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M. L. MOGRA (*)

On a class of univalent functions whose derivatives have a positive real part (**)

1 - Introduction

Let f be analytic in a convex domain E. If f satisfies the condition

$$\operatorname{Re}\left(f'(z)\right) > 0 \tag{1.1}$$

for all $z \in E$, then it is wellknown (see [**9**], [**13**] and others) that f is univalent in E. MacGregor [**7**]$_1$ investigated the properties, e.g., coefficient estimates, radius of convexity etc. for functions f analytics in $\Delta \equiv \{z\colon |z| < 1\}$ having power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

and satisfying (1.1) for $z \in \Delta$. We denote the class of such functions by R. Analogous properties have also been obtained in [**7**]$_1$ for analytic functions with initial zero coefficients in (1.2) and satisfying (1.1) for $z \in \Delta$. Ezrohi [**3**] and Martynov [**8**] obtained the radius of convexity along with the other properties for the class R_α of functions $f(z)$ that are analytic and satisfy

$$\operatorname{Re}\left(f'(z)\right) > \alpha \tag{1.3}$$

(*) Indirizzo: School of Mathematical Sciences, University of Khartoum, P.O. Box 321, Khartoum, Sudan.
(**) Ricevuto: 10-X-1979.

for $0 \leqslant \alpha < 1$, $z \in \Delta$. Several other subclasses of R have also been studied by Caplinger and Causey [1], Goel $[4]_{1,2}$, MacGregor $[7]_2$, Padmanabhan [10], Shaffer [11] and others.

In the present paper, we propose a unified approach to the study of various subclasses of univalent functions whose derivatives have a positive real part in Δ. Thus, we introduce the class $R_k(\alpha, \beta)$ which, for different values of the parameters α, β $(0 \leqslant \alpha < 1, 0 < \beta \leqslant 1)$, not only gives rise to the classes studied by the above mentioned workers but also gives rise to many new subclasses of univalent functions. Thus we have the following

Definition. *Let* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *be analytic in the unit disc* Δ. *Then* $f \in R_k(\alpha, \beta)$ *if the condition*

$$|(f'(z) - 1)/\{2\beta(f'(z) - \alpha) - (f'(z) - 1)\}| < 1 \tag{1.4}$$

is satisfied for some α, β $(0 \leqslant \alpha < 1, 0 < \beta \leqslant 1)$ *and for all* $z \in \Delta$.

It is easy to check that $R_1(\alpha, 1)$ is the class R_α studied by Ezrohi [3], Martynov [8] etc.; $R_1(0, 1) \equiv R$, $R_1(0, \frac{1}{2})$, $R_k(0, 1)$ and $R_k(0, 1 - \delta)$, where $0 \leqslant \delta < 1$, give rise to the classes introduced and studied by MacGregor $[7]_{1,2}$ and Shaffer [11], while the cases $(\alpha, \beta) = (0, (2\delta - 1)/2\delta)$, $\delta > \frac{1}{2}$ and $(\alpha, \beta) = ((1 - \gamma)/(1 + \gamma), (1 + \gamma)/2)$, $0 < \gamma \leqslant 1$ with $k = 1$ lead respectively to the classes studied earlier by Goel [4], Padmanabhan [10], Caplinger and Causey [1] etc.; also $k = 1$ and a replacement of α by $1 - \alpha$ and β by $\frac{1}{2}$ in (1.4) gives the class introduced by Goel $[4]_2$.

From the definition given above it is clear that $R_k(\alpha, \beta)$ is a subclass of the class of functions whose derivatives have a positive real part in Δ. Also $R_k(\alpha, \beta) \subset R_k(\alpha, \beta')$ for $\beta \leqslant \beta'$. It is easily seen that for $f \in R_k(\alpha, \beta)$, the values $f'(z)$ lie inside the circle in the right half plane with centre at $(1 + \alpha - 2\alpha\beta)/2(1 - \beta)$ and radius $(1 - \alpha)/2(1 - \beta)$. Further, it follows from Schwarz lemma that if $f \in R_k(\alpha, \beta)$, then $f'(z) = (1 + (2\alpha\beta - 1) z^k \varphi(z))/(1 + (2\beta - 1) z^k \varphi(z))$ where φ is analytic in Δ and satisfies $|\varphi(z)| < 1$ for $z \in \Delta$.

