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Some generalizations of Laguerre polynomials (III) (**)

Introduction

In the first paper of this series of papers [1]$_2$ we studied the function ${}_n\Phi^m$ defined as

(1) $${}_n\Phi^m \equiv {}_n\Phi^m(b_1, \ldots, b_n; c; x_1, \ldots, x_n)$$
$$= \sum_{s_1,\ldots,s_n}^{0,\ldots,\infty} \frac{(b_1)_{s_1} \ldots (b_n)_{s_n}}{(c)_{ms_1+\ldots+ms_n}} \cdot \frac{x_1^{s_1}}{s_1!} \ldots \frac{x_n^{s_n}}{s_n!},$$

which is a generalization of Erdélyi's [3] function ${}_n\Phi$; incidentally, the particular case obtained by taking $m = 1$ in (1) has recently been studied by Exton [4]. In the second paper [1]$_3$ we took $b_1 = -m_1, \ldots, b_n = -m_n$, where $m_1, \ldots, m_n$ are positive integers in which case ${}_n\Phi^m$ reduces to polynomials; we found some interesting properties of this generalization of the Laguerre polynomials defined for $\mu = m$ a positive integer by

(2) $$\mathfrak{L}^{\alpha,\mu}_{m_1,\ldots,m_n}(x_1, \ldots, x_n)$$
$$= \frac{(\alpha+1)_{m_1+\ldots+m_n}}{m_1! \ldots m_n!} \, {}_n\Phi^m(-m_1, \ldots, -m_n; \alpha+1; x_1, \ldots, x_n),$$

and which obviously is an extension of the generalized Laguerre polynomials in n arguments studied by Erdélyi [3]. For general $\mu > 0$ we could also define

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our polynomials by the operational image

$$(3) \qquad t^{\alpha} \mathfrak{L}^{\alpha,\mu}_{m_1,\dots,m_n}(x_1 t^{\mu}, \dots, x_n t^{\mu})$$

$$\supset \frac{\Gamma(\alpha + m_1 + \dots + m_n + 1)}{\Gamma(m_1+1)\dots\Gamma(m_n+1)} \cdot \frac{1}{p^{\alpha}} \left(1 - \frac{x_1}{p^{\mu}}\right)^{m_1} \dots \left(1 - \frac{x_n}{p^{\mu}}\right)^{n_n},$$

where $f(t) \supset \Phi(p)$ and $\Phi(p) = p\int_0^{\infty} e^{-pt} f(t)\,dt$.

In this third paper of the series we generalize the operational image of Laguerre polynomials obtaining a different generalization of these polynomials and study its properties by techniques similar to that used by the author [**1**]$_1$ in 1954. We are also able to connect this generalization of Laguerre polynomials with that of the Bessel-Maitland function [**1**]$_1$ and to extend it to 2 variables; generalization of our class of polynomials to n variables can easily be done.

1 - Definition of a class of polynomials $\mathscr{L}^{\alpha_1,\dots\alpha_n}_{m;\mu_1,\dots\mu_n}(x)$, m a positive integer

Humbert [**5**] studied a generalization of the classical Laguerre polynomials to two variables by means of their operational image in 2 variables, viz.

$$(1.1) \qquad \mathfrak{L}_m(x, y) \Supset \left(1 - \frac{1}{p} - \frac{1}{q}\right)^m .$$

Delerue [**2**] investigated a class of polynomials in n variables given by their image in n variables, viz.

$$(1.2) \qquad x_1^{\alpha_1} \dots x_n^{\alpha_n} \mathfrak{L}^{\alpha_1 \dots \alpha_n}_m (x_1, \dots, x_n)$$

$$\supset_n \frac{\Gamma(m + \alpha_1 + 1) \dots \Gamma(m + \alpha_n + 1)}{(m!)^n} \times \frac{1}{p_1^{\alpha_1} \dots p_n^{\alpha_n}} \left(1 - \frac{1}{p_1} - \dots - \frac{1}{p_n}\right)^m .$$

Later, Srivastava [**7**] studied yet another generalization of the Laguerre polynomials which we have already generalized to n variables [**1**]$_3$. Srivastava has taken as his starting point the operational relation

$$(1.3) \qquad x^{\alpha} \mathscr{L}^{\alpha}_{m,\mu}(x^{\mu}) \supset \frac{\Gamma(m\mu + \alpha + 1)}{\Gamma(m\mu + 1)} \cdot \frac{1}{p^{\alpha}} \left(1 - \frac{1}{p^{\mu}}\right)^m .$$

