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On injective and p-injective modules (**)

Introduction

C. Faith ([**4**]$_1$, theorem 14) proved that right PCI rings (rings whose proper cyclic right modules are injective) are either semi-simple Artinian or right semi-hereditary simple domains. In the first section of this note, we prove the following p-injective analogue of Faith's theorem. Left PCP rings (rings whose proper cyclic left modules are p-injective) are either von Neumann regular or simple domains. Next, commutative hereditary Noetherian rings are characterised as rings whose divisible modules are injective. If A is a semi-prime indecomposable ring such that any divisible left or right A-module is injective, then A/I is an Artinian serial ring for every non-zero ideal I of A (this is motivated by a well-known theorem of Eisenbud-Griffith-Robson). Letf Ore domain are characterised in terms of indecomposable CS-rings [**3**]. Finally, characterisations of semi-simple and simple Artinian rings are given.

Throughout, A represents an associative ring with identity and A-modules are unitary. Z and S denote respectively the left singular ideal and the left socle of A. Recall that a left A-module M is p-injective (resp. f-injective) if, for any principal (resp. finitely generated) left ideal I of A and any left A-homomorphism $g: I \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in I$. A is von Neumann regular iff every left A-module is p-injective (f-injective). An element a of A is called von Neumann regular iff Aa is a direct summand of ${}_AA$. An ideal (two-sided) T of A is called von Neumann

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regular iff every element of T is von Neumann regular. As usual, A is called a left V-ring if every simple left A-module is injective [**4**]$_2$. Call A a left $p-V$-ring [**11**]$_1$ if every simple left A-module is p-injective. Left $p-V$-rings (which are fully left idempotent) generalise both regular rings and left V-rings (since Faith has shown that regular rings need not be left V-rings and Cozzens has constructed a left PCI V-domain which is not regular). Since several years, regular rings and V-rings raise a great deal of interest and research activity (cfr. for example [**4**]$_2$, [**5**], [**7**], [**9**], [**11**]$_{1,2}$).

1 - CPP and PCP rings

In ([**13**]$_2$, theorem 2), it is proved that left CPP rings (rings whose cyclic left modules are either projective orp-injective) are fully left idempotent, left p.p.-rings. We here call A a left PCP ring if every cyclic left A-module which is not isomorphic to $_AA$ is p-injective. It is well-known (J. H. Cozzens) that simple left PCI domains need not be division rings. Consequently, left PCP rings generalise both regular rings and left PCI rings. Our first result improves ([**13**]$_2$, theorem 2).

Theorem 1.1. *Let A be a left* CPP *ring. Then A is either a left p.p. left $p-V$-ring with a non-zero von Neumann regular socle S or a regular ring with zero socle or a simple domain.*

Proof. Since A is fully left idempotent [**13**]$_2$, then A is semi-prime. We first suppose that $S \neq 0$. If U is a minimal left ideal of A, then $A = U \oplus M$, where M is a maximal left ideal of A. If $M \subseteq S$, then $A = S$ is semi-simple Artinian. Suppose $M \not\subseteq S$. Then M contains a proper essential left subideal L. Since M/L and A/L are both p-injective left A-modules (A/L projective leads to $L = M$ which is a contradiction), and $U \approx A/M \approx (A/L)(M/L)$, then $A/L \approx (M/L) \oplus K$, where $(A/L)(M/L) = K$ is p-injective. This proves U p-injective. Since any simple projecrive left A-module is isomorphic to a minimal left ideal of A, then A is a left $p-V$-ring. If $s \in S$, then $As = U_1 \oplus \ldots \oplus U_n$, where each U_i is a p-injective minimal left ideal. Then As is a finitely generated p-injective left ideal which is therefore a direct summand of $_AA$. This proves that S is a von Neumann regular ideal. Now suppose that $S = 0$. If A is an integral domain, then A being fully left idempotent [**13**]$_2$ implies A simple. If A is not an integral domain, there exist $0 \neq b$, $c \in A$ such that $bc = 0$. Since A is a left p.p.ring [**13**]$_2$, then $A = l(c) \oplus B$ for some non-zero left ideal B of A. Since $S = 0$, then the above argument shows that both $l(c)$ and B are cyclic p-injective left A-modules which proves that A

is a left p-injective ring. Then any cyclic projective left A-module is p-injective which implies that every cyclic left A-module is p-injective. This proves that A is regular in this case.

