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Para-Kählerian manifolds having the self-orthogonal Killing property (**)

Let $M(\Omega, g, \mathscr{U})$ be a para-Kählerian manifold, that is a neutral C^∞ pseudo-Riemannian manifold having a Kählerian structure [5].

The structure tensors are: a canonical symplectic form Ω, a para-Hermitian metric tensor g, exchangeable with Ω, the para-complex operator $\mathscr{U}$ ($\mathscr{U}^2 = +1$). If $T_p(M)$ is the tangent space at $p \in M$ to M, one has the decomposition $T_p(M) = S_p(M) \oplus S_p^*(M)$. If $\dim M = 2n$, $S_p(M)$ and $S_p^*(M)$ are two self-orthogonal (abr. s.o.) vectorial sub-spaces of dimension n and the pair $(S_p(M), S_p^*(M))$ defines an involutive automorphism.

Let $\mathscr{R}(M)$ be the bundle of real Witt frames over M and $\mathscr{R} = \{h_A\}$ $(1 \leqslant A, B, C \leqslant 2n)$, where h_A are null real vector fields, an element of $\mathscr{R}(M)$. If in the neighborhood of each point $p \in M$ there exists an $\mathscr{R}$ such that the vector fields $h_a \in S_p(M)$ $(a = 1, \dots, n;\ a^* = a + n)$ are Killing vector fields we say that M has the s.o. Killing property. In this case it is proved that the s.o. vectorial sub space $S_p^*(M) = \{h_{a^*}\}$ is *auto-parallel* and the whole manifold M has the *divergence property* [1]. Moreover:

(i) any vector field $X^* \in S_p^*(M)$ is an *isovector* of the Pfaffian system defined by the covectors corresponding to the dual of $S_p(M)$;

(ii) if $X \in S_p(M)$ is any Killing vector field, then the following two properties are equivalent

(a) X is a *local Hamiltonian* of Ω;

(b) X is a *parallel vector* field.

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Next the following proper and improper immersions in M are considered:

(i) The proper immersion $x: M_I \to M$ where M_I is an *invariant* submanifold of M. If the codimension of M_I is $2r$, then x is of *geodesic index* r.

(ii) The proper immersion $x: M_A \to M$ where M_A is an *anti-invariant* manifold of dimension n; then M_A is always *minimal.*

(iii) The improper immersion $x: M_C \to M$ where M_C is a co-isotropic hypersurface whose normal h_a is in the s.o. Killing sub-space $S_p(M)$; this immersion is geodesic. Moreover the normal covector ω^a defines with the restriction $\Omega|_{M_C}$ a *semi-cosymplectic structure.*

(iv) The improper immersion $x: M_L \to M$ where M_L is a *mixed Lagrangian* sub manifold of M; this immersion is *totally geodesic.*

1 - Let $M(\Omega, g, \mathscr{U})$ be a para-Kählerian manifold [**5**] of dimension $2n$. Let $\mathscr{R}(M)$ be the bundle of real Witt frames over M and let $\mathscr{R} \in \mathscr{R}(M)$ be an element of $\mathscr{R}(M)$. Denote by $\{h_A\}$ $(1 \leqslant A, B, C \leqslant 2n)$ the null real vectorial basis of $\mathscr{R}$ and by $\{\omega^A\}$ its dual basis. If $T_p(M)$ is the tangent space at $p \in M$ to M one has the decomposition of Libermann [**5**]

$$T_p(M) = S_p(M) \oplus S_p^*(M)\,. \tag{1.1}$$

In (1.1) $S_p(M)$ and $S_p^*(M)$ are two self-orthogonal [**5**] (abr s.o.) sub-spaces defined by h_a $(1 \leqslant a \leqslant n)$ and h_{a^*} $(a^* = a + n)$ respectively. $\mathscr{U}$ is the paracomplex operator [**5**] and the pair (S_p, S_p^*) defines an involutive automorphism such that

$$\mathscr{U} h_a = h_a\,, \qquad \mathscr{U} h_{a^*} = -h_{a^*}\,. \tag{1.2}$$

The line element dp on $R(M)$ is

$$dp = \omega^A \oplus h_A \tag{1.3}$$

and $\mathscr{R}$ being normed one has $\langle h_a, h_b \rangle = \delta_{ab}$. The metric tensor g of M has the para-Hermitian form