In the present paper, we determine sharp coefficient estimates, radius of convexity etc. for functions in $R_k(\alpha, \beta)$. A sufficient condition for a function to be in $R_k(\alpha, \beta)$ has also been obtained. For different values of the parameters α, β $(0 \leqslant \alpha < 1, 0 < \beta \leqslant 1)$ our results sharpen and generalize the corresponding results obtained by Caplinger and Causey [1], Ezrohi [3], Goel $[4]_{1,2}$, Kaczmarski [6], MacGregor $[7]_{1,2}$, Martynov [8], Padmanabhan [10], Shaffer [12] etc.

Remark. The function $f(z)$ given by (1.2) and satisfying

$$(1.5) \qquad \big(f'(z)-1\big)/\big(f'(z)+1-2\alpha\big)\,| < \beta$$

for some α, β $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1)$, $z \in \Delta$ is obtained by replacing α by $(1-\beta+2\alpha\beta)/(1+\beta)$ and β by $(1+\beta)/2$ in (1.4). The class of functions satisfying (1.5) was introduced and studied in [**5**].

2 - Coefficient estimates

Theorem 1. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is in* $R_k(\alpha, \beta)$, *then* $|a_n| \leqslant 2\beta(1-\alpha)/n$ *for* $n \geqslant k+1$, $k = 1, 2, \ldots$. *The bounds are sharp for the functions*

$$f_n(z) = \int_0^z \frac{1+(1-2\alpha\beta)\,t^{n-1}}{1-(2\beta-1)\,t^{n-1}}\,dt$$

for $n \geqslant k+1$ *and* $z \in \Delta$.

The proof of the above theorem is similar to that of Clunie [**2**], and hence is omitted.

Remark. Different values of the parameters α, β and $k=1$ in Theorem 1 lead to the coefficient estimates obtained earlier by Caplinger and Causey [**1**], Goel [**4**], MacGregor $[\mathbf{7}]_{1,2}$, Padmanabhan [**10**] etc.

3 - A sufficient condition for a function to be in $R_k(\alpha, \beta)$

Theorem 2. *Let* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *be analytic in* Δ. *If for some* α, β $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant \frac{1}{2})$,

$$(3.1) \qquad \sum_{n=k+1}^{\infty} (1-\beta)\,n\,|a_n| \leqslant (1-\alpha)\beta\,,$$

then $f(z)$ *belongs to* $R_k(\alpha, \beta)$.

Proof. Suppose (4.1) holds for some α, β $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant \frac{1}{2})$ and that

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n,$$

then for $z \in \Delta$,

$$|f'(z)-1| - |2\beta(f'(z)-\alpha)-(f'(z)-1)|$$

$$\leqslant \sum_{n=k+1}^{\infty} n|a_n|r^{n-1} - \{2\beta(1-\alpha) + \sum_{n=k+1}^{\infty}(1-2\beta)n|a_n|r^{n-1}\}$$

$$< \sum_{n=k+1}^{\infty} n|a_n| - 2\beta(1-\alpha) + \sum_{n=k+1}^{\infty}(1-2\beta)n|a_n|$$

$$= 2[\sum_{n=k+1}^{\infty}(1-\beta)n|a_n| - \beta(1-\alpha)] = 0\,,$$

by (3.1). Hence it follows that $|(f'(z)-1)/\{2\beta(f'(z)-\alpha)-(f'(z)-1)\}|<1$, so that $f \in R_k(\alpha, \beta)$. Hence the theorem.