Corollary 1.2. *A left* CPP *ring with zero socle is either regular or a simple domain.*

Corollary 1.3. *A left* CPP *ring is a left $p-V$-ring.*

Proof. If A is a left CPP simple domain which is not a division ring, then there exists no simple projective left A-module. Thus A is a left $p-V$-domain. The corollary then follows from Theorem 1.1.

The first part of the proof of Theorem 1.5 yields the following result concerning CPP rings.

Proposition 1.4. *An indecomposable left* CPP *ring is either regular or a simple domain.*

If A is a left CPP ring, then for any ideal T of A, A/T is a left CPP ring. The next theorem is motivated by ([**4**]$_1$, theorem 14).

Theorem 1.5. *A left* PCP *ring is either von Neumann regular or a simple domain.*

Proof. Let A be a left PCP ring. Then A is a left CPP ring and if $S = 0$, by Corollary 1.2, A is either regular or a simple domain. Now let $S \neq 0$. Then S is a von Neumann regular ideal (Theorem 1.1) and by Zorn's Lemma the set of von Neumann regular ideals containing S has a maximal member T. Suppose A is not regular. Since $B = A/T$ contains no non-zero von Neumann regular ideal, if ${}_AB$ is p-injective, then B is a left p-injective left CPP ring which is therefore regular (cfr. the proof of Theorem 1.1). This contradiction proves that ${}_AB$ is projective which implies $A = T \oplus D$, where $T = Ae$, $e = e^2 \in A$ and $D = A(1-e)$ is also an ideal of A (since $(1-e)A \subseteq r(AeA) = l(AeA) \subseteq l(e) = A(1-e)$). Thus T is a regular ring with identity and D contains no minimal left ideal of A. If $S \subset T$, then ${}_AT$ contains a proper essential left submodule and the proof of Theorem 1.1 shows that D is a left p-injective left CPP ring which is therefore regular. Then $A = T \oplus D$ is regular which is a contradiction. Thus $T = S$ is a direct sum of minimal left ideals which are p-injective (Corollary 1.3). Since a direct sum of left A-modules is p-injective iff each direct summand is p-injective, then ${}_AT$ is p-injective. Since A is a left PCP ring, if $f\colon {}_AA \to {}_AD$ is an isomorphism, then for any minimal left ideal U of A, $f(U)$ is a minimal left ideal of A contained in D which contradicts $T \cap D = 0$. Thus ${}_A D$ is p-injective which implies A is a left p-injective left CPP ring. This again contradicts our hypothesis that A is not regular. This proves that $S \neq 0$ implies A regular.

Corollary 1.6. (1) *A is von Neumann regular iff A is a left* PCP *ring containing a non-zero divisible left or right ideal.*

(2) *If A is a left* PCP *ring whose left ideals are either divisible or projective, then A is either regular or a simple left hereditary domain.*

Proof. An integral domain containing a non-zero divisible left or right ideal is a division ring.

Rings whose complement left ideals are direct summands, called left CS-ring, are studied in [**3**]. Left continuous rings (in the sense of Utumi [**12**]) are obviously left CS-rings but the converse is not true.

Corollary 1.7. *A left* PCP, *left* CS-*ring is either left continuous regular or a simple left Ore domain.*

Proof. A regular left CS-ring is left continuous. If A is a left CS-domain, then ${}_A A$ is uniform which implies A a left Ore domain.

Corollary 1.8. (1) *A left* PCP *ring with maximum condition on left annihilators is either semi-simple Artinian or a simple domain.*

(2) *A left* PCP *ring with maximum condition on complement left ideals is either semi-simple Artinian or a simple left Ore domain.*

A left ideal of A is called reduced if it contains no non-zero nilpotent element.

Corollary 1.9. *A left* PCP *ring whose complement left ideals are ideals is either strongly regular or a simple left Ore domain.*

Proof. Since $Z = 0$, A is reduced by [**13**]$_1$, lemma 1. Since an integral domain is left Ore iff 0 and A are the only complement left ideals, the corollary then follows from Theorem 1.5.

Since a left $p - V$-ring whose maximal left ideals are ideals is strongly regular, the next corollary then follows.

Corollary 1.10. *A prime left* PCP *ring whose maximal essential left ideals are ideals is a primitive regular ring with non-zero socle.*

(It is now known that prime regular rings need not be primitive (O. I. Domanov), which settles in the negative a question raised by I. Kaplansky).

2 - Injective and divisible modules

This section is motivated by the following result of Levy ([**8**], theorem 3.4). If A is a left hereditary ring with a two-sided classical quotient ring which is semi-simple Artinian, then every divisible left A-module is injective. We here consider rings over which the notions of injectivity, p-injectivity and divisibility coincide. The next lemma shows that this happens iff divisible modules are injective.