We propose in this paper to consider a class of polynomials $\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\mu_1,\dots,\mu_n}(x)$ (m being a positive integer) such that

$$x_1^{\alpha_1}\dots x_n^{\alpha_n}\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\mu_1,\dots,\mu_n}(x_1^{\mu_1}\dots x_n^{\mu_n}) \supset_n \prod_{r=1}^{n}\frac{\Gamma(m\mu_r+\alpha_r+1)}{\Gamma(m\mu_r+1)}\cdot\frac{1}{p_1^{\alpha_1}\dots p_n^{\alpha_n}}\left(1-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}\right)^m. \tag{1.4}$$

Here $\mu_1, \dots, \mu_n$ are all >0 and $\alpha_1, \dots, \alpha_n > -1$. On interpretation, we get

$$\prod_{r=1}^{n}\frac{\Gamma(m\mu_r+1)}{\Gamma(m\mu_r+\alpha_r+1)}\cdot\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\mu_1,\dots,\mu_n}(x)=\sum_{s=0}^{m}\binom{m}{s}(-x)^s\prod_{r=1}^{n}\frac{1}{\Gamma(s\mu_r+\alpha_r+1)}. \tag{1.5}$$

2 - Recurrence relations and differential equation satisfied by $\mathscr{L}^{\alpha_1,\dots\alpha_n}_{m;\mu_1,\dots\mu_n}(x)$

Applying the symbolic relation [**6**], that if $f(x_1, \dots, x_n) \supset_n \Phi(p_1, \dots, p_n)$, then $x_1(\partial f/\partial x_1) \supset_n -p_1(\partial\Phi/\partial p_1)$, to (1.4), we easily have the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\mu_1,\dots\mu_n}(x)=-m\prod_{r=1}^{n}\frac{\Gamma(m\mu_r-\mu_r+1)}{\Gamma(m\mu_r+1)}\mathscr{L}^{\alpha_1+\mu_1,\dots,\alpha_n+\mu_n}_{m-1;\mu_1,\dots,\mu_n}(x). \tag{2.1}$$

Again, we know that [**6**], if

$$f(x_1, \dots, x_n)\supset_n \Phi(p_1, \dots, p_n) \quad\text{then}\quad -x_1 f\supset_n -p_1\frac{\partial}{\partial p_1}\left(\frac{\Phi}{p_1}\right).$$

This gives, on using (1.5)

$$\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\mu_1,\dots,\mu_n}(x)-\frac{\alpha_1+1}{m\mu_1+\alpha_1+1}\mathscr{L}^{\alpha_1+1,\alpha_2,\dots,\alpha_n}_{m;\mu_1,\dots,\mu_n}(x) \tag{2.2}$$

$$=-\frac{\mu_1 mx}{m\mu_1+\alpha_1+1}\prod_{r=1}^{n}\frac{\Gamma(m\mu_r-\mu_r+1)}{\Gamma(m\mu_r+1)}\cdot\mathscr{L}^{\alpha_1+\mu_1+1,\alpha_2+\mu_2,\dots,\alpha_n+\mu_n}_{m-1;\mu_1,\dots,\mu_n}(x).$$

Again, we have

$$\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}\left(1-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}\right)^{m-1}=\left(1-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_1}}\right)^{m-1}-\left(1-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}\right)^{m},$$

which gives the recurrence relation

$$(2.3)\qquad x\mathscr{L}^{\alpha_1+\mu_1,\dots,\alpha_n+\mu_n}_{m-1;\,\mu_1,\dots,\mu_n}(x)+\prod_{r=1}^{n}\frac{\Gamma(m\mu_r+1)}{\Gamma(m\mu_r-\mu_r+1)}\cdot\mathfrak{L}^{\alpha_1,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}(x)$$

$$=\prod_{r=1}^{n}\frac{\Gamma(m\mu_r+\alpha_r+1)}{\Gamma(m\mu_r-\mu_r+\alpha_r+1)}\cdot\mathfrak{L}^{\alpha_1,\dots,\alpha_n}_{m-1;\,\mu_1,\dots,\mu_n}\,x.$$

Relation (2.2) and (2.3) give yet another recurrence relation, which is interesting since it contains constant coefficients, viz.