Remark. Since $f \in R_k(\alpha, \frac{1}{2})$ implies $f \in R_k(\alpha, \beta)$ for $\frac{1}{2} \leqslant \beta \leqslant 1$, the condition (3.1) for $\beta = \frac{1}{2}$, that is, the condition

$$\sum_{n=k+1}^{\infty} n|a_n| \leqslant (1-\alpha) \tag{3.2}$$

can also be used as a sufficient condition for a function to be in $R_k(\alpha, \beta)$ for $0 \leqslant \alpha < 1$, $\frac{1}{2} \leqslant \beta \leqslant 1$. The condition (3.2) with $k=1$, $\alpha = 0$ may be found in [**14**] as a sufficient condition for a function to be in R.

4 - The radii of convexity for functions in $R_1(\alpha, \beta)$

Let B denote the class of analytic functions $\omega(z)$ in Δ which satisfy the conditions (i) $\omega(0)=0$ and (ii) $|\omega(z)|<1$ for $z \in \Delta$. We require the following lemmas.

Lemma 1 [**12**]. *If $\omega \in B$, then for $z \in \Delta$*

$$|z\omega'(z) - \omega(z)| \leqslant \frac{|z|^2 - |\omega(z)|^2}{1-|z|^2}\,. \tag{4.1}$$

Lemma 2. *Let $\omega \in B$. Then we have*

$$\operatorname{Re}\left\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(2\alpha\beta-1)\omega(z))}\right\} \leqslant -\frac{1}{4\beta^2(1-\alpha)^2}\operatorname{Re}\{(2\beta-1)p(z)$$

$$+\frac{(2\alpha\beta-1)}{p(z)}-2(\beta+\alpha\beta-1)\}+\frac{r^2|(2\beta-1)p(z)-(2\alpha\beta-1)|^2-|1-p(z)|^2}{4\beta^2(1-\alpha)^2(1-r^2)|p(z)|},$$

where $p(z)=(1+(2\alpha\beta-1)\omega(z))/(1+(2\beta-1)\omega(z))$, $r=|z|$ *and* $0\leqslant\alpha<1$, $0<\beta\leqslant 1$.

The proof of the above lemma follows from (4.1) immediately. So we omit it.

Remark. The transformation

$$p(z)=(1+(2\alpha\beta-1)\omega(z))/(1+(2\beta-1)\omega(z))$$

maps the circle $|\omega(z)|\leqslant r$ onto the circle

$$\left|p(z)-\frac{1-(2\alpha\beta-1)(2\beta-1)r^2}{1-(2\beta-1)^2r^2}\right|\leqslant\frac{2\beta(1-\alpha)r}{1-(2\beta-1)^2r^2}.$$

Theorem 3. *Let $f\in R_1(\alpha,\beta)$. For a given β $(0<\beta\leqslant 1)$ let*

$$\alpha_0(\beta)=\{-(1+10\beta)+\sqrt{(1+12\beta+36\beta^2+32\beta^3)}\}/4\beta(4\beta^2-8\beta-1).$$

Further, let

$$V=\{(\alpha,\beta): 0\leqslant\alpha<1,\ 0<\beta\leqslant 1\},\qquad \Gamma_1=\{(\alpha,\beta): 0\leqslant\alpha<\frac{1}{10},\ 0<\beta\leqslant 1\},$$

$$\Gamma_2=\{(\alpha,\beta): \frac{1}{10}\leqslant\alpha<\alpha_0(\beta),\ 0<\beta\leqslant 1\},\qquad \Gamma_3=V-(\Gamma_1\cup\Gamma_2).$$