Lemma 2.1. *A p-injective left A-module is divisible.*

Proof. Let M be a p-injective left A-module. If c is a non-zero-divisor of A, for any $y \in M$, define a left A-homomorphism $g\colon Ac \to M$ by $g(ac) = ay$ for all $a \in A$. Then there exists $u \in M$ such that $g(ac) = acu$ for all $a \in A$. In particular, $y = g(c) = cu \in cM$ which implies $M = cM$ and proves M divisible.

Lemma 2.2. *If A is an integral domain, then any divisible left A-module is p-injective.*

Proof. Let D be a divisible left A-module, $P = Ab$, $0 \neq b \in A$, and $f\colon P \to D$ a left A-homomorphism. Since $f(b) \in bD$, $f(b) = bd$ for some $d \in D$ which implies $f(ab) = af(b) = abd$ for all $a \in A$. This proves D p-injective.

Remark. Divisible modules over integral domains need not be injective. If K is a commutative field, $A = K[y, z]$, $F = K(y, z)$, $I (= Ay + Az)$ the left ideal of A generated by y and z, then the left A-module F/I is divisible but not injective.

Corollary 2.3. *If A is a left* PCP *ring, then a left A-module is p-injective iff it is divisible.*

Commutative rings whose singular modules are injective are hereditary regular but not necessarily semi-simple Artinian [**1**]. Therefore, the rings considered in the next result need not be left or right Noetherian. However, we show that Matlis' conjecture on decomposable modules holds for such rings.

Theorem 2.4. *Let A be a ring whose divisible singular left modules are injective and such that every maximal essential right ideal is f-injective. Then A is a regular left hereditary ring such that every direct summand of any completely decomposable left or right A-module is completely decomposable.*

Proof. It is well-known that A is left hereditary iff every homomorphic image of any injective left A-module is injective. For any left A-module M, if $\hat{M}$ is an injective hull of M, then $\hat{M}/M$ is a divisible singular left A-module (since any homomorphic image of a divisible left A-module is divisible) which is therefore injective. The proof of ([**13**]$_2$, proposition 4) then shows that A is left hereditary. Now let F be a finitely generated proper right ideal of A and R a maximal right ideal containing F. If R is essential in A_A (and therefore f-injective), the canonical injection $F \to R$ yields $b = ub$ for some $u \in R$ and every $b \in F$ which implies $l(F) \neq 0$. Otherwise, R is a direct summand of A_A which again implies $l(F) \neq 0$. Then a well-known theorem of H. Bass implies that any finitely generated projective submodule of a projective left A-module is a direct summand. Since A is left semi-hereditary, then every finitely generated left ideal is a direct summand of $_AA$ which proves A regular. By Lemma 2.1, every singular left A-module is injective and by ([**6**]$_1$, corollary 3.7), every singular right A-module is injective. Then every direct summand of any completely decomposable left or right A-module is completely decomposable ([**13**]$_3$, corollary 4).

It is well-known that a commutative integral domain is a Dedekind ring iff every divisible module is injective. Commutative hereditary Noetherian rings may be similarly characterised. Call A a left (resp. right) DI-ring if every divisible left (resp. right) A-module is injective.

Lemma 2.5. *If A is a left* DI-*ring, then A is a left hereditary left Noetherian ring.*

Proof. We note from the proof of Theorem 2.4 that rings whose disisible singular left modules are injective are left hereditary. Since a direct sum of left A-modules is p-injective iff each direct summand is p-injective, then Lemma 2.1 implies that any direct sum of injective left A-modules is injective and by a well-known theorem ([**4**]$_2$, p. 111), A is left Noetherian.

Applying Small's theorem [**10**], we get

Proposition 2.6. *A left* DI-*ring has a classical left quotient ring which is left hereditary left Artinian.*

Applying Chatter's theorem [**2**], we get

Proposition 2.7. *If A is a left and right* DI-*ring, then A is a finite direct sum of rings each of which is either hereditary Artinian or prime hereditary Noetherian.*

Applying ([**8**], theorem 4.3), we get

Proposition 2.8. *If A is a semi-prime left* DI-*ring, then A is a finite direct sum of prime left hereditary left Noetherian rings.*

Proposition 2.9. *The following conditions are equivalent for a semi-prime ring A:*

(1) *A is hereditary Noetherian (both left and right);*
(2) *A is a left and right* DI-*ring.*

Proof. Apply ([**8**], theorem 3.4) to Lemma 2.5.

