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A note on the initial-value problem in the linearized theory of general relativity (**)

1 - Preliminaries

In a previous paper [**3**] we discussed the application of ADM method [**2**] to the analysis of the dynamics of the gravitational field in the linear approximation. The results obtained in [**3**] are now used in order to write down the field equations in Hamiltonian form and state the initial-value problem.

In the space-time manifold V_4 (initially carrying a pseudo-euclidean metric) we introduce a family Σ od spatial parallel hyperplanes σ [**3**]. Let $B = \{x^i\}$ be a pseudo-cartesian co-ordinate system adapted to Σ [**4**], and let $\{\partial_i, \mathrm{d}x^j\}$ be the corresponding basis of the tensor algebra over V_4; then the pseudo-euclidean fundamental form is

$$\Phi_0 = \eta_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad \eta_{00} = -1\,, \qquad \eta_{0\alpha} = 0\,, \qquad \eta_{\alpha\beta} = \delta_{\alpha\beta}\ (^1)\,.$$

In the linearized theory of gravitation, the space-time is supposed to carry a fundamental form

$$\Phi = (\eta_{ij} + h_{ij})\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad |h_{ij}| \ll 1\,,$$

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(1) Latin indices run from 0 to 3, greek ones from 1 to 3; Einstein summation convention is used throughout. Tensor indices are raised and lowered using the flat-space metric.

the perturbation h_{ij} being a solution of the linearized Einstein equations. The spatial resolution of Φ relative to the family of hyperplanes Σ gives rise to a spatial fundamental form [**3**]

$$\tilde{\Phi} = (\delta_{\alpha\beta} + \theta_{\alpha\beta})\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta, \qquad |\theta_{\alpha\beta}| \ll 1 ,$$

to a spatial vector field $N^\mu \partial_\mu$ and to a scalar field N, with the identifications

$$N_\mu = h_{0\mu} , \qquad N = 1 - \tfrac{1}{2} h_{00} . \tag{1}$$

The Einstein equations may be identified with the Hamilton equations associated with the Hamiltonian

$$H = -\int_{R^3} \mathrm{d}^3x [N R^{*1} + 2N^\mu \partial_\nu \pi^\mu{}_\nu + (\tfrac{1}{2}\delta^{\alpha\beta}\theta^\mu{}_\mu - \theta^{\alpha\beta})\, R^{*1}_{\alpha\beta} + R^{*2} - \tfrac{1}{2}\pi^2 + \pi_{\alpha\beta}\pi^{\alpha\beta}] ,$$

where the variables $\pi^{\alpha\beta}$ are the momenta conjugate to $\theta_{\alpha\beta}$, while the quantities $R^{*1}_{\alpha\beta}$, R^{*1}, R^{*2} are defined according to the equations [**3**]

$$R^{*1}_{\alpha\beta} = \partial_\beta \left\{ {\mu \atop \alpha\ \mu} \right\} - \partial_\mu \left\{ {\mu \atop \alpha\ \beta} \right\} , \qquad R^{*1} = \delta^{\alpha\beta} R^{*1}_{\alpha\beta} ,$$

$$R^{*2}_{\alpha\beta} = \left\{ {\mu \atop \alpha\ \varrho} \right\} \left\{ {\varrho \atop \beta\ \mu} \right\} - \left\{ {\mu \atop \alpha\ \beta} \right\} \left\{ {\varrho \atop \mu\ \varrho} \right\} , \qquad R^{*2} = \delta^{\alpha\beta} R^{*2}_{\alpha\beta} .$$

The resulting equations may be written

$$\partial_0 \theta_{\alpha\beta} = \delta_{\alpha\beta} \pi^\mu{}_\mu - 2\pi_{\alpha\beta} + \partial_\alpha N_\beta + \partial_\beta N_\alpha , \tag{2a}$$

$$\partial_0 \pi_{\alpha\beta} = - R^{*1}_{\alpha\beta} + \tfrac{1}{2}\delta_{\alpha\beta} R^{*1} - \partial_\alpha \partial_\beta N + \delta_{\alpha\beta} \partial_\mu \partial^\mu N , \tag{2b}$$

$$\partial_\mu \pi^{\mu\nu} = 0 , \qquad R^{*1} = 0 . \tag{3}$$

Eqs. (2) are true dynamical equations for the variables $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$, whilst eqs. (3) express four constraints among the variables themselves, essentially equivalent to the linearized version of the equations $G_{0i} = 0$.

2 - Gauge conditions and initial-value problem

Since eqs. (2) do not involve the time derivatives of N, N^μ, the time evolution of these fields is not fixed by the theory, but may be chosen arbitrarily [**7**];

on the other hand, the time evolution of the remaining variables $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$ is subordinate to this choice, so that the quantities N, N^μ play the role of gauge fields. As it is well known [**2**], the occurrence of such fields is strictly related to the invariance of the theory under arbitrary co-ordinate transformations.