Then

(i) *f is convex in $|z|<r_1$, for $(\alpha,\beta)\in\Gamma_1\cup\Gamma_2$,*
(ii) *f is convex in $|z|<r_2$, for $(\alpha,\beta)\in\Gamma_3$,*

where $r_1=[(1-2\alpha\beta)+\sqrt{2\beta(1-\alpha)(1-2\alpha\beta)}]^{-1}$,

$$r_2=\left[\frac{\alpha}{\alpha\beta+\sqrt{\alpha(1-2\alpha\beta+\alpha\beta^2)}}\right]^{\frac{1}{2}}.$$

The bounds for $|z|$ in (i) *and* (ii) *are sharp.*

Proof. Since $f \in R_1(\alpha, \beta)$, we have by Schwarz lemma

$$f'(z) = \frac{1 + (2\alpha\beta - 1)\,\omega(z)}{1 + (2\beta - 1)\,\omega(z)}, \tag{4.2}$$

where $\omega \in B$. Differentiating (4.2) logarithmically, we get

$$1 + z\frac{f''(z)}{f'(z)} = 1 - 2\beta\,(1-\alpha)\left\{\frac{z\omega'(z)}{(1 + (2\beta-1)\,\omega(z))(1 + (2\alpha\beta - 1)\omega(z))}\right\}. \tag{4.3}$$

An application of Lemma 2 to the above equation gives

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geqslant \frac{1}{2\beta(1-\alpha)}\Big[\operatorname{Re}\left\{(2\beta-1)\,p(z) + \frac{(2\alpha\beta-1)}{p(z)}\right\} - \frac{r^2\,|(2\beta-1)\,p(z) + 1 - 2\alpha\beta\,|^2 - |1 - p(z)\,|^2}{(1-r^2)\,|p(z)\,|}\Big] + \frac{1-2\alpha\beta}{\beta(1-\alpha)}, \tag{4.4}$$

where $p(z) = \big(1 + (2\alpha\beta - 1)\,\omega(z)\big)/\big(1 + (2\beta - 1)\,\omega(z)\big)$. Setting $p(z) = A + \xi + i\eta$, $R^2 = (A + \xi)^2 + \eta^2$ where $A = \{1 - (2\alpha\beta - 1)(2\beta - 1)\,r^2\}/\{1 - (2\beta - 1)^2\,r^2\}$ and denoting the expression on the right hand side of (4.4) by $S(\xi, \eta)$, we get

$$S(\xi, \eta) = \frac{1-2\alpha\beta}{\beta(1-\alpha)} + \frac{1}{2\beta(1-\alpha)}\Big[(2\beta-1)(A + \xi) + (2\alpha\beta - 1)(A+\xi)\,\bar{R}^{-2} - \frac{1 - (2\beta-1)^2 r^2}{1 - r^2}\,(D^2 - \xi^2 - \eta^2)\,\bar{R}^{-1}\Big], \tag{4.5}$$

where $D = 2\beta(1-\alpha)\,r/\{1 - (2\beta - 1)^2 r^2\}$. Differentiating (4.5) partially w.r.t. η, we get

$$\frac{\partial S}{\partial \eta} = \frac{1}{2\beta(1-\alpha)}\,\eta\bar{R}^{-4}\,T(\xi, \eta) \tag{4.6}$$

where

$$T(\xi, \eta) = 2(1 - 2\alpha\beta)(A + \xi) + \frac{1 - (2\beta-1)^2 r^2}{(1 - r^2)}\,R + 2\,\frac{1 - (2\beta - 1)^2\,r^2}{(1-r^2)}\,R^3.$$