Since a commutative ring is semi-prime iff it is nonsingular, Proposition 2.9 yields the following

Corollary 2.10. *A commutative ring is hereditary Noetherian iff it is* DI.

For any left ideal I, the closure of I in A is $Cl(I) = \{b \in A / Lb \subseteq I$ for some essential left ideal L of $A\}$. $I + Z$ is always essential in $Cl(I)$ [**13**]$_3$ and if $Z = 0$, then $Cl(I)$ is a complement left ideal of A. Obviously, I an ideal of A implies $Cl(I)$ an ideal of A.

Proposition 2.11. *Let A be an indecomposable left* CPP *left* CS-*ring. Then A is either primitive left self-injective regular or a simple left Ore domain.*

Proof. By Proposition 1.4, A is either regular or a simple domain. If A is a simple left CS-domain, any non-zero complement left ideal coincides with A which implies A is a left Ore domain. Now suppose A is regular. Then A is left continuous [**12**] and for any non-zero ideal T of A, $Cl(T)$ is a direct summand of ${}_AA$. Then $A = Cl(T) \oplus K$, where $Cl(T) = Ae$, $e = e^2 \in A$ and $K = A(1-e)$ is also an ideal of A (cfr. the proof of Theorem 1.5). Since A is indecomposable, then $K = 0$ and ${}_AT$ is essential in ${}_AA$. Therefore $T_1T_2 \neq 0$ for any non-zero ideals T_1, T_2 of A which implies A prime. Then A is a prime left self-injective regular ring from ([**12**], p. 604) and is therefore primitive by a theorem of Goodearl ([**6**]$_2$, p. 181).

A well-known theorem of Eisenbud-Griffith-Robson ([**4**]$_2$, p. 244) states that if A is a hereditary Noetherian prime ring, then A/I is an Artinian serial ring for every non-zero ideal I.

Theorem 2.12. *Let A be a semi-prime indecomposable left and right* DI-*ring. Then A/I is an Artinian serial ring for every non-zero ideal I of A.*

Proof. By Lemma 2.5, A is a (left and right) hereditary Noetherian ring. Since A is semi-prime, then A has a two-sided classical quotient ring Q which is semi-simple Artinian. Therefore Q is the regular maximal left and right quotient ring of A. Since A is left and right non-singular, then every complement left (resp. right) ideal is a left (resp. right) annihilator ([**6**]$_2$, Theorem 2.38). Then A being hereditary Noetherian implies that A is a left and right CS-ring ([**4**]$_2$, Lemma 20.27). By ([**3**], theorem 6.14), A is either Artinian or prime. If A is Artinian, then by Proposition 2.11, A is prime. Thus A is a prime hereditary Noetherian ring and by ([**4**]$_2$, theorem 25.5.1), A/I is an Artinian serial ring for every non-zero ideal I of A.

We conclude this section with a characteristic property of left Ore domains in terms of indecomposable left CS-rings.

Theorem 2.13. *The following conditions are equivalent:*

(1) *A is a left Ore domain;*

(2) *A is an indecomposable left* CS-*ring with a reduced essential left ideal.*

Proof. (1) implies (2) obviously.

Assume (2). Since A contains a reduced essential left ideal E, then A is semi-prime. Now suppose that $Z \neq 0$. Then $Z \cap E$ is essential in ${}_AZ$. If $0 \neq z \in Z$, there exists $b \in A$ such that $0 \neq bz \in Z \cap E$. Then there exists $c \in A$ such that $0 \neq cbz \in l(z)$. Since $cbz \in Z \cap E$, $(zcbz)^2 = 0$ implies $zcbz = 0$ and $(cbz)^2 = 0$ implies $cbz = 0$, which is a contradiction. This proves $Z = 0$. Since A is semi-prime, the proof of Proposition 2.11 shows that any non-zero ideal of A is an essential left ideal of A which implies A prime. Suppose there exist non-zero $s, t \in A$ such that $st = 0$. Then $0 \neq us \in E$ for some $u \in A$ and since A is prime, $taus \neq 0$ for some $a \in A$. But $(taus)^2 = 0$ and since $taus \in E$, then $taus = 0$, a contradiction. Thus A is an integral domain and since A is left CS, then A is a left Ore domain which proves that (2) implies (1).

3 - PLD rings

In this section, we consider a class of rings which generalise both left duo rings and semi-simple Artinian rings.