$$(2.4)\qquad \mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}(x)-\mathscr{L}^{\alpha_1-1,\alpha_2,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}(x)$$

$$=\frac{m\mu_1}{m\mu_1+\alpha_1}\prod_{r=1}^{n}\frac{\Gamma(m\mu_r+\alpha_r+1)\,\Gamma(m\mu_r-\mu_r+1)}{\Gamma(m\mu_r+1)\,\Gamma(m\mu_r-\mu_r+\alpha_r+1)}\cdot\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m-1;\,\mu_1,\dots,\mu_n}(x)\,.$$

In order to obtain the differential equation satisfied by the polynomials $y=\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}(x)$ we will use the method developed by the author $[\mathbf{1}]_4$. (2.1) and (2.2) give

$$(2.5)\qquad (\mu_1 x\mathrm{D}+\alpha_1)y=(m\mu_1+\alpha_1)\,\mathscr{L}^{\alpha_1-1,\alpha_2,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}(x)\,,\qquad\text{where}\qquad \left(\mathrm{D}\equiv\frac{\partial}{\partial x}\right).$$

Now let us put (for μ_r positive and integral)

$$F^{\mu_r}_{\alpha_r}(\mathrm{D})\equiv(\mu_r x\mathrm{D}+\alpha_r-\mu_r+1)\,\dots\,(\mu_r x\mathrm{D}+\alpha_r-1)(\mu_r x\mathrm{D}+\alpha_r)\,.$$

Then, we have

$$(2.6)\qquad \{F^{\mu_n}_{\alpha_n}(\mathrm{D})\,\dots\,F^{\mu_1}_{\alpha_1}(\mathrm{D})\}\,y=\prod_{r=1}^{n}\frac{\Gamma(m\mu_r+\alpha_r+1)}{\Gamma(m\mu_r+\alpha_r-\mu_r+1)}\cdot\mathscr{L}^{\alpha_1-\mu_1,\dots,\alpha_n-\mu_n}_{m;\,\mu_1,\dots,\mu_n}(x)\,.$$

Differentiating this once more, we have

$$a_{\sigma_n+1}x^{\sigma_n+1}y^{(\sigma_n+1)}+a_{\sigma_n}x^{\sigma_n}y^{(\sigma_n)}+\dots+a_1xy'$$

$$=-\,mx\prod_{r=1}^{n}\frac{\Gamma(m\mu_r-\mu_r+1)\,\Gamma(m\mu_r+\alpha_r+1)}{\Gamma(m\mu_r+1)\,\Gamma(m\mu_r-\mu_r+\alpha_r+1)}\,\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m-1;\,\mu_1,\dots,\mu_n}(x)\,,$$

where $\sigma_n=\mu_1+\dots+\mu_n$ and $a_{\sigma_n+1},\dots,a_2,a_1$ are functions of $\alpha_1,\dots,\alpha_n$ and

$\mu_1, \ldots, \mu_n$ and can easily be calculated. If we now use (2.2) and (2.3) we get

$$a_{\sigma_n+1} x^{\sigma_n+1} y^{(\sigma_n+1)} + \ldots + a_1 x y' + m x y = x^2 y', \tag{2.7}$$

which obviously is the differential equation satisfied by our class of polynomials $\mathscr{L}^{\alpha_1,\ldots,\alpha_n}_{m;\mu_1,\ldots,\mu_n}(x)$. We notice that this is of the order $1 + \mu_1 + \ldots + \mu_n (\equiv 1 + \sigma_n)$.

3 - A generating function and some integral representations

In 1954 the author [1]₁ studied the function $J^{\mu_1,\ldots,\mu_n}_{\alpha_1,\ldots,\alpha_n}(x)$ given by its image as

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n} J^{\mu_1,\ldots,\mu_n}_{\alpha_1,\ldots,\alpha_n}(z x_1^{\mu_1} \ldots x_n^{\mu_n}) \supset_n p_1^{-\alpha_1} \ldots p_n^{-\alpha_n} \exp\left(-\frac{z}{p_1^{\mu_1} \ldots p_n^{\mu_n}}\right). \tag{3.1}$$

From this we easily get

$$\sum_{m=0}^{\infty} \frac{c(m, n)}{m!} \mathscr{L}^{\alpha_1,\ldots,\alpha_n}_{m;\mu_1,\ldots,\mu_n}(x) z^m = e^z J^{\mu_1,\ldots,\mu_n}_{\alpha_1,\ldots,\alpha_n}(xz), \tag{3.2}$$

where

$$c(m, n) = \prod_{r=1}^{n} \frac{\Gamma(m\mu_r + 1)}{\Gamma(m\mu_r + \alpha_r + 1)}.$$

It is interesting to observe that the generalized Bessel-Maitland function $J^{\mu_1,\ldots,\mu_n}_{\alpha_1,\ldots,\alpha_n}(x)$ plays the same role with respect to the polynomials $\mathscr{L}^{\alpha_1,\ldots,\alpha_n}_{m;\mu_1,\ldots,\mu_n}(x)$, which the Bessel function does in relation to the classical Laguerre polynomials $\mathfrak{L}^{(\alpha)}_m(x)$.