The freedom in the choice of N, N^μ enables one to impose four supplementary conditions among the dynamical variables. These four gauge conditions, together with the four constraint equations, constitute a set of eight relations linking the fields N, N^μ on one hand, to the fields $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$ on the other hand. In this way, fixing explicitly the fields N, N^μ, we obtain eight equations which, at least formally, allow to express eight of the components $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$ in terms of the remaining four.

The problem is then to decide whether some choices of the gauge conditions are more convenient than others; as it will be seen, in this respect the Hilbert gauge [**5**], completed with York's decomposition of symmetric tensors of rank two [**8**], plays a distinguished rôle, as it gives rise to a complete decoupling of the eight equations mentioned above.

In three-dimensional notation, the Hilbert condition reads

(4a) $$\partial_0 N + \partial_\beta N^\beta = \tfrac{1}{2}\partial_0 \theta^\beta{}_\beta\,,$$

(4b) $$\partial_0 N_\alpha + \partial_\alpha N = \partial_\mu(\theta^\mu{}_\alpha - \tfrac{1}{2}\delta^\mu{}_\alpha \theta^\beta{}_\beta)\,.$$

Eqs. (4), together with the constraints (3)

(5a,b) $$R^{*1} = 0\,, \qquad \partial_\mu \pi^{\mu\nu} = 0\,,$$

provide eight relations among the twelve variables $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$, in agreement with the fact that only four of these variables are to be related to the true degrees of freedom of the field.

Following York, let us now introduce the decomposition

(6) $$\theta_{\alpha\beta} = \theta^{TT}_{\beta\alpha} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \tfrac{2}{3}\delta_{\alpha\beta}\partial_\mu \xi^\mu + \tfrac{1}{3}\delta_{\alpha\beta}\theta\,,$$

where $\theta = \theta^\mu{}_\mu$, the functions ξ_β are solutions of the equation

(7) $$\partial_\beta(\partial_\alpha \xi^\beta + \partial^\beta \xi_\alpha - \tfrac{2}{3}\delta_\alpha{}^\beta \partial_\mu \xi^\mu) = \partial_\beta(\theta^\beta{}_\alpha - \tfrac{1}{3}\delta^\beta{}_\alpha \theta)\,,$$

and $\theta^{TT}_{\alpha\beta}$ is defined by eq. (6) itself. It turns out that $\theta^{TT}_{\alpha\beta}$ is traceless and transverse (divergenceless), and that the longitudinal term

(8) $$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \tfrac{2}{3}\delta_{\alpha\beta}\partial_\mu \xi^\mu \overset{\text{def}}{=} \theta^L_{\alpha\beta}$$

is traceless. This decomposition is covariant, unique and orthogonal [**8**].

A first insight into the physical significance of this decomposition is given by the following fact: let $\theta_{\alpha\beta}$ be a spatial perturbation satisfying the constraint (5a); then $\theta^{TT}_{\alpha\beta} = 0$ if and only if $\theta_{\alpha\beta} = \partial_\alpha \chi_\beta + \partial_\beta \chi_\alpha$, i.e. if and only if $\theta_{\alpha\beta}$ is the spatial projection of a Weyl solution [1]. This implies that $\theta^{TT}_{\alpha\beta}$ is a gauge-independent quantity; we may therefore expect $\theta^{TT}_{\alpha\beta}$ to obey some sort of gauge-independent dynamical equation. As we shall see, this is indeed the case.

In a similar way we decompose $\pi^{\alpha\beta}$

$$\pi^{\alpha\beta} = \pi^{\alpha\beta}_{TT} + \partial^\alpha \psi^\beta + \partial^\beta \psi^\alpha - \tfrac{2}{3}\delta^{\alpha\beta}\partial_\mu \psi^\mu + \tfrac{1}{3}\delta^{\alpha\beta}\pi \,.$$

Taking into account eq. (7), the analogous relation for the functions ψ^β, and the equation

$$\pi = \partial_0 \theta - 2\partial_\mu N^\mu, \tag{9a}$$

which follows at once from eq. (2a), the eight eqs. (4), (5) become a set of equations for the eight functions θ, ξ_β, π, ψ^β, thus allowing to express them in terms the fields N, N^μ, and, in principle, also of $\theta^{TT}_{\alpha\beta}$ and $\pi^{\alpha\beta}_{TT}$. Setting for simplicity $D_{\beta\nu} = \delta_{\beta\nu}\partial_\mu\partial^\mu + \frac{1}{3}\partial_\beta\partial_\nu$, eq. (7), and the analogous relation for ψ^β may be written more briefly

$$D_{\beta\nu}\xi^\nu = \partial_\nu(\theta^\nu{}_\beta - \tfrac{1}{3}\delta^\nu{}_\beta\theta) \,, \tag{9b}$$

$$D_{\beta\nu}\psi^\nu = \partial_\nu(\pi^\nu{}_\beta - \tfrac{1}{3}\delta^\nu{}_\beta\pi) \,. \tag{9c}$$