$$g = 2\Sigma_a \omega^a \otimes \omega^{a^*} \tag{1.4}$$

and it is exchangeable with the symplectic form

$$\Omega = \Sigma_a \omega^a \wedge \omega^{a^*}\,. \tag{1.5}$$

If $\theta^A_B = l^A_{BC}\,\omega^C$ $(l^A_{BC} \in C^\infty(M))$ are the connection forms on $\mathscr{R}(M)$ and ∇ is the covariant derivation operator defined by g, the structure equations (F. Cartan) in compact notation are:

$$\nabla h = \theta \otimes h\,; \tag{1.6}$$

$$\mathrm{d}\wedge\omega = -\theta\wedge\omega\,, \quad \mathrm{d}\wedge\text{: exterior differentiation}\,; \tag{1.7}$$

$$\mathrm{d}\wedge\theta = -\theta\wedge\theta + \Theta\,, \quad \Theta\text{: curvature 2-forms}\,. \tag{1.8}$$

The para-Hermitian metric of M gives, making use of (1.6)

$$\theta^b_a + \theta^{a*}_{b*} = 0 \tag{1.9}$$

and the Kählerian structure of M implies that the connection matrix $\mathscr{M}_\theta$ is a Chern-Libermann matrix, that is

$$\mathscr{M}_\theta = \begin{pmatrix} \theta^a_b & 0 \\ 0 & \theta^{a*}_{b*} \end{pmatrix}. \tag{1.10}$$

2 – According to D'Atri and Nickerson [**1**] we give the following

Definition. We say that a manifold M has the *s.o. Killing property* if in the neighbourhood of each point $p \in M$ there exists a frame $\mathscr{R}(S_p, S^*_p)$ such that the null vectors of one or the s.o. sub spaces $S_p(M)$ or $S^*_p(M)$ are Killing vector fields. Suppose that the vector fields $h_a \in S_p(M)$ are Killing vector fields. This condition is expressed in intrinsic manner by

$$\langle \nabla_Z h_a, Z' \rangle + \langle \nabla_{Z'} h_a, Z \rangle = 0\,, \quad \forall Z, Z' \in T_p(M)\,. \tag{2.1}$$

Taking account of the metric (1.4) and making use of (1.6), equations (2.1) give

$$l^b_{ab} = 0\,, \quad l^a_{cb} + l^b_{ca} = 0\,, \quad l^a_{bc*} = 0, \ \forall C \tag{2.2}$$

and this implies, as it is known, $\operatorname{div} h_a = 0$. But if $\eta = \omega^1\wedge\ldots\wedge\omega^n\wedge\omega^{n+1}\wedge\ldots\wedge\omega^{2n}$ is the volume element on M, one has $\mathscr{L}_{h_{a*}}\eta = (\operatorname{div} h_{a*})\eta$ ($\mathscr{L}_X$: Lie derivative in the direction of the vector field $X \in T(M)$). With the help of (1.7) and (1.10) one finds

$$\operatorname{div} h_{a*} = \sum_b l^a_{bb*}\,, \tag{2.3}$$

and by (2.2) one has

$$\mathrm{div}\, h_{a*} = 0 \ . \tag{2.4}$$

So referring to [**1**] one may say that if a para-Kählerian manifold M has the s.o. Killing property then it has the *divergence property*. Moreover, by (1.10) and (2.2) one easily gets

$$\nabla_{h_{a*}} h_{b*} = 0 \qquad \forall a^*, b^* \in \{n+1, \ldots, 2n\} \ , \tag{2.5}$$

and this shows that S_p^* is *auto-parallel.*

In the following we shall call $S_p(M)$ and $S_p^*(M)$ the *Killing s.o. space* and the *auto-parallel s.o. space* respectively.

We shall now *point out some properties* of the Lie Algebra defined by the real vector space $X^* \in S_p^*$. By means of (1.7) and (2.2) one finds

$$\{\forall X^* \in S_p^* ; \quad \Sigma : \omega^b = 0 ; \quad b = 1, \ldots, n ; \quad \mathscr{L}_{X^*}\omega^b = 0\} \ . \tag{2.6}$$

The above equations prove that any vector field $X^* \in \cup S_p^*$ is an *isovector* of the Pfaffian system Σ (one may also say that X^* are local *invariant sections* of ω^b [**6**]).

Putting $\nabla_{h_a} h_a = -A_{h_a} h_a$ [**3**] one deduces from (1.6), (2.2) and (1.5)

$$\Omega(A_{h_a} h_a, A_{h_b} h_b) = 0 \ . \tag{2.7}$$

Hence the vector fields $A_{h_a} h_a$ are in *involution* with respect to the symplectic form Ω.

