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Continuous selections and multivalued differential equations (**)

Introduction

The present paper deals with differential equations of the form

$$x' \in F(t, x)\,, \tag{E}$$

where F is a multivalued mapping defined in $[0, 1]\times R^n$, whose values are nonempty closed convex sets of R^n.

Here we prove selection theorems from which the existence of a solution of (E) can be deduced as a direct consequence of an existence theorem for solutions of the classical differential equation $x' = f(t, x)$ [**3**].

Selection theorems have been previously used for similar purposes as in [**4**], [**1**], [**2**].

In this paper we consider multivalued mappings under weaker hypotheses than the classical ones (continuity and lower-continuity) and we associate with them a mapping K from continuous to measurable functions. Applying a well known result by Michael [**6**] we prove the existence of a selection theorems also determine the properties of the selection.

1 - Let I denote the closed interval $[0, 1]$, and let S be a subset of I such that $\mu(I - S) = 0$. Let $C(I)$ denote the space of R^n-valued continuous functions with the topology of uniform convergence, let $\mathscr{M}(I)$ denote the space

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of R^n-valued measurable functions with the metric of convergence in measure and let $\mathscr{L}^1(I)$ be the Banach space of R^n-valued integrable functions.

We shall denote by $B[A, \varepsilon]$ the open ball of radius $\varepsilon > 0$ about the set $A \subset R^n$.

In this paper we consider multivalued mappings of a topological measure space X into R^n, for this purpose we shall start by stating some properties of multivalued closed measurable mappings (for every open set $A \subset R^n$, the set $\{x \in X / F(x) \subset A\}$ is measurable in X), and of lower semi-continuous mappings (for every open set $A \subset R^n$ the set $\{x \in X / F(x) \cap A \neq \emptyset\}$ is open in X).

(1) *Let* $\{F_i\}_{i\in N}$ *be a family of measurable closed mappings* $F_i\colon X \to P(R^n)$. *Then the mappings* $F\colon t \to \bigcap_{i\in N} F_i(t)$ *and* $G\colon t \to \bigcup_{i\in N} F_i(t)$ *are measurable.*

(2) *Let* F *be a measurable closed mapping, then* $\mathrm{cl}\,\big(B[F(t), \varepsilon]\big)$ *is measurable.*

(3) *Let* F *be a measurable non empty closed mapping, then there exists a countable family* K *of measurable functions* $f\colon X \to R^n$ *such that*

$$F(t) = \mathrm{cl}\,\{f(t), f \in K\} .$$

For the proof of properties 1 and 3 see for istance [7]. To prove property 2 it is sufficient to observe that for any closed set $A \subset R^n$

$$\{x / \mathrm{cl}\,\big(B[F(x), \varepsilon]\big) \cap A \neq \emptyset\} = \{x / F(x) \cap \mathrm{cl}\,\big(B[A, \varepsilon]\big) \neq \emptyset\} .$$

(4) *If* X *is a metric (first countable) space and* F *a multivalued mapping from* X *into* R^n *then* F *is lower semi-continuous if and only if it is lower semi-continuous for sequences* i.e. *if for all sequences* $\{x_n\}$ *which converge to* x_0 *in* X *and for all* $\varepsilon > 0$ *there exists an* $\bar{n}$ *such that for* $n > \bar{n}$ $F(x_0) \subset B[F(x_n), \varepsilon]$.

Proof. The proof of the necessity is trivial. To prove the sufficiency: assume F not lower semi-continuous: then there exist $x_0 \in X$ and $\varepsilon > 0$ such that for all $\delta > 0$ there exists x_δ such that $|x_\delta - x_0| < \delta$ and $F(x_0) \not\subset B[F(x_\delta), \varepsilon]$ i.e. there exists $y \in F(x_0)$: for all $\bar{y} \in F(x_\delta)$, $|y - \bar{y}| > \varepsilon$.

Thus we can choose a sequence $\{x_n\}$ with $|x_n - x_0| < 1/n$ such that $F(x_0) \not\subset B[F(x_n), \varepsilon]$; hence a contradiction.

With any given mapping $F\colon I \times R^n \to P(R^n)$ such that for any $x \in C(I)$ $F\big(t, x(t)\big)$ is a measurable non empty closed mapping, we can associate a mapping $K\colon C(I) \to P\big(\mathscr{M}(I)\big)$ defined as follows

$$K\big(t, x(t)\big) = \{f\big(t, x(t)\big) \in \mathscr{M}(I) / f\big(t, x(t)\big) \in F\big(t, x(t)\big) \text{ a.e. in } I\} .$$

By (3) this mapping is non empty.

We shall show now that K inherits some of the properties of F.