Substituting eq. (4a) into (9a), eq. (4b) into (9b) and eq. (5b) into (9c), we obtain

$$D_{\beta\nu}\xi^\nu = \partial_\beta N + \partial_0 N_\beta + \tfrac{1}{6}\partial_\beta\theta \,, \tag{10}$$

$$\pi = 2\partial_0 N \,, \qquad D_{\beta\nu}\psi^\nu = -\tfrac{1}{3}\partial_\beta\pi = -\tfrac{2}{3}\partial_0\partial_\beta N \,.$$

Also, by eq. (5a), $0 = R^{*1} = \partial_\mu\partial^\mu\theta - \partial_\mu\partial_\beta\theta^{\mu\beta} = \frac{2}{3}\partial_\mu\partial^\mu\theta - \partial^\beta D_{\beta\nu}\xi^\nu$. Substituting eq. (10) to eliminate the last term, we have $\partial_\mu\partial^\mu\theta = 2(\partial_\mu\partial^\mu N + \partial_0\partial_\mu N^\mu)$. To sum up, we have obtained the following set of equations

$$\pi = 2\partial_0 N \,, \tag{11a}$$

$$\nabla^2\theta = 2(\nabla^2 N + \partial_0\partial_\mu N^\mu) \,, \tag{11b}$$

$$D_{\beta\nu}\psi^\nu = -\tfrac{2}{3}\partial_0\partial_\beta N \,, \tag{11c}$$

$$D_{\beta\nu}\xi^\nu = \partial_\beta N + \partial_0 N_\beta + \tfrac{1}{6}\partial_\beta\theta \,. \tag{11d}$$

A remarkable feature of eqs. (11) is that they do not involve the TT terms of $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$. This makes it very simple to state the initial-value problem on a hyperplane $\sigma \in \Sigma$. Indeed, after having fixed the fields N, N^{μ} arbitrarily in a neighbourhood of σ, and having assigned the « true » degrees of freedom $\theta^{TT}_{\alpha\beta}$ and $\pi^{\alpha\beta}_{TT}$ on σ, eqs. (11) enable to complete the fields $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$ on σ; in fact they fix the longitudinal terms and the traces of the latters. By construction, the fields so obtained satisfy the gauge conditions (4) and the constraints (5) and therefore from a consistent initial-value set for eqs. (2) in the Hilbert gauge.

Both fields $\theta^{TT}_{\alpha\beta}$, $\pi^{\alpha\beta}_{TT}$ are characterized by two functions, for they must be transverse and traceless. Therefore, the number of degrees of freedom is two.

3 - Splitting of field equations

The introduction of the decomposition of the tensors $\theta_{\alpha\beta}$, $\pi^{\alpha\beta}$ into the field equations provides a further insight into the physical significance of the decomposition itself. Splitting both sides of eqs. (2), making use of the uniqueness property, and passing to second-order equations, we obtain

$$\partial_0 \partial_0 \theta^{TT}_{\alpha\beta} = \nabla^2 \theta^{TT}_{\alpha\beta}\,, \tag{12a}$$

$$\partial_0 \partial_0 \xi_\beta = -\tfrac{1}{3}\partial_\beta \partial_\mu \xi^\mu + \tfrac{1}{6}\partial_\beta \theta + \partial_\beta N + \partial_0 N_\beta\,, \tag{12b}$$

$$\partial_0 \partial_0 \theta = \tfrac{1}{3}\nabla^2 \theta - \tfrac{2}{3}\nabla^2 \partial_\mu \xi^\mu + 2(\nabla^2 N + \partial_0 \partial_\mu N^\mu)\,. \tag{12c}$$

In view of eq. (8), eq. (12b) is a dynamical equation for the longitudinal term of $\theta_{\alpha\beta}$.

Eq. (12a) is a wave equation for the dynamical variables $\theta^{TT}_{\alpha\beta}$; it does not involve the gauge fields N, N^{μ}. In this sense, the decomposition (6) has set apart the variables the time evolution of which is described in a completely gauge-independent way. On the contrary, on the ground of eqs. (12b, c), the time evolution of the functions θ, ξ_β depends explicitly on the choice of the gauge fields [7]; in particular, these relations reduce to the wave equations $\partial_0 \partial_0 \xi_\beta = \nabla^2 \xi_\beta$; $\partial_0 \partial_0 \theta = \nabla^2 \theta$ only after inserting eqs. (11) in them. By choosing a gauge different from Hilbert's one, the time evolution of the functions θ, ξ_β is modified; so only the variables $\theta^{TT}_{\alpha\beta}$ and the conjugated momenta $\pi^{\alpha\beta}_{TT}$ have full dynamical meaning. This is consistent with the identification of these variables with the true degrees of freedom of the gravitational field.

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Sunto

Utilizzando un formalismo analogo al metodo ADM, *si scrivono in forma hamiltoniana le equazioni di campo linearizzate della Relatività Generale e viene formulato il corrispondente problema dei dati iniziali. Si analizzano infine gli effettivi gradi di libertà dinamici del campo gravitazionale.*

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