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# Remarks on the growth of an entire function and the degree of approximation (**)

## 1 - The main results

Set $I=[-1,1]$. Let $F: I \to \boldsymbol{C}$ be a continuous function (not a polynomial), $P_n$ be the class of the polynomials of degree at most $n$, $p_n = p_n(F) \in P_n$ be the polynomial of best approximation for $F$ (in $\mathscr{C}(I)$) and $e_n = e_n(F)$ be the $n$-th degree of approximation of $F$ (1). We have $e_n \neq 0 \ \forall_n$.

Let's consider the function $g$ of the complex variable $z$

$$g(z) = g(z; F) = \sum_0^\infty e_n z^n \quad e_n = e_n(F) .$$

The following Proposition holds.

Proposition. *$F$ is the restriction to $I$ of an entire function $f$ if and only if $g$ is an entire function.*

Indeed, $f$ is entire if and only if $e_n^{1/n} \to 0$ for $n \to \infty$ (S. N. Bernstein, [**1**]$_2$).

Let $\varphi$ be an entire function. By $\varrho(\varphi)$, $\lambda(\varphi)$, $\tau(\varrho, \varphi)$ and $\nu(\varrho, \varphi)$ we denote, as usual, respectively the order, the lower order, and for every $\varrho\,(0 < \varrho < \infty)$ the type and the lower type of the order $\varrho$ of $\varphi$ (2).

In section **3** we prove the following

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(1) I.e. $e_n = \operatorname{Max}_{x \in I} |F(x) - p_n(x)| = \operatorname{Min}_{p \in P_n} \operatorname{Max}_{x \in I} |F(x) - p(x)|$.

(2) See, e.g., [**2**], p. 8.

Theorem 1. *If $F$ is the restriction to $I$ of an entire function $f$, then*

$$\varrho(f) = \varrho(g)\,; \quad \lambda(f) = \lambda(g)\,; \quad \tau(\varrho, f) = 2^{\varrho}\tau(\varrho, g)\,; \quad \nu(\varrho, f) = 2^{\varrho}\nu(\varrho, g)\,.$$

Theorem 1 and the relations between the sequence of the coefficients of an entire function $\varphi$ and the order (3), the lower order (4), the type (3) and the lower type (5) of $\varphi$ give at once

Theorem 2. *If $F$ is the restriction to $I$ of an entire function $f$, then*

$$\varrho(f) = \limsup_{n\to\infty} \frac{n \log n}{-\log e_n}\,, \tag{1.1}$$

$$\lambda(f) = \operatorname*{Max}_{\{n_h\}} \liminf_{h\to\infty} \frac{n_h \log n_{h-1}}{-\log e_{n_h}}\,, \tag{1.2}$$

$$\tau(\varrho, f) = \frac{2^{\varrho}}{e\varrho} \limsup_{n\to\infty} n e_n^{\varrho/n}\,, \tag{1.3}$$

$$\nu(\varrho, f) = 2^{\varrho} \operatorname*{Max}_{\{n_h\}} \liminf_{h\to\infty} \varphi_{n_h}\, e^{-\varrho\varphi_{n_h}/n_h}\, {e_{n_h}}^{\varrho/n_h}\,, \tag{1.4}$$

*where*

$$\varphi_{n_h} = \frac{n_h \log e_{n_{h-1}} - n_{h-1} \log e_{n_h}}{n_h - n_{h-1}}\,.$$

So by a simple proceeding we obtain contemporarily the results of S. N. Bernstein ((1.1) and (1.3): see $[\mathbf{1}]_{1,2}$ see also [**8**]) and J. P. Singh ((1.2): see [**7**]), moreover we obtain the lower type of $f$ via the degrees of approximation.

## 2 - Consequences and remarks

Corollary 1. *If at least one of the right hand sides of* (1.1), (1.2), (1.3) *is finite, then $F$ is the restriction to $I$ of an entire function $f$ and* (1.1) *to* (1.4) *hold.*

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(3) See, e.g., [**2**], pp. 9-11.
(4) See [**5**].
(5) See [**4**].

Indeed, as $e_n \to 0$ for $n \to \infty$, the assumption gives $e_n^{1/n} \to 0$, hence $g$ is entire. Then Corollary follows from Proposition and Theorem 1.