Proposition 1. *If $F(t, x(t))$ is closed for all $(t, x) \in S \times R^n$ then, for all $x \in C(I)$ the set $K(t, x(t))$ is closed in the topology of $\mathscr{M}(I)$.*

Proof. For any sequence $f_n(t, x(t)) \in K(t, x(t))$ which converges to $f(t, x(t))$ there is a subsequence converging a.e. to $f(t, x(t))$; since $F(t, x(t))$ is closed it follows $f(t, x(t)) \in F(t, x(t))$ a.e. in I. Then there is a $\psi \in K$ which coincides with f a.e. in I.

Proposition 2. *From $F(t, x)$ convex for every $t \in S$ and $x \in R^n$ it follows $K(t, x(t))$ convex for every $x \in C(I)$.*

Proof. Every linear combination of measurable functions is measurable.

Proposition 3. *If $F(t, x)$ is closed for every $(t, x) \in S \times R^n$ and, for every $t \in S$, it is lower semi-continuous in x, then K is a lower semi-continuous mapping.*

Proof. Let $\{x_n\}$ converge to x_0 in $C(I)$. Given $\varepsilon > 0$ let us define the following sequences of measurable sets

$$A(n, \varepsilon) = \{t / F(t, x_0(t)) \subset B[F(t, x_n(t)), \varepsilon/2]\},$$

$$C(k, \varepsilon) = \bigcap_{n \geqslant k} A(n, \varepsilon) = \{t / F(t, x_0(t)) \subset B[F(t, x_n(t)), \varepsilon/2], n \geqslant k\}.$$

The sequence $\{C(k, \varepsilon)\}$ is increasing in k and $\bigcup_k C(k, \varepsilon) = S$. For any decreasing sequence of positive numbers $\{\varepsilon_i\}$ which converges to zero we can find a sequence $\{k_i\}$ such that for $k > k_i$, $\mu(I - C(k, \varepsilon_i)) < \varepsilon_i/2$.

For every measurable $y_0(t) \in F(t, x_0(t))$ and $n > k_i$ the mapping cl $(B[y_0(t), \varepsilon_i/2] \cap F(t, x_n(t))$ is non empty, closed-valued and measurable on $C(k_i, \varepsilon_i)$, hence there exists in $C(k_i, \varepsilon_i)$ a measurable selection $\varphi_n(t)$. If $y_n(t) \in F(t, x_n(t))$ is a measurable extension of $\varphi_n(t)$ then

$$\int_I |y_0 - y_n|(1 + |y_0 - y_n|)^{-1} \, d\mu$$

$$= \int_{C(k_i, \varepsilon_i)} |y_0 - y_n|(1 + |y_0 - y_n|)^{-1} \, d\mu + \int_{I - C(k_i, \varepsilon_i)} |y_0 - y_n|(1 + |y_0 - y_n|)^{-1} \, d\mu$$

$$< \varepsilon_i/2\mu(I) + \mu(I - C(k_i, \varepsilon_i)) < \varepsilon_i.$$

Since y_0 is an arbitrary element of $K(x_0)$ and y_n is in $K(x_n)$ it follows that

$$K(x_0) \subset B[K(x_n), \varepsilon_i] \qquad (n \geqslant k_i)$$

and then $K\colon C(I) \to \mathscr{M}(I)$ is lower semi-continuous.

Theorem 1. *Let F be a mapping from $I \times R^n$ to $P(R^n)$ such that the following hold:*

(i) *for every $x \in C(I)$, the mapping $F(t, x(t))$ is measurable;*
(ii) *for every $(t, x) \in S \times R^n$ the set $F(t, x)$ is closed and convex;*
(iii) *$F(t, x)$ is lower semi-continuous in R^n for every $t \in S$;*
(iv) *there exists a function $\beta \in \mathscr{L}^1(I)$ such that every $y \in F(t, x)$ satisfies $|y| \leqslant \beta(t)$.*

Then there exists a continuous function $f\colon C(I) \to \mathscr{L}^1(I)$ such that $f(t, x(t)) \in F(t, x(t))$ for every $x \in C(I)$.

Proof. The property (iv) implies that the mapping K associated with F takes its values in $\mathscr{L}^1(I)$. From propositions (1), (2), (3) it follows that K is lower semi-continuous, closed and convex-valued in $\mathscr{M}(I)$ and then by (iv) also in $\mathscr{L}^1(I)$.

By Michael's theorem ([**6**] p. 367) there exists a continuous selection for K from $C(I)$ to $\mathscr{L}^1(I)$.

