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Graph-covering and Ramsey's numbers by conversion matrices (**)

1 – Let be B_n the set of not-isomorphic simple graphs with n vertices, partially ordered by the relation $G_1 < G_2 \Leftrightarrow G_1$ is a subgraph of G_2 (in the terminology of Harary [**6**]); let's be $G \in B_n$ and $G_i \in B_n$, $i \in \{1, 2, ..., k\}$. Let's denote $G = (V, X)$ and $G_i = (V, X_i)$, utilizing, for sake of simplicity, the same set of vertices for all graphs of B_n. In the following we shall denote with (G, H) the number of subgraphs of H that are isomorphic to G.

In extremal graph theory (see Bollobas [**3**]) we can find two covering problems.

Cover 1: a vertex and an edge are said to cover each other if they are incident. A vertex-cover is a set of vertices covering all edges of a graph G. Vertex-covering-number of G, $\alpha_0(G)$, is the cardinality of a minimal vertex-cover. Analogously can be defined the edge-cover (a set of edges covering all vertices of G) and the edge-covering-number $\alpha_1(G)$.

Cover 2: a cover of G is a set $\{G_1, ..., G_k\}$ of subgraphs of G if $\bigcup_{i=1}^{k} X_i = X$.

We can now define from Cover 2 the following enumeration problem: « graph-covering-number » of G by $(G_1, G_2, ..., G_k)$ is the number of distinct k-uples $(G'_1, G'_2, ..., G'_k)$ that Cover 2 G such that $\forall i$, G'_i is isomorphic to G_i.

Remark. Two k-uples $(G'_1, ..., G'_k)$ and $(G''_1, ..., G''_k)$ are distinct if $\exists i \ni' G'_i \neq G''_i$.

Afterwards let's denote with $[G_1, G_2, ..., G_k/G]$ the graph-covering number of G by $(G_1, G_2, ..., G_k)$.

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If K_n is the complete graph with n vertices, we set

$$[G_1, \ldots, G_k/K_n] = [G_1, \ldots, G_k]\,. \tag{1}$$

If $\sum_{i=1}^{k} |X_i| = |X|$ we shall call the graph-covering-number the « graph-partition-number ». If $\hat{G}$ is the complementary graph of G, it is easy to show that

$$(G, K_n)[G_1, \ldots, G_k/G] = [G_1, \ldots, G_k, \hat{G}]\,. \tag{2}$$

In this paper we will show the solution of the graph-covering enumeration for complete graphs (and from (2) for any graph) by conversion matrices; our opinion is that such matrices, though hard to compute, could be useful tools for many problems in graph theory; in the last section we will show an application of our results to Ramsey-numbers.

2 - Conversion matrices and graph-covering

Let's write $B_n = \{g_1, \ldots, g_M\}$, where $M = |B_n|$, by a total extension of the partial order defined on B_n. We can construct R (n-th order conversion matrix), setting $R_{ij} = (g_i, g_j)$.

For the theory of conversion matrices see: Sykes [**8**], Domb [**5**], Kennedy and Gordon [**7**], Borzacchini [**4**].

If A is a square matrix, following Zykov [**9**], Barton and David [**2**], we can introduce the « graph symmetric function » of a graph H, with n vertices and m edges, relative to A as

$$[H]_A = \sum_{i_1 < \ldots < i_n} \; \sum_{\text{d.w.l.}\,H} A_{u_1 v_1} \ldots A_{u_m v_m}\,,$$

i.e. a function that comprises:

(i) an outer sum over all sets of n integers $(i_1, \ldots, i_n)$ with $i_1 < i_2 < \ldots < i_n \leqslant k$, where k is the dimension of A;

(ii) an inner sum, whose summand is $A_{u_1 v_1} \ldots A_{u_m v_m}$, in which there is one term for each distinguishable way of labelling (d.w.l.) the vertices of H with $i_1, \ldots, i_n$ and where $\{(u_1, v_1), \ldots, (u_m, v_m)\}$ is the set of edges of a graph isomorphic to H.

Obviously if $\tilde{G}$ is the adjacency matrix of a graph G, then

$$(g, G) = [g]_{\tilde{G}}, \tag{3}$$

where $[g]_{\tilde{G}}$ is the graph symmetric function of g relative to $\tilde{G}$.