On the contrary, if the right hand side of (1.4) is finite and that ones of (1.1), (1.2), (1.3) are infinite, it does not seem possible to ensure the existence of the entire function $f$. Indeed there exist non entire power series $\sum e_n z^n$ with $e_n \downarrow 0$ such that the right hand side of (1.4) is finite for every positive $\varrho$. E.g. consider the function $h(z) = \sum_1^\infty ((n+1)/2n)^n z^n$. We have $\varphi_{n_h} e^{-\varrho\varphi_{n_h}/n_h} e_{n_h}^{\varrho/n_h} \to 2^{-\varrho}$ hence the right hand side of (1.4) is positive finite for every $\varrho > 0$, while, as $h$ is non entire, the right hand sides of (1.1), (1.2), (1.3) are infinite.

Corollary 2. *If the right hand side of* (1.3) *is positive finite for a certain* $\varrho$, $0 < \varrho < \infty$, *then $f$ is entire and* $\varrho = \varrho(f)$.

Indeed in such a case the right hand side of (1.1) is $\varrho$.

A similar statement does not hold for (1.4). Indeed there exists $F$ restriction to $I$ of an entire function $f$ such that the right hand side of (1.4) is positive finite for some $\varrho \neq \varrho(f)$. E.g. consider

$$f(z) = \sum_{n=3}^\infty a_n z^{[\lambda_n]} \text{ where } \lambda_1 = 1\,, \quad \lambda_n = \exp(\lambda_{n-1}) \text{ and } a_n = (\lambda_{n-1}\lambda_{n-2})^{-[\lambda_n]}\,.$$

$f$ is an entire function of order $\infty$ (and of lower order 1). For every $\{n_h\}\uparrow\infty$ we have $\varphi_{n_h} = \lambda_{n_{h-1}}\lambda_{n_h} - 2\,(1 + o(1))$ and then $\nu(1, f) = 1$. By Theorem 1 the same must happen to the right hand side of (1.4).

Remark 1. If the right hand side of (1.3) is zero, then $\varrho(f) \leqslant \varrho$ and it can be, obviously, $\varrho(f) < \varrho$ too (6). A similar statement does not hold when the right hand side of (1.4) is zero. Indeed there exists $F$ restriction to $I$ of an entire function $f$ such that the right hand side of (1.4) is zero for some $\varrho > \varrho(f)$. E.g. consider $f(z) = \sum_{n=1}^\infty \lambda_n^{-[\lambda_n]} z^{[\lambda_n]}$, $\lambda_1 = 1$, $\lambda_n = \exp(\lambda_{n-1})$. $f$ is an entire function, $\varrho(f) = 1$ and, for every $\varrho > 0$, $\nu(\varrho, f) = 0$.

Remark 2. Let's now consider the case $\{e_n/e_{n+1}\}$ non decreasing. If $F$ is the restriction to $I$ of an entire function $f$, then (1.2) and (1.4) can be replaced by the more simple formulas (7)

(6) In spite of what said in [**8**] Theorem 2.

(7) See $[\mathbf{6}]_{1,2}$, [**4**] corollary 4.

$$\lambda(f) = \liminf_{n\to\infty} \frac{n \log n}{-\log e_n}\,, \tag{2.1}$$

$$\nu(\varrho, f) = \frac{2^\varrho}{e\varrho} \liminf_{n\to\infty} n e_n{}^{\varrho/n}\,. \tag{2.2}$$

Vice versa, as in the case we are considering there exists

$$\lim_{n\to\infty} e_n^{1/n} = \lim_{n\to\infty} e_{n+1}/e_n < 1\,,$$

if the right hand side of (2.1) is finite, then $F$ is the restriction to $I$ of an entire function; if the right hand side of (2.2) is positive finite, then $F$ is the restriction to $I$ of an entire function of lower order $\varrho$ (8).

### 3 - Proof of Theorem 1

For every $r > 1$ let $E_r$ be the ellipse

$$|z-1| + |z+1| \leqslant r + r^{-1}$$

and let

$$B_r = \operatorname*{Max}_{z\in E_r} |f(z)|\,.$$

From $|p_n(x) - p_{n+1}(x)| \leqslant 2e_n$ for $x \in I$, we obtain (see e.g. [3], theor. 7, p. 42)

$$|p_n(z) - p_{n+1}(z)| \leqslant 2 e_n r^{n+1} \qquad \forall z \in E_r\,.$$

From the above bound it follows that the series

$$p_0(z) + \sum_0^\infty \big(p_{n+1}(z) - p_n(z)\big)$$

(converging to $f(z)$ for $z \in I$) converges to $f(z)$ for every complex $z$ and, moreover, that

$$\forall z \in E_r \wedge \forall r > 1 \qquad |f(z)| \leqslant e_0 + \sum_0^\infty 2 e_n r^{n+1} \leqslant e_0 + 2 r g(r) \leqslant K r g(r)$$

(8) Indeed the right hand side of (2.1) is $\varrho$.