It is easy to check that $T(\xi, \eta) > 0$ and so (4.6) gives that the minimum of $S(\xi, \eta)$ inside the disc $\xi^2 + \eta^2 \leqslant D^2$ is attained on the diameter $\eta = 0$. On putting $\eta = 0$ in (4.5), we obtain

$$U(R) \equiv S(\xi, 0) = \frac{1-2\alpha\beta}{\beta(1-\alpha)} + \frac{1}{2\beta(1-\alpha)}\Big[\Big(2\beta - 1 + \frac{1-(2\beta-1)^2 r^2}{1-r^2}\Big) R$$
$$+ \frac{2\alpha\beta(1-(2\alpha\beta-1)r^2)}{1-r^2} R^{-1} - 2A\frac{1-(2\beta-1)^2 r^2}{1-r^2}\Big],$$

where $R = A + \xi$ and $A - D \leqslant R \leqslant A + D$. Thus the absolute minimum of $U(R)$ in $(0, \infty)$ is attained at

$$R_0 = \Big[\frac{\alpha(1-(2\alpha\beta-1)r^2)}{1-(2\beta-1)r^2}\Big]^{\frac{1}{2}} \tag{4.7}$$

and the value of this minimum is

$$U(R_0) = \frac{1}{\beta(1-\alpha)(1-r^2)}\Big[\sqrt{(4\alpha\beta^2(1-(2\beta-1)r^2)(1-(2\alpha\beta-1)r^2)} - (1-(2\alpha\beta-1)(2\beta-1)r^2) + (1-2\alpha\beta)(1-r^2)\Big]. \tag{4.8}$$

It is easily seen that $R_0 < A + D$, but R_0 is not always greater than $A - D$. In such a case when $R_0 \notin [A - D, A + D]$, the minimum of $U(R)$ on the segment $[A - D, A + D]$ is attained at $R_1 = A - D$ since $U(R)$ increases with R on this segment. The value of this minimum equals

$$U(R_1) \equiv U(A-D) = \frac{1-2(1-2\alpha\beta)r + (2\beta-1)(2\alpha\beta-1)r^2}{(1+(2\beta-1)r)(1+(2\alpha\beta-1)r)}. \tag{4.9}$$

It follows from what has been said that the bound r' of convexity for the class $R_1(\alpha, \beta)$ is determined either from the equation

$$U(R_0) = 0, \tag{4.10}$$

or from the equation

$$U(R_1) = 0. \tag{4.11}$$

Also, $U(R_0) = U(R_1)$ for those $(\alpha, \beta) \in V$ for which

$$R_0 = R_1. \tag{4.12}$$

Equations (4.10) and (4.11) may be reduced to the following equations

$$(4.13) \qquad (2\beta-1)(2\alpha\beta-1)r^2-2(1-2\alpha\beta)r+1=0\,,$$

$$(4.14) \qquad (1-2\alpha\beta)r^4+2\alpha\beta r^2-\alpha=0\,.$$

From (4.13) and (4.14), we get

$$(4.15) \qquad r'=r_1=[(1-2\alpha\beta)+\sqrt{(2\beta(1-\alpha)(1-2\alpha\beta))}]^{-1},$$

$$(4.16) \qquad r'=r_2=\Big[\frac{\alpha}{\alpha\beta+\sqrt{(\alpha(1-2\alpha\beta+\alpha\beta^2))}}\Big]^{\frac{1}{2}}\,.$$

To obtain the points $(\alpha, \beta) \in V$ which determine the transition from formula (4.15) to formula (4.16) we eliminate r from (4.12) and (4.13) and get

$$(4.17) \qquad Q(\alpha, \beta) \equiv 2\beta(4\beta^2-8\beta-1)\alpha^2+(1+10\beta)\alpha-1=0\,.$$

For a given β $(0<\beta\leqslant 1)$, the smallest positive root of the equation (4.17), which is quadratic in α is given by

$$\alpha_0(\beta)=\frac{-(1+10\beta)+\sqrt{(1+12\beta+36\beta^2+32\beta^2)}}{4\beta(4\beta^2-8\beta^2-1)}\,.$$

It is evident that $\alpha_0(1)=1/10$ and $\alpha_0(0)=1$ as β tends to zero.