To get an integral representation let

$$g(x_1, \ldots, x_n) = e^{-x_1-\ldots-x_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \cdot \sum_{r=0}^{m} (-1)^r \binom{m}{r}$$

$$\times \prod_{k=1}^{n} \frac{\Gamma(r\mu_k + 1)\Gamma(m\mu_k + \alpha_k + 1)}{\Gamma(r\mu_k + \alpha_k + 1)\Gamma(m\mu_k + 1)} \mathfrak{L}^{(\alpha_1)}_{\mu_1 r}(x_1) \ldots \mathfrak{L}^{(\alpha_n)}_{\mu_n r}(x_n),$$

where $\mathfrak{L}^{(\alpha)}_{\mu}(x)$ denotes the well-known classical generalized Laguerre polynomials. Then

$$g(x_1, \ldots, x_n) \supset_n \sum_{r=0}^{m} (-1)^r \binom{m}{r} \frac{1}{c(m, n)} \cdot \frac{p_1^{\mu_1 r+1}}{(p_1+1)^{\alpha_1+\mu_1 r+1}} \cdots \frac{p_n^{\mu_n r+1}}{(p_n+1)^{\alpha_n+\mu_n r+1}}. \tag{3.3}$$

Since, we have $[Rl(\alpha) > -1]$

$$(3.4)\qquad e^{-x}x^{\alpha}\,\mathfrak{L}_n^{(\alpha)}(x) \supset \frac{\Gamma(n+\alpha+1)}{n!}\,p^{n+1}(p+1)^{-n-\alpha-1},$$

we get, on using a generalization of Tricomi's formula [**8**] to n variables by Chak [**1**]$_1$, viz. if $\Phi(p_1, p_2, \dots, p_n) \subset_n f(x_1, x_2, \dots, x_n)$, then

$$(3.5)\qquad \frac{1}{p_1^{\lambda_1}\dots p_n^{\lambda_n}}\,\Phi\left(\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}\right)$$

$$\subset_n \int_0^\infty x_1^{\lambda_1+\mu_1}\dots x_n^{\lambda_n+\mu_n}\,J^{\mu_1,\dots,\mu_n}_{\lambda_1+\mu_1,\dots,\lambda_n+\mu_n}(xx_1^{\mu_1}\dots x_n^{\mu_n})f(x)\,\mathrm{d}x\,,$$

the following integral representation of our class of polynomials $\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}(x)$

$$(3.6)\qquad e^{-x_1-\dots-x_n}\,x_1^{\alpha_1/2}\dots x_n^{\alpha_n/2}\,\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}(x_1^{\mu_1}\dots x_n^{\mu_n})$$

$$= \int_0^\infty\dots\int_0^\infty J_{\alpha_1}(2\sqrt{t_1x_1})\dots J_{\alpha_n}(2\sqrt{t_nx_n})\,t_1^{-\alpha_1/2}\dots t_n^{-\alpha_n/2}\,g(t_1,\dots,t_n)\,\mathrm{d}t_1\dots\mathrm{d}t_n\,.$$

For $n=1$, we get the following result of Srivastava [**7**]

$$(3.7)\qquad e^{-x}x^{\alpha/2}\,\mathfrak{L}^{(\alpha)}_{m,\mu}(x^{\mu}) = \int_0^\infty e^{-t}t^{\alpha/2}J_{\alpha}(2\sqrt{xt})\,H(t)\,\mathrm{d}t\,,$$

where $\displaystyle H(t) = \frac{\Gamma(m\mu+\alpha+1)}{\Gamma(m\mu+1)}\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{\Gamma(m\mu_r+1)}{\Gamma(m\mu_r+\alpha_r+1)}\,\mathfrak{L}^{(\alpha)}_{\mu_r}(t)$

and $\mathfrak{L}^{(\alpha)}_{\mu_r}(t)$ is the generalized Laguerre polynomial.