Corollary 1. *If $F\colon I \times R^n \to P(R^n)$ satisfies the hypotheses of Theorem 1, then there exists $u\colon I \to R^n$ such that $u(0) = 0$ and $u'(t) \in F(t, u(t))$ for almost every $t \in I$.*

Proof. Let $G = \{g \in \mathscr{L}^1(I) / |g(t)| \leqslant \beta(t)\}$. The function f, whose existence has been proved in Theorem 1, is G-regular (according to the definition of [**3**]) hence there exists (see [**3**]) a solution of Cauchy's problem

$$x' = f(t, x), \qquad x(0) = 0 .$$

Since the Hausdorff distance h between two compact-valued measurable functions is a measurable function, we can give the following definition.

Definition. A sequence $\{F_n(t)\}$ of compact valued measurable functions is said to *converge in measure to a compact valued measurable function* $F(t)$ if for every $\varepsilon > 0$ there exists a $\bar{n}$ such that for $n > \bar{n}$

$$\mu\{t \,/\, h[F_n(t), F(t)] \geqslant \alpha\} \leqslant \varepsilon .$$

Theorem 2. *Let F be a function defined in $I \times R^n$ whose values are compact convex subsets of R^n such that* (i) *and* (iv) *of Theorem* 1 *hold and furthermore for every sequence $\{x_n\}$ converging to x_0 in $C(I)$ the sequence $\{F(t, x_n(t))\}$ converges in measure to $F(t, x_0(t))$.*

Under these hypotheses there exists a continuous function $f\colon C(I) \to \mathscr{L}^1(I)$ such that $f(t, x(t) \in F(t, x(t))$ a.e. *in I for every $x \in C(I)$.*

Proof. First we shall prove that if F converges in measure then K is continuous from $C(I)$ to $\mathscr{M}(I)$. This is equivalent to proving that given a sequence $\{x_n\}$ converging to x_0 in $C(I)$ then for every $\varepsilon > 0$ there exists a $\bar{n}$ such that for $n > \bar{n}$

$$\bar{h}[K(t, x_n(t)), K(t, x_0(t))] < \varepsilon ,$$

where $\bar{h}$ is the Hausdorff distance in the space of closed subsets of $\mathscr{M}(I)$. To prove the above it is sufficient to show that for $n > \bar{n}$ the following hold simultaneously

(a) for all $g \in K(t, x_0(t))$ there exists $f \in K(t, x_n(t))$ such that

$$\int_I |f-g|(1+|f-g|)^{-1}\,\mathrm{d}\mu < \varepsilon ,$$

(b) for all $f \in K(t, x_n(t))$ there exists $g \in K(t, x_0(t))$ such that

$$\int_I |f-g|(1+|f-g|)^{-1}\,\mathrm{d}\mu < \varepsilon .$$

Let $H(n, \alpha) = \{t / h[F(t, x_0(t)), F(t, x_n(t))] \geqslant \alpha\}$.

For every $g \in K(t, x_0(t))$ the multivalued function $t \to \mathrm{cl}\,(B[g(t), \alpha]) \cap F(t, x_n(t))$ is defined on $I - H(n, \alpha)$ is non empty, closed valued and measurable and then there exists a measurable selection φ.

Let $f\colon I \times R^n \to R^n$ be a measurable selection for $F(t, x_n(t))$ which coincides with φ in $I - H(n, \alpha)$.

For $\alpha < \varepsilon/2$ and n such that $\mu H(n, \alpha) < \varepsilon/2$ it is

$$\int_I |f-g|(1+|f-g|)^{-1}\,\mathrm{d}\mu$$

$$= \int_{H(n,\alpha)} |f-g|(1+|f-g|)^{-1}\,\mathrm{d}\mu + \int_{I-H(n,\alpha)} |f-g|(1+|f-g|)^{-1}\,\mathrm{d}\mu \leqslant \mu H(n, \alpha) + \mu(I)\,\varepsilon/2 .$$

This proves condition (a); the proof of condition (b) is similar.

By the continuity of K in $\mathscr{M}(I)$, hypothesis (iv) and Michael's theorem our proof is complete.

Corollary 2. *If F verifies the hypothesis of Theorem 2 then there exists a function $u\colon I \to R^n$ such that $u(0) = 0$ and $u' \in F(t, u(t))$ for almost every $t \in I$.*

This is proved analogously to Corollary 1.

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Summary

In questo lavoro si considera il problema $x' \in F(t, x)$, dove F è una funzione multivoca in $[0, 1] \times R^n$ a valori in $P(R^n)$, sulla quale si fanno ipotesi più deboli di quelle classiche (continuità e semicontinuità). Si dànno teoremi di esistenza di selettori attraverso i quali si giunge a stabilire l'esistenza di soluzioni del problema suddetto.

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