) + \lambda(\nu+k-1)x + \lambda$$

$$= f_{k,\nu}(x) + \lambda \sum_{j=1}^{k} f_{k-j,\nu+j}(x) ,$$

which completes the proof of the theorem.

We have as an immediate corollary by inversion the formula

$$f_{n,\nu}(x) = g_{n,\nu}^{(\lambda)}(x) - \lambda \sum_{j=1}^{n} (1-\lambda)^{j-1} g_{n-j,\,\nu+j}^{(\lambda)}(x) . \tag{4.3}$$

The recurrence (4.1) can be used to define $g_{n,\nu}^{(\lambda)}(x)$ when n is a negative integer. In particular, it is easy to verify using induction and recurrences (2.1) and (4.1) that for n a positive integer

$$g_{-n,\nu}^{(\lambda)}(x) = g_{n-2,2-\nu}^{(-\lambda)}(x) - \lambda f_{n-2,2-\nu}(x) .$$

If $n > 2$ we apply Theorem 4.1 and get

$$g_{-n,\nu}^{(\lambda)}(x) = (1-\lambda) f_{n-2,2-\nu}(x) - \lambda \sum_{j=1}^{n-2} f_{n-2-j,2-\nu+j}(x) . \tag{4.4}$$

For $\lambda = 0$, (4.4) reduces to Graf's result [4] $f_{-n,\nu}(x) = f_{n-2,2-\nu}(x)$. For $\nu = 1$, $\lambda = 2$, (4.4) becomes by Theorem 4.1, $w_{-n}(x) = -w_{n-2}(x)$, a result obtained by Carlitz [**1**].

If we apply (2.3) to Theorem 4.1, we get

$$g_{n,\nu}^{(\lambda)}(x) = (-1)^n f_{n,-\nu-n+1}(x) + \lambda \sum_{j=1}^{n} (-1)^{n-j} f_{n-j,-\nu-n+1}(x) .$$

Taking $\nu = 1$, $\lambda = 2$ we have the following relationship between the modified Ménage polynomials and the modified Lommel polynomials

$$w_n(x) = (-1)^n f_{n,-n}(x) + 2 \sum_{j=1}^{n} (-1)^{n-j} f_{n-j,-n}(x) . \tag{4.5}$$

5 - Other recurrences

The following lemma will be used to derive further recurrences for the $g_{n,\nu}^{(\lambda)}(x)$.

Lemma 5.1. *For* $\lambda \neq 0$,

$$\sum_{j=1}^{n} j f_{n-j,\nu+j}(x) = \sum_{j=1}^{n} \lambda^{-1} (1-(1-\lambda)^{j}) g_{n-j,\nu+j}^{(\lambda)}(x) .$$

Proof. $\lambda \sum_{j=1}^{n} j f_{n-j,\nu+j}(x) = \lambda \sum_{k=1}^{n} \sum_{j=1}^{k} f_{k-j,\nu+n-k+j}$

$$= \sum_{k=1}^{n} [g_{k,\nu+n-k}^{(\lambda)}(x) - f_{k,\nu+n-k}(x)] ,$$

by Theorem 4.1. Using formula (4.3) we get

$$\sum_{k=1}^{n} \lambda \sum_{j=1}^{k} (1-\lambda)^{j-1} g_{k-j,\nu+n-k+j}^{(\lambda)}(x) ,$$

changing the order of summation by defining $r = n - k + j$, we get

$$\sum_{r=1}^{n} \sum_{j=1}^{r} (1-\lambda)^{j-1} g_{n-r,\nu+r}^{(\lambda)}(x) .$$

Since $\lambda \neq 0$, the inner sum is $\lambda^{-1}(1-(1-\lambda)^{r})$ which completes the proof.