Let's now denote with $A(G)$ the order of the automorphisms group of a graph G; if $P(G)$ is the number of distinguishable ways of labelling the n vertices of G, then

$$A(G) = \frac{n!}{P(G)} = \frac{n!}{(G, K_n)} \tag{4}$$

(see Harary [**6**]).

In Borzacchini [**4**] the following lemma has been demonstrated.

Lemma. *If* $g \in B_{n,m}$, *where* $B_{n,r}$ *is the set of not-isomorphic graphs with* n *vertices and* r *edges*

$$\sum_{G \in B_{n,r}} \frac{(g, G)}{A(G)} = \frac{1}{A(g)} \binom{\binom{n}{2} - m}{\binom{n}{2} - r}. \tag{5}$$

Hence we can show the following theorem.

Theorem. *If* $G_i \in B_n$, $i \in \{1, 2, ..., k\}$

$$\sum_{H \in B_n} \frac{(-1)^{m_H}}{A(H)} \prod_{i=1}^{k} (G_i, H) = \frac{(-1)^{\binom{n}{2}}}{n!} [G_1, ..., G_k].$$

Proof. Plainly any product of graph symmetric functions can be written as a linear sum of graph symmetric functions, where the coefficient of the R-graph symmetric function is the graph-covering-number of R (see also Barton and David [**2**]).

From (3) we have then

$$\prod_{i=1}^{k} (G_i, H) = \prod_{i=1}^{k} [G_i]_{\tilde{H}} = \sum_{R \in B_n} [G_i, ..., G_k/R][R]_{\tilde{H}}.$$

From (5), for any $R \neq K_n$

$$\sum_{H \in B_n} (-1)^{m_H} \frac{[R]_{\tilde{H}}}{A(H)} = \sum_{H \in B_n} (-1)^{m_H} \frac{(R, H)}{A(H)}$$

$$= \sum_{r=0}^{\binom{n}{2}} (-1)^r \sum_{H \in B_{n,r}} \frac{(R, H)}{A(H)} = \frac{1}{A(R)} \sum_{r=0}^{\binom{n}{2}} (-1)^r \binom{\binom{n}{2} - m_R}{\binom{n}{2} - r} = 0,$$

for $R = K_n$, $(K_n, H) = 1$ if $H = K_n$, $(K_n, H) = 0$ elsewhere and hence the thesis.

3 - Conversion matrices and Ramsey's numbers

Our theorem can be applied to any existence-problem of a cover (or partition) of a graph by graphs, or to any enumeration-problem of such covers (or partitions).

An interesting example can be the theory of Ramsey's numbers $N(k; l_1, ..., l_r)$ (see Aigner [1]).

Let's be S and n-set (i.e. a set with $|S| = n$) and $S^{(k)}$ the set of all k-subsets of S. $N(k; l_1, ..., l_r)$ is the smallest integer, depending only on k, r, $l_1, ..., l_r$, such that $\forall n \geqslant N(k; l_1, ..., l_r)$, if $A_1, ..., A_r$ is a partition of $S^{(k)}$, then exists an l_i-subset T of S such that $T^{(k)} \subseteq A_i$.

The simplest case is $k = 1$. When $k = 2$ the problem admits a convenient interpretation in graph theory: S can be the set of vertices and $S^{(2)}$ the set of edges of K_n. The partition $A_1, ..., A_r$ is an r-coloring of the edges; then $N(2; l_1, ..., l_r)$ is the smallest integer such that, $\forall n \geqslant N(2; l_1, ..., l_r)$, for any r-coloring of K_n, for some i there is a complete monochromatic subgraph on l_i vertices.

In other terms: if $G_1, ..., G_r$ are the subgraphs of K_n induced by the partition $A_1, ..., A_r$, there must be an i such that K_{l_i} is a subgraph of G_i, i.e. $(K_{l_i}, G_i) \neq 0$; because this condition must come true for any r-coloring, we have

$$\prod_{\substack{G_1, ..., G_r \\ G_i \in B_n}} ([G_1, ..., G_r] \cdot \sum_{j=1}^{r} (K_{l_j}, G_j)) \neq 0 ,$$

where the first product is over all r-uples $(G_1, ..., G_r)$ such that $\forall i \; G_i \in B_n$ and $\sum_{i=1}^{r} |X_i| = \binom{n}{2}$ (i.e. $(G_1, ..., G_r)$ is a graph-partition of K_n).