Again, on using the product theorem for n variables [**6**], we get

$$x_1^{\alpha_1}\dots x_n^{\alpha_n}\,\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{2m;\,\mu_1,\dots,\mu_n}(x_1^{\mu_1}\dots x_n^{\mu_n})$$

$$= \prod_{r=1}^{n}\frac{\Gamma(m\mu_r+1)\,\Gamma(2m\mu_r+\alpha_r+1)}{\Gamma(m\mu_r+\alpha_r)\,\Gamma(2m\mu_r+1)}\int_0^{x_1}\dots\int_0^{x_n}\xi_1^{\alpha_1-1}\dots\xi_n^{\alpha_n-1}\,\mathscr{L}^{\alpha_1-1,\dots,\alpha_n-1}_{m;\,\mu_1,\dots,\mu_n}(\xi_1^{\mu_1}\dots\xi_n^{\mu_n})$$

$$\times\,\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\,\mu_1,\dots,\mu_n}\{(x_1-\xi_1)^{\mu_1}\dots(x_n-\xi_n)^{\mu_n}\}\cdot\mathrm{d}\xi_1\dots\mathrm{d}\xi_n\,.$$

4 - Extension of the class of polynomials $\mathscr{L}^{\alpha_1,\dots,\alpha_n}_{m;\mu_1,\dots,\mu_n}(x)$ to 2 variables

It is possible to define and study a generalization of our polynomials to 2 variables on the same lines as Humbert [**5**] in 1936 did for the classical Laguerre polynomials. We shall only obtain a few properties by the methods of symbolic calculus of several variables.

We define the polynomials $\mathscr{L}^{\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_k}_{m;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y)$ by means of their image given by (here m is a positive integer and $\nu_1, \dots, \nu_k$; $\alpha_1, \dots, \alpha_n$; $\beta_1, \dots, \beta_k$; $\mu_1, \dots, \mu_n$ any numbers)

$$(4.1)\qquad x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_k^{\beta_k} \mathscr{L}^{\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_k}_{m;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x_1^{\mu_1} \dots x_n^{\mu_n}, y_1^{\nu_1} \dots y_k^{\nu_k})$$

$$\supset_{n+k} \frac{1}{c(m, n)\, d(m, k)} \cdot \frac{1}{p_1^{\alpha_1} \dots p_n^{\alpha_n}} \cdot \frac{1}{q_1^{\beta_1} \dots q_k^{\beta_n}} \cdot \Big(1 - \frac{1}{p_1^{\mu_1} \dots p_n^{\mu_n}} - \frac{1}{q_1^{\nu_1} \dots q_k^{\nu_k}}\Big)^m ,$$

where
$$d(m, k) = \prod_{r=1}^{k} \frac{\Gamma(m\nu_r + 1)}{\Gamma(m\nu_r + \beta_r + 1)} .$$

Using the methods of **2** we easily get the following recurrence relations:

$$(4.2)\qquad \frac{\partial}{\partial x} \mathscr{L}^{\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_k}_{m;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y) = -\, mA\, \mathscr{L}^{\alpha_1+\mu_1,\dots,\alpha_n+\mu_n;\beta_1,\dots,\beta_k}_{m-1;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y) ,$$

$$(4.3)\qquad \frac{m\mu_1 x}{m\mu_1 + \alpha_1 + 1} A\, \mathscr{L}^{\alpha_1+\mu_1+1,\alpha_2+\mu_2,\dots,\alpha_n+\mu_n;\beta_1,\dots,\beta_k}_{m-1;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y)$$

$$= \frac{\alpha_1 + 1}{m\mu_1 + \alpha_1 + 1} \mathscr{L}^{\alpha_1+1,\alpha_2,\dots,\alpha_n;\beta_1,\dots,\beta_k}_{m;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y) - \mathscr{L}^{\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_k}_{m;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y) ,$$

$$(4.4)\qquad Ax\, \mathscr{L}^{\alpha_1+\mu_1,\dots,\alpha_n+\mu_n;\beta_1,\dots,\beta_k}_{m-1;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y) + By\, \mathscr{L}^{\alpha_1,\dots,\alpha_n;\beta_1+\nu_1,\dots,\beta_k+\nu_k}_{m-1;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y)$$

$$= C\, \mathscr{L}^{\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_k}_{m-1;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y) - \mathscr{L}^{\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_k}_{m;\mu_1,\dots,\mu_n;\nu_1,\dots,\nu_k}(x, y) ,$$