If we apply Theorem 4.1 to $g_{n+1,\nu-1}^{(\lambda)}(x)$ and use (2.2), we have

$$g_{n+1,\nu-1}^{(\lambda)}(x) = (\nu-1)\, x g_{n,\nu}^{(\lambda)}(x) + g_{n-1,\nu+1}^{(\lambda)}(x) + \lambda + \lambda x \sum_{j=1}^{n} j f_{n-j,\nu+j}(x) .$$

By Lemma 5.1 we have the following generalization of (2.2)

$$g_{n+1,\nu-1}^{(\lambda)}(x) = (\nu-1)\, x g_{n,\nu}^{(\lambda)}(x) + g_{n-1,\nu+1}^{(\lambda)}(x) + \lambda + x \sum_{j=1}^{n} (1-(1-\lambda)^{j})\, g_{n-j,\nu+j}^{(\lambda)}(x) . \tag{5.1}$$

If we add $(n+1)\, xg^{(\lambda)}_{n,\nu}(x)$ to both sides of (5.1) and use the fundamental recurrence (4.1) we get

$$g^{(\lambda)}_{n+1,\nu}(x) - g^{(\lambda)}_{n-1,\nu}(x) + g^{(\lambda)}_{n-1,\nu+1}(x) - g^{(\lambda)}_{n+1,\nu-1}(x)$$

$$= (n+1)\, xg^{(\lambda)}_{n,\nu}(x) - x\sum_{j=1}^{n}\big(1-(1-\lambda)^j\big)\, g^{(\lambda)}_{n-j,\nu+j}(x)\,.$$

We will now obtain some recurrences involving derivatives.

Lemma 5.2. $$g^{(\lambda)}_{n+1,\nu-1}(x) + g^{(\lambda)}_{n-1,\nu+1}(x)$$

$$= 2g^{(\lambda)}_{n,\nu}(x) + (\nu-1)\, xf_{n,\nu}(x) + (\lambda-2)[f_{n,\nu}(x) - f_{n-1,\nu+1}(x)]\,.$$

Proof. Apply Theorem 4.1 to both $g^{(\lambda)}_{n+1,\nu-1}(x)$ and $g^{(\lambda)}_{n-1,\nu+1}(x)$ and collect terms to get

$$g^{(\lambda)}_{n+1,\nu-1}(x) + g^{(\lambda)}_{n-1,\nu+1}(x)$$

$$= f_{n+1,\nu-1}(x) + f_{n-1,\nu+1}(x) + \lambda f_{n,\nu}(x) + \lambda f_{n-1,\nu+1}(x) + 2\lambda\sum_{j=2}^{n} f_{n-j,\nu+j}(x)\,.$$

Add and subtract $\lambda f_{n-1,\nu+1}(x) + 2f_{n,\nu}(x)$ and use (2.2) and Theorem 4.1 to complete the proof.

Differentiating the formula in Theorem 4.1 and using (2.4) we get the following recurrence, where the prime denotes a derivative with respect to x

$$x^2 g^{(\lambda)'}_{n,\nu}(x) \tag{5.2}$$

$$= -\, xg^{(\lambda)}_{n,\nu}(x) + g^{(\lambda)}_{n+1,\nu}(x) + g^{(\lambda)}_{n-1,\nu}(x) - g^{(\lambda)}_{n-1,\nu+1}(x) - g^{(\lambda)}_{n+1,\nu-1}(x)\,.$$

If we use the fundamental recurrence (4.1) to replace $g^{(\lambda)}_{n+1,\nu}(x)$ and $g^{(\lambda)}_{n-1,\nu}(x)$ respectively, we get the alternative forms

$$x^2 g^{(\lambda)'}_{n,\nu}(x) \tag{5.3}$$

$$= (n+\nu-1)\, xg^{(\lambda)}_{n,\nu}(x) + 2g^{(\lambda)}_{n-1,\nu}(x) + \lambda - g^{(\lambda)}_{n-1,\nu+1}(x) - g^{(\lambda)}_{n+1,\nu-1}(x)\,,$$

$$x^2 g^{(\lambda)'}_{n,\nu}(x) \tag{5.4}$$

$$= -\,(n+\nu+1)\, xg^{(\lambda)}_{n,\nu}(x) + 2g^{(\lambda)}_{n+1,\nu}(x) - \lambda - g^{(\lambda)}_{n-1,\nu+1}(x) - g^{(\lambda)}_{n+1,\nu-1}(x)\,.$$