Definition. A is called a PLD (pseudo left duo) ring if for any essential left ideal E of A different from A, every left subideal is an ideal of E.

Begore characterizing semi-simple and simple Artinian rings, let us mention, without proof, a useful Lemma.

Lemma 3.1. *If A is a prime* PLD *ring, then A is either simple Artinian or a left Ore domain.*

It is known that: (1) (Goodearl) simple left and right self-injective rings need not be Artinian ($[4]_2$, p. 104); (2) prime left Noetherian, left hereditary, left V-rings need not be Artinian ($[4]_2$, p. 175). We here give a few nice characteristic properties of simple Artinian rings in terms of regular rings and V-rings. Q will denote the regular maximal left quotient ring of A whenever $Z = 0$.

Theorem 3.2. *The following conditions are equivalent:*

(1) *A is simple Artinian;*

(2) *A is a prime* PLD *regular ring;*

(3) *A is a prime* PLD *left f-injective ring;*

(4) *A is a prime regular ring such that every essential left ideal of Q is an ideal of Q;*

(5) *A is a prime regular ring such that Q is* PLD;

(6) *A is a prime* PLD *left V-ring;*

(7) *A is a prime* PLD *ring with an injective simple left module.*

Proof. (1) implies (2) through (7) obviously.

Since any left module over a regular ring is f-injective, (2) implies (3).

Assume (3). Then A left f-injective implies that every principal right ideal is a right annihilator ([7], theorem 1). If A is an integral domain, then A is a division ring. Thus (3) implies (1) by Lemma 3.1.

Since a well-known theorem of Jain, Mohamed and Singh states that prime left self-injective rings whose essential left ideals are ideals are simple Artinian, then (4) implies (5).

Assume (5). Since Q is a prime left self-injective regular ring, then Q is simple Artinian by (2) or (3). A theorem of Sandomierski then implies that A satisfies the maximum condition on complement left ideals ($[4]_2$, p. 83). Since A is prime regular, then (5) implies (1).

Assume (6). Suppose A is not simple Artinian. Then by Lemma 3.1, A is left Ore domain. If L is a proper essential left ideal of A, then L contains a maximal left subideal M (since A is a left V-ring). Since L cannot be a minimal left ideal of A, then $M \neq 0$ and if $0 \neq b \in M$, define left A-homomorphism $f: Lb \to L/M$ by $f(ab) = a + M$ for all $a \in L$. Since L/M is injective, there exists $c \in L$ such that $f(ab) = abc + M$ for all $a \in L$ which implies $a - abc \in M$. Then A PLD implies $abc \in M$ and hence $L \subseteq M$, a contradiction. Thus A contains no proper essential left ideal and since A is a left Ore domain, then A is a division ring. This contradiction proves that (6) implies (1).

Assume (7). Suppose A is not simple Artinian. Then A is a left Ore domain (Lemma 3.1) and A is not a left V-ring by (6). Let U ($\approx A/M$) be an injective simple left A-module, where M is a maximal left ideal of A. Then M is essential in ${}_AA$. Suppose that every proper essential left ideal of A is contained in M. Then any maximal left ideal, being essential, coincides with M which implies that every simple left A-module is isomorphic to A/M. This contradicts the hypothesis that A is not a left V-ring. Thus there exists a proper essential left ideal L which is not contained in M. If $g: L \to A/M$ is the left A-homomorphism defined by $g(b) = b + M$ for all $b \in L$, then $L/K \approx A/M$, where $K = \ker g$ is a maximal left subideal of L. If $0 \neq c \in K$, define a left A-homomorphism $f: Lc \to L/K$ by $f(ac) = a + K$ for all $a \in L$. Since K is an ideal of L, this leads to a contradiction as in the proof of « (5) implies (1) ». Thus (7) implies (1).

Finally, ([**4**]$_1$, theorem 14), ([**4**]$_2$, lemma 20.27 and Ex. 14 (p. 24)), ([**7**], theorem 1) and Theorem 2.3 (6) yield

Theorem 3.3. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian;*
(2) *A is a* PLD *left* PCI *ring;*
(3) *A is a left p-injective left* DI-*ring;*
(4) *A is a right p-injective left* DI-*ring.*

In a recent paper (Hiroshima Math. J. **9** (1979), 137-149), Hirano and Tominaga extend our results in [**13**]$_2$ to s-unital rings and prove the following theorem concerning CPP rings. If A is an s-unital right CPP ring which is not regular, then $A = S \oplus T$, where S is a right (and left) completely reducible semi-prime ring and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.

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