($K$ suitable positive constant). Hence for $r \to \infty$

$$\log g(r) \geqslant \log B(r) + O(\log r)\,. \tag{3.1}$$

From the well known upper bound (see, e.g. [3] (6) p. 78)

$$e_n \leqslant \frac{2B(\varrho)}{\varrho - 1}\, \varrho^{-n} \quad \forall \varrho > 1 \wedge \forall n\,,$$

we obtain for every $r > 1$

$$g(r) = \sum_0^\infty e_n r^n \leqslant \frac{2B(r+1)}{r} \sum_0^\infty \left(\frac{r}{r+1}\right)^n = 2\,\frac{r+1}{r}\, B(r+1) < 4B(r+1)$$

and then, for $r \to \infty$

$$\log g(r) \leqslant \log B(r+1) + O(1)\,. \tag{3.2}$$

From (3.1) and (3.2) it follows

$$\begin{matrix}\varrho(g) \\ \lambda(g)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log B(r)}{\log r}\,; \qquad \begin{matrix}\tau(\varrho, g) \\ \nu(\varrho, g)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log B(r)}{r^\varrho}\,. \tag{3.3}$$

Let $M(r) = \operatorname*{Max}_{|z|=r} |f(z)|$. Obviously for every $r > 1$

$$M\left(\frac{r - r^{-1}}{2}\right) \leqslant B(r) \leqslant M\left(\frac{r + r^{-1}}{2}\right).$$

As (for $r \to \infty$)

$$\log \frac{r \pm r^{-1}}{2} \sim \log r\,, \qquad \left(\frac{r \pm r^{-1}}{2}\right)^\varrho \sim 2^{-\varrho}\, r^\varrho \quad \forall \varrho > 0,$$

we obtain

$$\begin{matrix}\varrho(f) \\ \lambda(f)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log B(r)}{\log r}\,; \qquad \begin{matrix}\tau(\varrho, f) \\ \nu(\varrho, f)\end{matrix} = 2^\varrho \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log B(r)}{r^\varrho}\,. \tag{3.4}$$

The Theorem follows from the comparison between (3.3) and (3.4).

## References

[1] S. N. BERNSTEIN: $[\cdot]_1$ *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné,* Mémoires publiées par la classe de sci. Acad. de Belgique (2) **4** (1912), 1-103; $[\cdot]_2$ *Leçons sur les propriétés extremales et la meilleure approximation des fonctions analytiques d'une variable réelle,* Gautier-Villars, Paris 1926.

[2] R. P. BOAS, *Entire functions,* Academic Press, New York 1954.

[3] G. G. LORENTZ, *Approximation of functions,* Holt, Rinehart & Winston, New York 1966.

[4] S. MASSA and D. ROUX, *On the lower type of entire functions,* Boll. Un. Mat. Ital. (to appear).

[5] D. ROUX, *Sull'ordine inferiore delle funzioni intere,* Boll. Un. Mat. Ital. (3) **20** (1965), 379-388.

[6] S. M. SHAH: $[\cdot]_1$ *On the lower order of integral functions,* Bull. Amer. Math. Soc. **52** (1946), 1046-1052; $[\cdot]_2$ *On the coefficients of an entire series of finite order,* J. London Math. Soc. **26** (1951), 45-46.

[7] J. P. SINGH, *Approximation of an entire function,* Yokohama Math. J., **19** (1971), 105-108.

[8] R. S. VARGA, *On an extension of a result of S. N. Bernstein,* J. Approximation Theory **1** (1968), 176-179.

## Sunto

*Data una funzione $F$ continua su $[-1, 1]$, sia $e_n$ lo scarto di $F$ dal polinomio di miglior approssimazione di grado al più $n$. Si costruisce mediante la successione $\{e_n\}$ una funzione analitica $g$ della variabile complessa $z$. Essa è intera se e solo se $F$ è la restrizione a $[-1, 1]$ di una funzione intera $f$. In questo caso si ottengono l'ordine, l'ordine inferiore, il tipo ed il tipo inferiore di $f$ da quelli di $g$. Si determina così il tipo inferiore di $f$ mediante l'andamento della successione $\{e_n\}$ e si riottengono altri risultati noti sull'accrescimento di $f$.*

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