If in (5.3) we use (4.1) to replace $g^{(\lambda)}_{n+1,\nu-1}(x)$ we get

(5.5) $$x^2 g^{(\lambda)'}_{n,\nu}(x)$$
$$= (n+\nu-1)\ x[g^{(\lambda)}_{n,\nu}(x) - g^{(\lambda)}_{n,\nu-1}(x)] + 2g^{(\lambda)}_{n-1,\nu}(x) - g^{(\lambda)}_{n-1,\nu+1}(x) - g^{(\lambda)}_{n-1,\nu-1}(x)\,.$$

Similarly, in (5.4) use (4.1) to replace $g^{(\lambda)}_{n-1,\nu+1}(x)$ to get

(5.6) $$x^2 g^{(\lambda)'}_{n,\nu}(x)$$
$$= (n+\nu+1)\ x[g^{(\lambda)}_{n,\nu+1}(x) - g^{(\lambda)}_{n,\nu}(x)] + 2g^{(\lambda)}_{n+1,\nu}(x) - g^{(\lambda)}_{n+1,\nu+1}(x) - g^{(\lambda)}_{n+1,\nu-1}(x)\,.$$

Formula (5.3) is a generalization of Riordan's formula [**3**] which in our notation is $x^2 w_n'(x) = (nx-2)\,w_n(x) + 2w_{n-1}(x) + 2$. To see this set $\lambda = 2$, $\nu = 1$ in (5.3) and use Lemma 5.2.

We also have the following two formulas relating the sum of derivatives. In (5.5) replace n by $n+2$, ν by $\nu-1$ and add to (5.4) to get

$$x^2[g^{(\lambda)'}_{n+2,\nu-1}(x) + g^{(\lambda)'}_{n,\nu}(x)]$$
$$= (n+2)\ xg^{(\lambda)}_{n+2,\nu-1}(x) - (n+2)\ xg^{(\lambda)}_{n,\nu}(x) - (\nu-1)\ xg^{(\lambda)}_{n,\nu}(x) + (\nu-2)\ xg^{(\lambda)}_{n+2,\nu-1}(x) - \lambda$$
$$- (n+\nu)\ xg^{(\lambda)}_{n+2,\nu-2}(x) + g^{(\lambda)}_{n+1,\nu-1}(x) + g^{(\lambda)}_{n+1,\nu}(x) - g^{(\lambda)}_{n-1,\nu+1}(x) - g^{(\lambda)}_{n+1,\nu-2}(x)\,.$$

Using the fundamental recurrence (4.1) we get

(5.7) $$x^2[g^{(\lambda)'}_{n+2,\nu-1}(x) + g^{(\lambda)'}_{n,\nu}(x)]$$
$$= (n+2)\ x[g^{(\lambda)}_{n+2,\nu-1}(x) - g^{(\lambda)}_{n,\nu}(x)] + g^{(\lambda)}_{n+1,\nu-1}(x) - g^{(\lambda)}_{n-1,\nu+1}(x)$$
$$- (\nu-1)\ xg^{(\lambda)}_{n,\nu}(x) - g^{(\lambda)}_{n+3,\nu-2}(x) + (\nu-2)\ xg^{(\lambda)}_{n+2,\nu-1}(x) + g^{(\lambda)}_{n+1,\nu}(x)\,.$$

If $\lambda = 0$, (5.7) becomes Nielsen's formula [**2**] which in our notations is

$$x[f'_{n,\nu}(x) + f'_{n+2,\nu-1}(x)] = (n+2)[f_{n+2,\nu-1}(x) - f_{n,\nu}(x)]\,.$$

The second formula is obtained by replacing n by $n+1$ in (5.3) and adding to (5.4). The result is

