Riv. Mat. Univ. Parma (4) 7 (1981), 217-223

LUIGI MANGIAROTTI (*)

Mechanical systems and holonomic constraints (**)

1 - Introduction

In this paper we study some geometric properties of dynamics of mechanical systems with (smooth) holonomic constraints. The framework is that of Classical Analytical Mechanics. The novelty of the treatment is the absolute approach of the theory, i.e. independent of any frame of reference. The need of a frame, as is well known, is required for operative physical reasons. For this, in Section **2** we introduce the frame configuration space in order to clarify how the traditional approach is related to our model (for example, we can then speak of a time independent constraint with respect to the frame, etc.).

In Section **2** also the concept of mechanical system is considered. The two other sections deal with the canonical connection and with the canonical force of constraint associated to an holonomic system, respectively. Let us emphasize again that we are able to characterize the force of constraint independently of any frame. This research is connected to other my works on the subject $[\mathbf{2}]_{1,2}$.

2 - Mechanical systems

All manifold, bundles and tensor fields will be C^∞. The notation is the standard one used in modern differential geometry. A *mechanical system* is a quadruplet $\mathcal{M}=(M, t, g, \nabla)$ where M is a differentiable manifold (of dimension $m+1$, $m\geqslant 1$), $t\colon M\to R$ is a surjective submersion, g is a (positive definite)

(*) Indirizzo: Istituto di Matematica, Università, Via V. Venanzi, 62032 Camerino (Macerata), Italy.

(**) Ricevuto: 26-X-1979.

Riemannian metric in $\Sigma_M = \operatorname{Ker} dt \subset TM$ and ∇ is a symmetric (linear) connection on M compatible with both t and g (i.e. such that $\nabla dt = 0$ and $\nabla g = 0$). Here TM denotes (as usual) the tangent space to M. The manifold M is the *event space*, the projection t is the *absolute time*, the fibers Σ_t of M are the *spaces of absolute symultaneity* and $\Sigma_M \to M$ is the bundle of the *spacelike vectors on* M [**2**]$_1$. This model of a mechanical system generalizes the well known one of the free-particle. In this case, M is an affine four-dimensional space and g is an euclidean metric (in which is emboded the mass of the particle).

In the sequel we shall always use charts (V, y^α) in M *adapted to the absolute time* t, i.e. such that $y^0 = t|V$ (while Greek indices run from 0 to m, Latin indices are used for spatial coordinates only). If $\Gamma^\gamma_{\alpha\beta}$ are the connection parameters for ∇ determined by the chart (V, y^α), locally $\nabla dt = 0$ is equivalent to $\Gamma^0_{\alpha\beta} = 0$, while $\nabla g = 0$ is equivalent to

$$(1) \qquad 2\Gamma_{k,ij} = \frac{\partial g_{ki}}{\partial y^j} + \frac{\partial g_{kj}}{\partial y^i} - \frac{\partial g_{ij}}{\partial y^k}, \qquad \frac{\partial g_{ij}}{\partial y^0} = \Gamma_{i,0j} + \Gamma_{j,0i},$$

where g_{ij} are the components of g and $\Gamma_{k,\alpha\beta} = g_{kh}\Gamma^h_{\alpha\beta}$. Note that the restrictions $\Gamma^k_{ij}|V_t$ are the Christoffel symbols of the Levi-Civita connections ∇_t of the metric $g|\Sigma_t$, where $V_t = \Sigma_t \cap V$.

An *holonomic constraint* for the mechanical system $\mathscr{M}$ is an *absolute time submanifold* $N \subset M$, i.e. a submanifold N such that also the restriction $t|N$ is a submersion (let $n+1$ be the dimension of N, $1 \leqslant n < m$). Note that the fibers of N are n-dimensional submanifolds of the corresponding hypersurface Σ_t in M. We have $\Sigma_N \subset \Sigma_M$ and $K_N \subset K_M$, where $K_M \subset TM$ is the *absolute kinematic space* of the mechanical system $\mathscr{M}$, that is the hypersurface of TM characterized by $dt|K_M = 1$.