Let now

$$X = \sum_a t^a h_a \in S_p(M) \ , \quad t^a \in C^\infty(M) \ , \tag{2.8}$$

be any vector field of the s.o. Killing vector space $S_p(M)$.

Suppose that X is a Killing vector field on M. Taking account of (2.2) we get

$$t^a_{;b} + t^b_{;a} = 0 \ , \quad t^a_{;b^*} = 0 \ , \tag{2.9}$$

where; indicates the Pfaffian derivative. On the other hand, taking account of (2.2), equations (1.7) take for the covectors ω^{a^*} of the dual of $S_p^*(M)$ the following form

$$d\wedge\omega^{a^*} = \sum_b \big(\sum_c l^b_{ac}\omega^c\big)\wedge\omega^{b^*} \quad (c \neq b) \ . \tag{2.10}$$

Let $\mu: X \to i_X \Omega$ (i_X interior product) be the bundle isomorphism defined by Ω. Then if $X \in S_p(M)$ is the Killing vector field defined by (2.8), the necessary and sufficient condition for X to be a *local Hamiltonian* of Ω is

$$\mathrm{d} \wedge i_X \Omega = 0 . \tag{2.11}$$

Since $X \in S_p(M)$, equation (2.11) gives with the help of (2.9) and (2.10)

$$t^a_{;b} + \sum_c t^c l^a_{cb} = 0 \qquad (b \neq a) . \tag{2.12}$$

Consider now the covariant derivative ∇X of X. One has, as it is known, $\nabla X = \nabla(t^a h_a) = \mathrm{d}t^a \otimes h_a + t^a \nabla h_a$, and referring to (2.12) one finds after a short calculation

$$\mathrm{d} \wedge i_X \Omega = 0 \Leftrightarrow \nabla X = 0 . \tag{2.13}$$

Theorem. *Let $M(\Omega, g, \mathscr{U})$ be a para-Kählerian manifold having the s.o. Killing property and let $S_p(M)$ and $S_p^*(M)$ be the two s.o. components of the tangent space $T_p(M)$. Then:*

(i) *M has the divergence property;*

(ii) *if $S_p(M)$ is the Killing s.o. vector space then its complementary $S_p^*(M)$ is auto parallel;*

(iii) *any vector field $X^* \in S_p^*(M)$ is an isovector of the Pfaffian system defined by the covectors corresponding to $S_p(M)$;*

(iv) *if $X \in S_p(M)$ is any Killing vector field, then the following two properties are equivalent:* (a) *X is a local Hamiltonian of the symplectic structure $S_p(n, R)$ defined by Ω;* (b) *X is a parallel vector field.*

3 – Let $x: M_I \to M$ be the immersion of any *invariant* [**4**], [**7**]$_1$ submanifold M_I of $M(\mathscr{U} T_{x(p)}(M_I) = T_{x(p)}(M_I))$.

If M is any para-Kählerian manifold we have proved [**9**] that M_I is *minimal*. Suppose that M_I is defined by the Pfaffian system

$$\omega^r = 0 , \quad \omega^{r^*} = 0 \quad (r = 1, ..., p;\ r^* = r + n) . \tag{3.1}$$

(we shall denote by the same letters the elements induced by x). Referring to (2.2), the exterior differentiation of (3.1), quickly gives

$$l^i_{rj} = 0 \quad (i \neq j) \quad (i, j = p + 1, ..., n;\ i^* = i + n) . \tag{3.2}$$

Let then

$$
(3.3)\qquad \begin{aligned} l_r &= -\langle \mathrm{d}x(p), \nabla h_r\rangle = -\Sigma \omega^{i^*} \otimes \theta^i_r\,, \\ l_{r*} &= -\langle \mathrm{d}x(p), \nabla h_{r*}\rangle = \Sigma \omega^i \otimes \theta^r_i\,, \end{aligned}
$$

be the second quadratic forms associated with x. It is readily shown by (3.2) that all quadratic forms l_r vanish; consequently the normal sections $h_r \in S_p(M)$ are *geodesic*. Since $l_{r*} \neq 0$, we shall say that the immersion $x\colon M_I \to M$ is of *geodesic index r*.