where

$$A = \prod_{r=1}^{n} \frac{\Gamma(m\mu_r - \mu_r + 1)}{\Gamma(m\mu_r + 1)} \cdot \prod_{r=1}^{k} \frac{\Gamma(m\nu_r + \beta_r + 1)\Gamma(m\nu_r - \nu_r + 1)}{\Gamma(m\nu_r + 1)\Gamma(m\nu_r + \beta_r - \nu_r + 1)} ,$$

$$B = \prod_{r=1}^{n} \frac{\Gamma(m\mu_r + \alpha_r + 1)\Gamma(m\mu_r - \mu_r + 1)}{\Gamma(m\mu_r + 1)\Gamma(m\mu_r - \mu_r + \alpha_r + 1)} \cdot \prod_{r=1}^{k} \frac{\Gamma(m\nu_r - \nu_r + 1)}{\Gamma(m\nu_r + 1)} ,$$

$$C = \prod_{r=1}^{n} \frac{\Gamma(m\mu_r + \alpha_r + 1)\Gamma(m\mu_r - \mu_r + 1)}{\Gamma(m\mu_r + 1)\Gamma(m\mu_r - \mu_r + \alpha_r + 1)} \cdot \prod_{r=1}^{k} \frac{\Gamma(m\nu_r - \nu_r + 1)\,\Gamma(m\nu_r + \beta_r + 1)}{\Gamma(m\nu_r + 1)\Gamma(m\nu_r - \nu_r + \beta_r + 1)} .$$

The calculations are quite tedious and are deliberately not given here.

From (4.2) and (4.3) we easily get $[z = \mathscr{L}^{\alpha_1,\dots,\alpha_n;\,\beta_1,\dots,\beta_k}_{m;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}(x, y)]$

$$(4.5)\qquad \frac{1}{\mu_1}\Big(\frac{\partial^{\mu_1+\dots+\mu_n}}{\partial x_1^{\mu_1}\dots\partial x_n^{\mu_n}}\Big)\Big(x_1\frac{\partial}{\partial x_1}-\alpha_1\Big)z = \frac{1}{\nu_1}\Big(\frac{\partial^{\nu_1+\dots+\nu_k}}{\partial y_1^{\nu_1}\dots\partial y_k^{\nu_k}}\Big)\Big(y_1\frac{\partial}{\partial y_1}-\beta_1\Big)z\,.$$

Again (4.2) and (4.3) easily give $(\mathrm{D}_x \equiv \partial/\partial x)$

$$(4.6)\qquad (\mu_1\times \mathrm{D}_x+\alpha_1)\,z = (m\mu_1+\alpha_1)\,\mathscr{L}^{\alpha_1-1,\alpha_2,\dots,\alpha_n;\,\beta_1,\dots,\beta_k}_{m;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}(y, x)\,.$$

To get the partial differential equation satisfied by our polynomials of 2 variables we follow the notation used in **2** and get for $\mu_1, \dots, \mu_n$ positive and integral

$$(4.7)\qquad \{F^{\mu_n}_{\alpha_n}(\mathrm{D}_x)\}\dots\{F^{\mu_1}_{\alpha_1}(\mathrm{D}_x)\}\,z$$

$$= \prod_{r=1}^{n}\frac{\Gamma(m\mu_r+\alpha_r+1)}{\Gamma(m\mu_r-\mu_r+\alpha_r+1}\cdot\mathscr{L}^{\alpha_1-\mu_1,\dots,\alpha_n-\mu_n;\,\beta_1,\dots,\beta_k}_{m;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}(x, y)\,.$$

Differentiating this once more partially with respect to x and using (4.2) and (4.3) we get

$$(4.8)\qquad y\Big(A_{\sigma_n+1}x^{\sigma_n+1}\frac{\partial^{\sigma_n+1}}{\partial x^{\sigma_n+1}}+\dots+A_1x\frac{\partial}{\partial x}+mx\Big)z$$

$$= -mxy\{Ax\,\mathscr{L}^{\alpha_1+\mu_1,\dots,\alpha_n+\mu_n;\,\beta_1,\dots,\beta_k}_{m-1;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}(x, y)+By\,\mathscr{L}^{\alpha_1,\dots,\alpha_n;\,\beta_1+\nu_1,\dots,\beta_k+\nu_k}_{m-1;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}(x, y)\}\,,$$

where A_{σ_n}'s are functions of $\mu_1, \dots, \mu_n, \alpha_1, \dots, \alpha_n$ which can easily be calculated and $\sigma_n = \mu_1+\dots+\mu_n$.