A *frame of reference* for the mechanical system $\mathscr{M}$, typically denoted by F, is a section of the kinematic space K_M (over M) [**2**]$_1$. Suppose (for the sake of brevity) that F is a complete vector field and let Q_F be the set of the orbits of F. Let $p_F\colon M \to Q_F$ be the canonical projection. Then there exists a unique differentiable manifold structure on Q_F such that p_F becomes a submersion [**1**]. The manifold Q_F (of dimension m) is the *frame configuration space* (of the mechanical system $\mathscr{M}$) canonically associated to the frame F. Note that we have the diffeomorphism

$$(2) \qquad (t, p_F)\colon M \to R \times Q_F,$$

from which we get the other

$$(3) \qquad (t, Tp_F|K_M)\colon K_M \to R \times TQ_F,$$

where $Tp_F\colon TM \to TQ_F$ is the tangent map of p_F [1]. The diffeomorphisms (2) and (3) induce on the absolute space M and K_M a (frame dependent) cartesian product structure. For example, by using charts (V, y^α) on M *adapted* to F, i.e. such that $F|V = \partial/\partial y^0$, we easily see that we get charts $(p_F(V), y^i)$ on Q_F where $y^i \circ p_F = y^i$. These are the usual lagrangian coordinates in configuration space.

We give an example of how this definition of frame is to be handled. The holonomic constraint $N \subset M$ is said to be *time independent with respect to* F iff the vector field F on M is tangent to the submanifold N (and $F|N$ is complete). Then it follows that the quotient space associated to $F|N$ is an n-dimensional submanifold of Q_F. This corresponds to the standard way by which time independent holonomic constraints are introduced (with respect to a fixed frame). By following this trace, the reader may pursue by himself to see how the traditional approach is related to the our absolute one.

3 - Canonical connection of an holonomic constraint

Let $N \subset M$ be an holonomic constraint for the mechanical system $\mathscr{M}$. Let h be the (positive definite) Riemannian metric in Σ_N induced by g and let $\nu_N \to N$ be the normal bundle of the restriction $\Sigma_M|N$ (note that $\Sigma_N \subset \Sigma_M|N$). It is easy to see that we have the canonical splitting

$$(4) \qquad TM|N = TN \oplus \nu_N\,, \qquad \Pi\colon TM|N \to TN\,, \qquad TN \subset TM|N\,,$$

where we have introduced the projection Π over TN. This canonical splitting is the basic fact that allows us an absolute treatment of the holonomic constraints (generalizing a procedure well known in the study of submanifolds of a Riemannian manifold [3]). Working on the restriction $TM|N$, we shall continue to denote by the same symbol ∇ also the connection induced on $TM|N$ by the original one ∇ on M.

In the sequel, we shall always use charts (U, x^α) on N and charts (V, y^α) on M such that

$$(5) \quad U \subset V\,, \qquad y^\alpha|U = x^\alpha \quad \text{if } 0 \leqslant \alpha \leqslant n\,, \qquad y^\alpha|U = 0 \quad \text{if } n+1 \leqslant \alpha \leqslant m\,.$$

Let $(p_N^{-1}(U), q^\alpha, \dot q^i)$ and $(p_M^{-1}(V), \eta^\alpha, \dot\eta^i)$ be the induced charts on K_N and K_M respectively (here $p_N\colon K_N \to N$ and $p_M\colon K_M \to M$ are the canonical projections). Then we have

$$(6) \quad \dot\eta^i|p_N^{-1}(U) = \dot q^i \quad \text{if } 1 \leqslant i \leqslant n\,, \qquad \dot\eta^i|p_N^{-1}(U) = 0 \quad \text{if } n+1 \leqslant i \leqslant m\,.$$

By using these charts, the components h_{ij} of h are given by

(7) $$h_{ij} = g_{ij}|U\,, \qquad 1 \leqslant i, j \leqslant n\,.$$

The local expression of Π is

(8) $$\Pi\Big(\frac{\partial}{\partial y^\alpha}\Big|_v\Big) = \frac{\partial}{\partial x^\alpha}\,, \qquad \Pi\Big(\frac{\partial}{\partial y^k}\Big|_v\Big) = h^{ij}(g_{jk}\,|U)\frac{\partial}{\partial x^i}\,,$$

where $0 \leqslant \alpha \leqslant n$, $n+1 \leqslant k \leqslant m$.