Consider now an *antivariant* submanifold $M_A \subset M$ of maximal dimension n. According to the general definition [**11**], $[7]_2$ one has $\mathscr{U} T_{x(p)}(M_A) = T^{\perp}_{x(p)}(M_A)$, ($T^{\perp}_{x(p)}(M_A)$: normal space to M_A at $x(p) \in M_A$). So M_A is defined by the Pfaffian system

$$
(3.4)\qquad \omega^a = \omega^{a^*}\,.
$$

Making use of (1.7) we derive from (3.1)

$$
(3.5)\qquad \sum_a \omega^a \wedge \theta^a_b + \omega^a \wedge \theta^b_a = 0\,.
$$

If e_a is the vectorial basis of $T_{x(p)}(M_A)$, then the mean curvature vectorial from Θ associated with x is as it is known

$$
(3.6)\qquad \Theta = \Sigma(-1)^{a-1}\omega^1\wedge\ldots\wedge\hat{\omega}^a\wedge\ldots\wedge\omega^n \otimes e_a
$$

($\wedge$ indicates the missing term; we denote the elements induced by x with the same letters). If $\eta = \omega^1\wedge\ldots\wedge\omega^n$ is the volume element of M, then

$$
(3.7)\qquad \mathrm{d}\wedge\Theta = nH\otimes\eta\,,
$$

where H is the *mean curvature vector* associated with x. Taking the exterior differentiation of (3.6) and making use of (3.7) and (1.7), one obtains

$$
(3.8)\qquad nH = \sum_a \Big(\sum_b l^a_{bb} + l^b_{ab}\Big)\, n_a\,,
$$

where n_a are the normal sections on M_A. But taking account of (2.2) we get from (3.2) $l^a_{bb} = 0$, and since by (2.2) one has $l^b_{ab} = 0$ it follows $H = 0$. Hence M_A is a *minimal* submanifold of M.

In the third place consider the improper immersion $x\colon M_C \to M$, where M_C a hypersurface defined by

$$\omega^{a^*} = 0 . \tag{3.9}$$

In this case one easily sees that $h_a \subset T_{x(p)}(M_C) \cap T^{\perp}_{x(p)}(M_C)$.

Therefore, the tangent vector field h_a being in the normal space $T_{x(p)}(M_C)$, the hypersurface M_C is called *co-isotropic* [**8**] (M_a may be also considered as a CR-manifold [**7**]$_1$).

Exterior differentiation of (3.9) gives, taking account of (2.2),

$$l^b_{ac} = 0 , \quad b \neq a . \tag{3.10}$$

But h_a being a normal section, the second quadratic form associated with x is

$$l_a = \langle \mathrm{d}p, \nabla h_a \rangle = \Sigma \theta^b_a \otimes \omega^{b^*} , \quad b \neq a . \tag{3.11}$$

So by (3.10) it follows $l_a = 0$ and this shows that the improper immersion $x\colon M_C \to M$ is *geodesic.*

Further since the induced forms $\Omega|_{M_C}$ are

$$\Omega|_{M_C} = \omega^1 \wedge \omega^{1*} + \ldots + \widehat{\omega^a \wedge \omega^{a^*}} + \ldots + \omega^n \wedge \omega^{n^*} \tag{3.12}$$

($\wedge$: missing terms), one easily sees that $\Omega|_{M_C}$ is of constant *class* $2n-2$. On the other hand after a straight forward calculation one finds

$$\mathrm{d}\wedge\omega^a = \sum_{b<c} (l^a_{bc} - l^a_{cb}) \omega^b \wedge \omega^c \qquad (b, c \neq a) . \tag{3.13}$$

One derives

$$\omega^a \wedge (\mathrm{d}\wedge\omega^a) \neq 0 , \qquad \big(\Lambda(\mathrm{d}\wedge\omega^a)\big)^n = 0 , \tag{3.14}$$

and the above relations prove that ω^a is of class 3 [**2**]. Obviously one has $(\Lambda\Omega|_{M_C})^{n-1} \wedge \omega^a \neq 0$; hence $\Omega|_{M_C}$ and ω^a define an *almost cosymplectic structure* (i.e. in the sense of G-structures, $\{1 \times S_p(n-1, \boldsymbol{R})\}$).

Definition. Let $1 \times S_p(n, R)$ be an almost cosymplectic structure defined by a 2-form Ω and a 1-form ω. We say that the pair (Ω, ω) defines a *semi-*

cosymplectic structure iff: (i) Ω is closed; (ii) ω is of odd class; (iii) the canonical vector field h of $\{1\times S_p(n,\boldsymbol{R})\}$ is an infinitesimal automorphism of both Ω and ω.