Now, let Γ denote the arc of the curve $Q(\alpha, \beta)=0$ lying in the region $G=\{(\alpha, \beta)\colon 1/10\leqslant\alpha<1,\ 0<\beta\leqslant 1\}\subset V$, that is, passing through the points $(1/10, 1)$ and $(1, 0)$. The curve Γ divides the region V into two subregions $H=\Gamma_1\cup\Gamma_2$ and Γ_3 where $\Gamma_1=\{(\alpha, \beta)\colon 0\leqslant\alpha<1/10,\ 0<\beta\leqslant 1\}$, $\Gamma_2=\{(\alpha, \beta)\colon 1/10\leqslant\alpha\leqslant\alpha_0(\beta),\ 0<\beta\leqslant 1\}$ and $\Gamma_3=V-(\Gamma_1\cup\Gamma_2)$. The curve Γ also gives transition from formula (4.15) to formula (4.16). It is obvious that Γ has void intersection with Γ_1 so that in Γ_1 we have to use either formula (4.15) or formula (4.16). But it is easily seen that it is impossible to use formula (4.16) for all the points (α, β) lying in $K=\{(\alpha, \beta)\colon \alpha=0,\ 0<\beta\leqslant 1\}\subset\Gamma_1$. So, formula (4.15) must be used for (α, β) lying in Γ_1. Now we consider the points (α, β) lying in the region $G=\Gamma_2\cup\Gamma_3$. Since the formula (4.15) cannot be used for the points (α, β) lying in $W'=\{(\alpha, \beta)\colon \alpha_0'(\beta)\leqslant\alpha<1,\ \frac{1}{2}\leqslant\beta\leqslant 1\}$, where $\alpha_0'(\beta)=\frac{1}{2\beta}\}\subset\Gamma_3$, it therefore follows that for $(\alpha, \beta)\in\Gamma_3$, we have to use formula (4.16) while formula (4.15) is to be used for $(\alpha, \beta) \in \Gamma_1 \cup \Gamma_2$. This proves (i) and (ii).

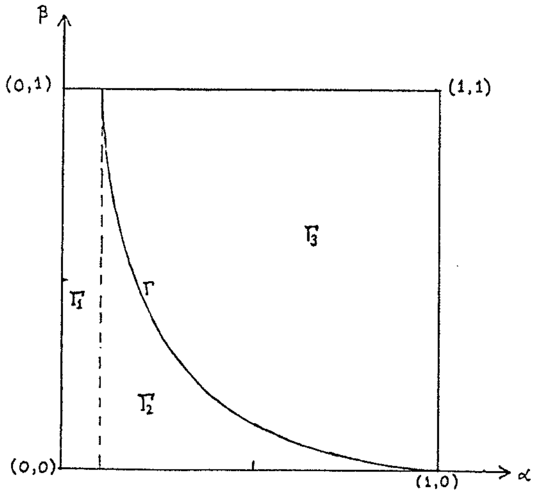

Figure 1

The above figure shows the transition curve Γ and the regions Γ_1, Γ_2, and Γ_3. The functions given by

$$f'(z) = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)z}, \qquad f'(z) = \frac{1-2\alpha\beta bz+(2\alpha\beta-1)z^2}{1-2\beta bz+(2\beta-1)r^2},$$

where b is determined by the relation

$$\frac{1-2\alpha\beta br+(2\alpha\beta-1)r^2}{1-2\beta br+(2\beta-1)r^2} = \sqrt{\frac{(\alpha(1-(2\alpha\beta-1)r^2))}{(1-(2\beta-1)r^2)}} = R_0 .$$

Show that the results obtained in the theorem are sharp.

Remark. Taking different values of the parameters α, β ($0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$) in Theorem 3, we get the radii of convexity for functions in different classes obtained earlier by Caplinger and Causey [**1**], Ezrohi [**3**], Goel [**4**]$_{1,2}$, Kaczmarski [**6**], MacGregor [**7**]$_{1,2}$, Martynov [**8**], Padmanabhan [**10**], Shaffer [**11**] and others.

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