The holonomic constraint N has a canonical structure of mechanical system as the following theorem shows.

Theorem 1. *Let X and Y be vector fields on N and put*

(9) $$D_X Y = \Pi \circ \nabla_X Y\,.$$

Then D is a symmetric (linear) connection on N compatible with both t and h.

Proof. Indeed it is clear that (9) defines a symmetric (linear) connection on N. Let $\binom{\gamma}{\alpha\ \beta}$ be the connection parameters for D (by using charts as in (5)). Then from (9), by using (8), we get that $\binom{0}{\alpha\ \beta} = 0$ and also

(10) $$\binom{k}{\alpha\ \beta} = \Gamma^k_{\alpha\beta}|U + h^{kj}(g_{jh}\Gamma^h_{\alpha\beta}|U)\,, \qquad 0 \leqslant \alpha,\ \beta \leqslant n,\ 1 \leqslant k \leqslant n\,,$$

where the sum over the indec h is from $n+1$ to m. From (10) it follows that

(11) $$(i, \alpha\beta) = h_{ij}\binom{j}{\alpha\ \beta} = \Gamma_{i,\alpha\beta}|U\,, \qquad 0 \leqslant \alpha,\ \beta \leqslant n,\ 1 \leqslant i \leqslant n\,.$$

From (1), (7) and (11) we see that also $Dh = 0$ is satisfied.

In the sequel, we shall denote by $\mathscr{N}$ the mechanical system (N, t, h, D). A *force* acting on $\mathscr{M}$, typically denoted by f, is a map $f\colon K_M \to \Sigma_M$ compatible with the projection $p_M\colon K_M \to M$. From the restriction $f|K_N\colon K_N \to \Sigma_M|N$, by using the projection Π, we get a force, denoted by f_Π, acting on $\mathscr{N}$, i.e. a map $f\colon K_N \to \Sigma_N$ compatible with the projection $p_N\colon K_N \to N$.

A *dynamics equation* for $\mathscr{M}$ is a second order equation on M, typically denoted by X, compatible with the absolute time t, i.e. a vector field X on K_M such that $Tp_M \circ X = j_{K_M}$, where j_{K_M} is the canonical injection $K_M \subset TM$ [**2**]$_1$. If X is a dynamics equation, there exists a unique force f such that

$$X = X_f = \xi_M + f\,, \tag{12}$$

where ξ_M is the (restriction to K_M of the) spray of the connection ∇ [**1**], [**2**]$_1$. Locally we have

$$\xi_M = \frac{\partial}{\partial \eta^0} + \dot\eta^i \frac{\partial}{\partial \eta^i} - (\Gamma^k_{00} + 2\Gamma^k_{0j}\dot\eta^j + \Gamma^k_{ij}\,\dot\eta^i\,\dot\eta^j)\,\frac{\partial}{\partial \dot\eta^k}\,. \tag{13}$$

It follows at once that there is a canonical way to restrict a dynamics equations $X = X_f$ to a dynamics equation Y for $\mathcal{N}$ by putting

$$Y = Y_{f_\Pi} = \xi_N + f_\Pi\,, \tag{14}$$

where ξ_N is the spray of the connection D. The principle of d'Alembert asserts that if N is smooth and f is force acting on $\mathcal{M}$, then the only motions dynamically admissible for $\mathcal{N}$ are exactly the solutions of the equation Y_{f_Π} [**2**]$_2$.