We can similarly get an analogous equation with $\mathrm{D}_y \equiv \partial/\partial y$, and $\nu_1, \dots, \nu_k$ positive integers, viz.

$$(4.9)\qquad x\Big(B_{\varrho_k+1}y^{\varrho_k+1}\frac{\partial^{\varrho_k+1}}{\partial y^{\varrho_k+1}}+\dots+B_1y\frac{\partial}{\partial y}+my\Big)z = \text{the right hand side of (4.8)}\,,$$

where B_{ϱ_k}'s are easily determined functions of $\nu_1, \dots, \nu_k, \beta_1, \dots, \beta_k$ and $\varrho_k = \nu_1+\dots+\nu_k$. Therefore the partial differential equation satisfied by $z = \mathscr{L}^{\alpha_1,\dots,\alpha_n;\,\beta_1,\dots,\beta_k}_{m;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}(x, y)$ is

$$(4.10)\qquad y\Big(A_{\sigma_n+1}\,x^{\sigma_n+1}\frac{\partial^{\sigma_n+1}}{\partial x^{\sigma_n+1}}+\dots+A_1x\frac{\partial z}{\partial x}\Big)$$

$$= x\Big(B_{\varrho_k+1}y^{\varrho_k+1}\frac{\partial^{\varrho_k+1}z}{\partial y^{\varrho_k+1}}+\dots+B_1y\frac{\partial z}{\partial y}\Big)\,.$$

It is interesting to note that the technique given by author [1]$_4$ can also be used to advantage in finding partial differential equations of some special functions of two or more variables.

Let us now take the identity

$$\frac{1}{p_1^{\alpha_1}\dots p_n^{\alpha_n} q_1^{\beta_1}\dots q_k^{\beta_k}}\Big(1-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}-\frac{1}{q_1^{\nu_1}\dots q_k^{\nu_k}}\Big)^m$$

$$=\frac{1}{p_1^{\alpha_1}\dots p_n^{\alpha_n} q_1^{\beta_1}\dots q_k^{\beta_k}}\Big[\Big(1-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}\Big)\Big(1-\frac{1}{q_1^{\nu_1}\dots q_k^{\nu_k}}\Big)-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n} q_1^{\nu_1}\dots q_k^{\nu_k}}\Big]^m$$

$$=\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\Big(1-\frac{1}{p_1^{\mu_1}\dots p_n^{\mu_n}}\Big)^s\Big(1-\frac{1}{q_1^{\nu_1}\dots q_k^{\nu_k}}\Big)^s$$

$$\times\frac{1}{p_1^{\mu_1(m-s)+\alpha_1}\dots p_n^{\mu_n(m-s)+\alpha_n} q_1^{\nu_1(m-s)+\beta_1}\dots q_k^{\nu_k(m-s)+\beta_k}}.$$

This on interpretation gives the interesting relation

$$(4.11)\qquad \sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}x^{m-s}y^{m-s}\prod_{r=1}^{n}\Gamma(s\,\mu_r+1)\prod_{r=1}^{k}\Gamma(s\,\nu_r+1)$$

$$\times\mathscr{L}_{s;\,\mu_1,\dots,\mu_n}^{\mu_1(m-s)+\alpha_1,\dots,\mu_n(m-s)+\alpha_n}(x)\cdot\mathscr{L}_{s;\,\nu_1,\dots,\nu_k}^{\nu_1(m-s)+\beta_1,\dots,\nu_k(m-s)+\beta_k}(y)$$

$$=\prod_{r=1}^{n}\Gamma(m\mu_r+1)\prod_{r=1}^{k}\Gamma(m\nu_r+1)\,\mathscr{L}_{m;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}^{\alpha_1,\dots,\alpha_n;\,\beta_1,\dots,\beta_k}(x,y)\,.$$

Lastly we obtain the development of $\mathscr{L}_{m;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}^{\alpha_1,\dots,\alpha_n;\,\beta_1,\dots,\beta_k}(x+\delta x,\,y+\delta y)$ in Taylor series in terms of polynomials of the same nature. Formula (4.2) together with its analogue for $\partial/\partial y$ permits us to obtain the successive derivatives of z very easily.