(5.8) $$x^2[g^{(\lambda)'}_{n+1,\nu}(x) + g^{(\lambda)'}_{n,\nu}(x)]$$
$$= (n+\nu)\ xg^{(\lambda)}_{n+1,\nu}(x) - (n+\nu+1)\ xg^{(\lambda)}_{n,\nu}(x) + 2g^{(\lambda)}_{n,\nu}(x) + 2g^{(\lambda)}_{n+1,\nu}(x)$$
$$- g^{(\lambda)}_{n,\nu+1}(x) - g^{(\lambda)}_{n+2,\nu-1}(x) - g^{(\lambda)}_{n-1,\nu+1}(x) - g^{(\lambda)}_{n+1,\nu-1}(x)\,.$$

If we set $\lambda = 2$, $\nu = 1$ and apply Lemma 5.2 to (5.8) we get Riordan's formula [**3**] which in our notation is

$$x[w'_{n+1}(x) + w'_n(x)] = (n+1)\,w_{n+1}(x) - (n+2)\,w_n(x)\,.$$

6 - Product and expansion formulas

Multiplying the fundamental recurrence (4.1) by $g^{(\lambda)}_{n,\nu}(x)$ and iterating we get the following product formula

Theorem 6.1. *For n a non-negative integer,*

$$g^{(\lambda)}_{n,\nu}(x)\,g^{(\lambda)}_{n+1,\nu}(x) = \sum_{k=0}^{n} [(\nu + k)\,x\big(g^{(\lambda)}_{k,\nu}(x)\big)^2 + \lambda g^{(\lambda)}_{k,\nu}(x)]\,.$$

Proof. The proof is by induction on n. For $n = 0$, the equality is clear. We assume the theorem for $n = m - 1$, and apply the recurrence

$$(4.1)\qquad g^{(\lambda)}_{m,\nu}(x)\,g^{(\lambda)}_{m+1,\nu}(x) = (m+\nu)\,x\big(g^{(\lambda)}_{m,\nu}(x)\big)^2 + g^{(\lambda)}_{m,\nu}(x)\,g^{(\lambda)}_{m-1,\nu}(x) + \lambda g^{(\lambda)}_{m,\nu}(x)\,.$$

Using the induction hypothesis on $g^{(\lambda)}_{m,\nu}(x)\,g^{(\lambda)}_{m-1,\nu}(x)$ completes the proof.

The following apparently new formulas are immediate consequences of Theorem 6.1. If $\lambda = 0$, $f_{n+1,\nu}(x)f_{n,\nu}(x) = x\sum_{k=0}^{n}(\nu+k)f^2_{k,\nu}(x)$, and for $\lambda = 2$, $\nu = 1$, $w_{n+1}(x)\,w_n(x) = \sum_{k=0}^{n}[(k+1)\,xw_k^2(x) + 2w_k(x)]$.

Lemma 6.1. *For m a non-negative integer,*

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\nu+n-m)_{m-j}(\nu+n+m+1-j)_j = \frac{(-1)^m(2m)!}{m!}\,.$$

Proof. Using the techniques of umbral calculus with $a^k \equiv (a)_k$, and noting that $(\nu+n+m+1-j)_j = (-1)^j(-\nu-n-m)_j$

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\;(\nu+n-m)^{m-j}(\nu+n+m+1-j)^j$$

$$= \big((\nu+n-m)+(-\nu-n-m)\big)^m = (-2m)^m \equiv (-2m)_m = \frac{(-1)^m(2m)!}{m!}\,.$$

If we solve recurrence (4.1) for $(\nu+n)\, x g_{n,\nu}^{(\lambda)}(x)$, multiply both sides by $(\nu+n+1)(\nu+n-1)$ and iterate the result we obtain an expansion formula.