Hence, from (6) we have

Corollary 1. *$N(2; l_1, ..., l_r)$ is the smallest integer n such that*

$$\prod_{\substack{G_1, ..., G_r \\ G_i \in B_n}} \Big(\sum_{H \in B_n} \frac{(-1)^{m_H}}{A(H)} \prod_{i=1}^{r} (G_i, H)\Big)\Big(\sum_{j=1}^{r} (K_{e_j}, G_j)\Big) \neq 0 .$$

A natural generalization of Ramsey's problem for $k = 2$ is the question whether, if $(L_1, ..., L_r)$ is an r-uple of graphs with n vertices and if G is a graph with n vertices, for any r-coloring of G, for some i there is a monochromatic subgraph (of G) isomorphic to L_i. We can say that the answer is affirmative if and only if

$$\prod_{\substack{G_1, ..., G_r \\ G_i \in B_n}} [G_1, ..., G_r / G] \cdot \sum_{j=1}^{r} (L_j, G_j) \neq 0 .$$

From (2) and (6) we can write

Corollary 2. *For any r-coloring of G, for some i there is a monochromatic subgraph isomorphic to L_i if and only if*

$$\prod_{\substack{G_1, \ldots, G_r \\ G_i \in B_n}} \Big(\sum_{H \in B_n} \frac{(-1)^{m_H}}{A(H)} \prod_{i=1}^{r} (G_i, H) \cdot (\hat{G}, H) \Big) \Big(\sum_{j=1}^{r} (L_j, G_j) \Big) \neq 0 \, .$$

Formulae of our corollaries are hard to compute, but Ramsey's numbers are really hard to compute. Corollary gives an answer for a generalization of Ramsey's problem that includes other proposed generalizations (see Aigner [**1**]).

These examples show that our theorem allows a general answer to the question: it's possible to cover a graph utilizing a set of graphs defined up to isomorphism, i.e. whose labelling is free? And, if the answer is affirmative: in how many ways it's possible such a cover? So the conversion matrices can be a central tool for graph theory, but their quick calculus is up today an open problem.

References

[**1**] M. Aigner, *Combinatorial theory*, Springer-Verlag, Berlin 1979.

[**2**] D. E. Barton and F. N. David, *The random intersection of two graphs*, « Research papers in statistics », F. N. David ed., J. Wiley 1966.

[**3**] B. Bollobas, *Extremal graph theory*, Academic Press, London 1978.

[**4**] I. Borzacchini, *Lattice constants. Matrical equations for conversion matrices*, Pubbl. IAC (3) **170** (1979).

[**5**] C. Domb, *Graph theory and embeddings*, « Phase transitions and critical phenomena », III Domb and Green eds., Academic Press 1974.

[**6**] F. Harary, *Graph theory*, Addison Wesley, Reading 1969.

[**7**] J. W. Kennedy and M. Gordon, *Graph contractions and a generalized Möbius inversion*, in « Second international conference on Combinatorial Mathematics », A. Gewitz and L. V. Quintas eds., The N. York Academy of Sciences, N. Y. 1979.

[**8**] M. F. Sykes, J. W. Essam, B. R. Heap and B. J. Hiley, *Lattice constants systems and graph theory*, J. Math. Phys. **7** (1966), 1557.

[**9**] A. A. Zykov, *Algebras of complexes*, Mat. Sb. N. S. (41) **159** (1957) (trans. E. Buchtenkirch, Trans. Amer. Math. Soc. (2) **15** (1960), 15-32).

Summary

If $G=(V,X)$ *and* $G_i=(V,X_i)$, $i\in\{1,2,\ldots,k\}$, *are graphs with* n *vertices, a cover of* G *is a set* $\{G_1,\ldots,G_k\}$ *of subgraphs of* G *if* $\bigcup_{i=1}^{k} X_i = X$. *Let's define « graph-covering-number » of* G *by* $G_1,\ldots,G_k$ *as the number of distinct* k*-uples* $(G'_1,\ldots,G'_k)$ *that cover* G *such that* $\forall i$: G'_i *is isomorphic to* G_i. *In this paper we show a theorem linking the graph-covering-numbers to conversion matrices and then some connections between Ramsey's numbers and conversion matrices.*

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