Since in the case under discussion the canonical vector field associated with $1\times S_p(n-1,\boldsymbol{R})$ is h_a, one readily finds by (3.12) and (3.14)

$$\mathscr{L}_{h_a}\Omega|_{M_G}=0\,,\qquad \mathscr{L}_{h_a}\omega^a=0\,. \tag{3.15}$$

On the other hand in our case $h_a\in S_p(M)$ is a Killing vector field.

So according to (3.15) we shall say that the triple $(\Omega|_{M_G}\,\omega^a,\,h_a)$ defines a *K-semi cosymplectic structure.* Finally we shall consider the improper immersion $x\colon M_L\to M$, where M_L is a *Lagrangian submanifold* [**2**] of M (if $T_{x(p)}(M_L)$ is the tangent space at $x(p)\in M_L$, then $T_{x(p)}(M_L)=T^{\perp}_{x(p)}(M_L)$).

Clearly the s.o. subspaces $S_p(M)$ and $S^*_p(M)$ are Lagrangian planes (a Lagrangian plane L is by definition n-dimensional and such that $L=L^{\perp}$, $\Omega|_L=0$ [**2**]).

We agree to call $S_p(M)$ and $S^*_p(M)$ the *principal Lagrangian planes* associated with a para-Hermitian metric. Any other Lagrangian plane will be denominated a *mixed Lagrangian plane* (the corresponding tangent manifold is then a mixed Lagrangian submanifold). Let then $x\colon M_L\to M$ be the improper immersion of a mixed Lagrangian submanifold M_L in M. Assume that M_L is defined by

$$\omega^r=0\,,\quad \omega^{s^*}=0\quad (s^*\neq r^*)\,, \tag{3.16}$$

with $r=1,\dots,p$; $s=p+1,\dots,n$. Denote by $\Sigma_p(M)=\{h_i;\,i\neq r\}$ (resp. $\Sigma^*_p(M)=\{h^*_j;\,j^*\neq s^*\}$) the complementary sub-space of $\{h_r\}$ in $S_p(M)$ (resp. of $\{h_{j^*}\}$ in $S^*_p(M)$). The line element $dx(p)$ of M_L is expressed by

$$dx(p)=\omega^i\otimes h_i+\omega^{j^*}\otimes h_{j^*}\,. \tag{3.17}$$

Then since by definition h_i and h_{j^*} are normal sections, one finds by (1.6)

$$l_i=\sum_j\theta^j_i\otimes\omega^{j^*}\,, \tag{3.18}$$

$$l_{j^*}=-\sum_i\theta^j_i\otimes\omega^i\,. \tag{3.19}$$

Next taking the exterior differentiation of (3.16) one gets after a straightforward calculation $\theta^j_i=0$. Consequently the improper immersion x is *totally geodesic.*

Theorem. *Let $M(\Omega,g,\mathscr{U})$ be a para-Kählerian manifold having the s.o. Killing property. One has the fallowing proper and improper immersions in M:*

(i) *The proper immersion* $x: M_I \to M$, *where* M_I *is an invariant submanifold of* M. *In this case all normal sections of the s.o. Killing sub-space* $S_p(M)$ *are geodesic and if* $2r$ *is the codimension of* M_I, *the immersion* x *is of geodesic index* r.

(ii) *The proper immersion* $x: M_A \to M$, *where* M_A *is an anti-invariant submanifold of dimension* n. *This immersion is always minimal.*

(iii) *The improper immersion* $x: M_C \to M$, *where* M_C *is a co-isotropic hypersurface whose normal* h_a *is in the s.o. Killing sub-space* $S_p(M)$. *This immersion is geodesic.*

Moreover the normal covector ω^a *defines, with the restriction on* M_C *of* Ω, *a K-semicosymplectic structure.*

(iv) *The improper immersion* $x: M_L \to M$ *where* M_L *is a mixed Lagrangian submanifold of* M. *This immersion is totally geodesic.*

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Résumé

Soit $M(\Omega, g, \mathcal{U})$ une variètè para-Kählerienne et soient $S_p(M)$ et $S_p^*(M)$ les deux sous espaces self-orthogonaux (abr. s.o.) de la decomposition canonique du l'espace tangent $T_p(M) = S_p(M) \oplus S_p^*(M)$.

Si les champs vectoriels (isotropes) de la base de l'un de ces sous-espaces sont des champs de Killing, on dit que M possède la propriété de Killing self-orthogonale. Dans ce cas toute la variété M possède la propriété de la divergence et différentes propriétés ayant trait à l'algèbre de Lie sur M sont étudiées. On considère ensuite différentes types d'immersions propres et impropres dans M.

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