Theorem 6.2. *For k, n non-negative integers,*

$$(\nu+n-k)_{2k+1} x^k g_{n,\nu}^{(\lambda)}(x)$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\nu+n-k)_{k-j}(\nu+n+k+1-j)_j(\nu+n+k-2j)\, g_{n+k-2j,\nu}^{(\lambda)}(x)\,.$$

$$+\lambda \sum_{j=0}^{k-1} \frac{(-1)^{k-j}(2k-2-2j)!}{(k-1-j)!} (\nu+n-k)_{j+1}(\nu+n+k-j))_{j+1} x^j.$$

Proof. The proof is by induction on k. For $k=1$, the formula reduces to recurrence (4.1). For $k=m+1$, we use the induction hypothesis to get

$$(\nu+n-m-1)(\nu+n+m+1)\, x[(\nu+n-m)_{2m+1} x^m g_{n,\nu}^{(\lambda)}(x)]$$

$$= \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\nu+n-m-1)_{m-j+1}(\nu+n+m+1-j)_{j+1}$$

$$\cdot (\nu+n+m-2j)\, x g_{n+m-2j,\nu}^{(\lambda)}(x)$$

$$+\lambda \sum_{j=0}^{m-1} (-1)^{m-j} \frac{(2m-2-2j)!}{(m-1-j)!} (\nu+n-m-1)_{j+2}(\nu+n+m-j)_{j+2}\, x^{j+1}.$$

Applying (4.1) in the first sum and shifting j to $j-1$ in the second sum gives

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} (\nu+n-m-1)_{m-j+1}(\nu+n+m+1-j)_{j+1}\, g_{n+m+1-2j,\nu}^{(\lambda)}(x)$$

$$-\sum_{j=0}^{m} (-1)^j \binom{m}{j} (\nu+n-m-1)_{m-j+1}(\nu+n+m+1-j)_{j+1} g_{n+m-1-2j,\nu}^{(\lambda)}(x)$$

$$-\lambda \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\nu+n-m-1)_{m-j+1}(\nu+n+m+1-j)_{j+1}$$

$$+\lambda \sum_{j=0}^{m} (-1)^{m-j+1} \frac{(2m-2j)!}{(m-j)!} (\nu+n-m-1)_{j+1}\cdot(\nu+n+m+1-j)_{j+1} x^j.$$

Shifi j to $j-1$ in the second sum and combine with the first. Apply Lemma 6.1 to the third sum and combine it with the fourth to complete the proof.

For $\lambda = 0$, we have

$$(6.1)\qquad (\nu + n - k)_{2k+1} x^k f_{n,\nu}(x)$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\nu + n - k)_{k-j} (\nu + n + k + 1 - j)_j (\nu + n + k - 2j) f_{n+k-2j,\nu}(x) .$$

If ν is not an integer we can divide both sides of (6.1) by $(\nu + n - k)_{2k+1}$ to get a generalization of Nielsen's formula [**2**]$_2$

$$(6.2)\qquad x^k f_{n,\nu}(x) = \sum_{j=0}^{k} \frac{(-1)^j \binom{k}{j} (\nu + n + k - 2j)\, f_{n+k-2j,\nu}(x)}{(\nu + n - j)_{k+1}} .$$

In particular, for $n = 0$, (6.2) becomes

$$x^k = \sum_{j=0}^{k} \frac{(-1)^j \binom{k}{j} (\nu + k - 2j)\, f_{k-2j,\nu}(x)}{(\nu - j)_{k+1}} .$$

Formulas of this type are interesting because the permit any polynomial to be written as a sum modified Lommel polynomials.

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Abstract

In this paper the polynomials defined by

$$g_{n+1,\nu}^{(\lambda)}(x) = (n+\nu)\times g_{n,\nu}^{(\lambda)}(x) + g_{n-1,\nu}^{(\lambda)}(x) + \lambda\,, \qquad g_{0,\nu}^{(\lambda)}(x) = 1\,, \qquad g_{1,\nu}^{(\lambda)}(x) = \nu x + \lambda$$

are shown to generalize modified forms of both the Ménage polynomials and the Lommel polynomials. Recurrences, expansion formulas and product formulas are obtained for the generalized polynomials.

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