4 - Canonical force of a constraint

The *canonical force of the constraint* $N \subset M$ is the map

$$r = \xi_N - \xi_M\,|\,K_N\colon K_N \to TK_M\,|\,K_N\,. \tag{15}$$

From (10) and (13) it follows easily that the local expression of r is

$$r = (\Gamma^k_{\alpha\beta}\,|\,U)\,\dot q^\alpha\,\dot q^\beta\,\nu_k\,, \tag{16}$$

where we have put $\dot q^0 = 1$ and where

$$\nu_k = \frac{\partial}{\partial y^k}\,|_U - h^{ij}(g_{jk}\,|\,U)\,\frac{\partial}{\partial x^i}\,, \quad n+1 \leqslant k \leqslant m\,, \tag{17}$$

gives a local basis for the normal fields on N (sections of $\nu_N \to N$). Note that the vertical bundle $\operatorname{Ker} Tp_M \subset TK_M$ (over K_M) is canonically isomorphic to the pull-back of Σ_M over K_M by means of the projection $p_M\colon K_M \to M$. Hence r is a quadratic map $r\colon K_N \to \nu_N$ compatible with the projection $p_N\colon K_N \to N$.

Clearly, if $\Pi: TM|N \to \nu_N$ is the projection over ν_N associated to the splitting (4), we have

$$\Pi\Big(\frac{\partial}{\partial y^\alpha}\Big|_v\Big) = 0 \quad \text{if } 0 \leqslant \alpha \leqslant n\,, \qquad \Pi\Big(\frac{\partial}{\partial y^k}\Big|_v\Big) = \nu_k \quad \text{if } n+1 \leqslant k \leqslant m\,. \tag{18}$$

Let $I \subset R$ be an open interval and let $\gamma: I \to N$ be a motion of $\mathscr{N}$ (we must have $t\circ\gamma = \mathrm{j}_I$ the canonical injection $I \subset R$). Let c be the corresponding motion of $\mathscr{M}$. Then we have

$$a_c = a_\gamma + r\circ\dot\gamma\,, \tag{19}$$

where a_c and a_γ are the accelerations of c and γ defined by means of the connections ∇ and D, respectively (clearly $\dot\gamma: I \to TN$ takes its values in $K_N \subset TN$). Indeed, (19) follows at once from the splittings

$$\ddot c = a_c + \xi_M\circ\dot c\,, \qquad \ddot\gamma = a_\gamma + \xi_N\circ\dot\gamma\,. \tag{20}$$

If f is a force acting on $\mathscr{M}$, the *force of the (smooth) constraint* $N \subset M$ is the map

$$r_f = r - f_\Pi: K_N \to \nu_N\,, \tag{21}$$

compatible with the projection p_N. Here f_Π is the map $K_N \to \nu_N$ (compatible with p_N) that we get from the restriction $f|K_N: K_N \to \Sigma_M|N$, by using the projection Π.

Theorem 2. *The force of constraint r_f is the unique map $K_N \to \nu_N$, compatible with p_N, such that we have*

$$a_c = (f|K_N + r_f)\circ\dot\gamma\,, \tag{22}$$

for any motion γ dynamically admissible for $\mathscr{N}$.

Proof. It follows at once from (19) and (21) since γ is dynamically admissible for $\mathscr{N}$ iff we have $a_\gamma = f_\Pi\circ\dot\gamma$.

Note that the constraint forces r and r_f (as well as f) are known functions of the coordinates $(t, q^\alpha, \dot q^i)$ on the (absolute) kinematic space K_N. Clearly, if we want to know these forces as functions of the time, we must integrate the equation $Y_{f\Pi}$.

It is interesting to note also that the fibers of N are auto-parallel submanifolds of (Σ_t, ∇_t) for any $t \in R$ iff r is an affine morphism over N (K_N is an affine bundle over N whose associated vector bundle is Σ_N). On the other hand, N itself is an auto-parallel submanifold of (M, ∇) iff we have $r = 0$.

References

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