On differentiating the relation (4.2) $(s-1)$ times, we obtain

$$(4.12)\qquad \frac{\partial^s z}{\partial x^s}=(-1)^s m(m-1)\dots(m-s+1)\,\mathscr{L}_{m-s;\,\mu_1,\dots,\mu_n;\,\nu_1,\dots,\nu_k}^{\alpha_1+s\mu_1,\dots,\alpha_n+s\mu_n;\,\beta_1,\dots,\beta_k}(x,y)$$

$$\times\prod_{r=1}^{n}\frac{\Gamma(m\mu_r-s\mu_r+1)}{\Gamma(m\mu_r+1)}\cdot\prod_{r=1}^{k}\frac{\Gamma(m\nu_r-s\nu_r+1)\,\Gamma(m\nu_r+\beta_r+1)}{\Gamma(m\nu_r+1)\,\Gamma(m\nu_r-s\nu_r+\beta_r+1)}.$$

Similarly

$$\frac{\partial^s z}{\partial y^s} = (-1)^s m(m-1) \ldots (m\ s = +1) \quad \mathscr{L}_{m-s;\, \mu_1,\ldots,\mu_n;\, \nu_1,\ldots,\nu\nu_k}^{\alpha_1,\ldots,\alpha_n;\, \beta_1+s s\nu_1,\ldots,\beta_k+s\nu_k}(x,y) \tag{4.13}$$

$$\times \prod_{r=1}^{n} \frac{\Gamma(m\mu_r - s\mu_r + 1)\,\Gamma(m\mu_r + \alpha_r + 1)}{\Gamma(m\mu_r + 1)\,\Gamma(m\mu_r - s\mu_r + \alpha_r + 1)} \cdot \prod_{r=1}^{k} \frac{\Gamma(m\nu_r - s\nu_r + 1)}{\Gamma(m\nu_r + 1)},$$

$$\frac{\partial^{s+s'} z}{\partial x^s\, \partial y^{s'}} \tag{4.14}$$

$$= (-1)^{s+s'} m(m-1) \ldots (m - s - s' + 1) \ \mathscr{L}_{m-s-s';\, \mu_1,\ldots,\mu_n;\, \nu_1,\ldots,\nu_k}^{\alpha_1+s\mu_1,\ldots,\alpha_n+s\mu_n;\, \beta_1+s'\nu_1,\ldots,\beta_k+s'\nu_k}(x, y)$$

$$\times \prod_{r=1}^{n} \frac{\Gamma(m\mu_r - \overline{s+s}'\mu_r + 1)\,\Gamma(m\mu_r + \alpha_r + 1)}{\Gamma(m\mu_r + 1)\,\Gamma(m\mu_r - \overline{s+s}'\mu_r + \alpha_r + 1)} \cdot \prod_{r=1}^{k} \frac{\Gamma(m\nu_r - \overline{s+s}'\nu_r + 1)\,\Gamma(m\nu_r + \beta_r + 1)}{\Gamma(m\nu_r + 1)\,\Gamma(m\nu_r - \overline{s+s}'\nu_r + \beta_r + 1)}.$$

In particular

$$\frac{\partial^m z}{\partial x^m} = (-1)^m m! \prod_{r=1}^{n} \frac{1}{\Gamma(m\mu_r + 1)} \prod_{r=1}^{k} \frac{\Gamma(m\nu_r + \beta_r + 1)}{\Gamma(m\nu_r + 1)\,\Gamma(\beta_r + 1)}, \tag{4.15}$$

$$\frac{\partial^m z}{\partial y^m} = (-1)^m m! \prod_{r=1}^{n} \frac{\Gamma(m\mu_r + \alpha_r + 1)}{\Gamma(m\mu_r + 1)\,\Gamma(\alpha_r + 1)} \prod_{r=1}^{k} \frac{1}{\Gamma(m\nu_r + 1)}, \tag{4.16}$$

and if $m = s + s'$

$$\frac{\partial^m z}{\partial x^s\, \partial y^{s'}} = (-1)^m m! \prod_{r=1}^{n} \frac{\Gamma(m\mu_r + \alpha_r + 1)}{\Gamma(m\mu_r + 1)\,\Gamma(m\mu_r - s'\mu_r + \alpha_r + 1)} \tag{4.17}$$

$$\times \prod_{r=1}^{k} \frac{\Gamma(m\nu_r + \beta_r + 1)}{\Gamma(m\nu_r + 1)\,\Gamma(m\nu_r - s\nu_r + \beta_r + 1)}.$$

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