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On the structure of classical dynamics (**)

Introduction

In the last years a new interest in an axiomatic and geometric approach to mechanics has been taken by several authors.

This paper gives a contribution in this direction, developing the point of view of connections, which seems to have some new aspects.

In particular the present approach to classical mechanics turns out to be very close to special and general relativity.

List of symbols

$\pi: TM \to M$ is the canonical projection,
$\varkappa \equiv T\pi: T^2M \to TM$,
$s: T^2M \to T^2M$ is the symmetrical endomorphism,
$\hat{}$ denotes the lift with respect to the tangent functor,
if $f: \boldsymbol{R} \to M$, then $\mathrm{d}f \equiv Tf_1: \boldsymbol{R} \to TM$,
if $f: M \to \boldsymbol{R}$, then $\dot{f} = \pi^2 \circ Tf: TM \to \boldsymbol{R}$,
T_i and d_i denote the i-th partial tangent map and differential.

1 - Absolute kinematics

1.1 – *The classical kinematical framework.* First we introduce the general kinematical framework for the classical mechanics.

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Definition. A *time bundle* is a 3-plet $\eta \equiv (E, t, \boldsymbol{T})$, where E and $\boldsymbol{T}$ are affine spaces, with dimension 4 and 1, respectively, and $t\colon E \to \boldsymbol{T}$ is an affine surjective map.

E and $\boldsymbol{T}$ are called the *event space* and the *time*.

We denote by $\bar{E}$ and $\bar{\boldsymbol{T}}$ the vector spaces of E and $\boldsymbol{T}$, respectively, and by j the inclusion $j \equiv (t, \mathrm{id}_E)\colon E \hookrightarrow \boldsymbol{T} \times E$.

η is an affine trivial bundle, but not a canonical product. The fibers $S_\tau \equiv t^{-1}(\tau)$, for $\tau \in \boldsymbol{T}$, are parallel affine subspaces of E, with dimension 3, whose vector space is $S \equiv \mathrm{D}t^{-1}(0)$. A vector $u \in \bar{E}$ is *space-like* or *time-like*, according as $u_0 \equiv \mathrm{Dt}(u) \underset{\neq}{=} 0$.

The sign $\check{}$ will denote the vertical quantities and operations with respect to the bundle η.

Definition. A *space-like conformal metric* is a euclidean conformal metric $\{\underset{\sim}{g}\}$ on S. A *time orientation* is an orientation θ on $\boldsymbol{T}$.

A time-like vector $u \in \bar{E}$ is *future* or *past-oriented*, according as $u^0 \gtrless 0$.

Definition. A *classical kinematical framework* is a 3-plet $ckf \equiv (\eta, \{g\}, \theta)$ as above.

Henceforth we assume such a *ckf* to be given.

A *Poincare's map* is an automorphism $G\colon E \to E$ of *ckf* and its derivative $\mathrm{D}G\colon \bar{E} \to \bar{E}$ is the associated *Galilei's map*.

The choice of a *space-like measure unity* is the choice of a metric $g\colon S \otimes S \to \boldsymbol{R}$, among $\{g\}$. The choice of a *time-like measure unity* is the choice of a metric $\mathring{g}\colon \bar{\boldsymbol{T}} \otimes \bar{\boldsymbol{T}} \to \boldsymbol{R}$.

Henceforth we assume such unities to be choosen.

We denote by $\tilde{g}\colon S^* \otimes S^* \to \boldsymbol{R}$ the contravariant metric tensor and we put $\underset{\sim}{u} \equiv \underset{\sim}{g}(\tilde{u})$, $\tilde{u} \equiv \tilde{g}(\underset{\sim}{u})$ and $u \cdot v \equiv \underset{\sim}{g}(\tilde{u}, \tilde{v}) \equiv \tilde{g}(\underset{\sim}{u}, \underset{\sim}{v})$. We denote by $\underset{\sim}{\check{o}}\colon S \otimes S \otimes S \to \boldsymbol{R}$ the space-like metric volume form, by $*\colon u \mapsto i_u \underset{\sim}{\check{o}}$ the Hodge isomorphism and by $(u, v) \mapsto u \times v \equiv i_{u \wedge v} \underset{\sim}{\check{o}}$ the wedge product (with respect to a choosen orientation of S).

$\mathring{g}$ and θ characterize a unitary future oriented vector $\mathring{u} \in \bar{\boldsymbol{T}}$, hence they determine an isomorphism $\bar{\boldsymbol{T}} \to \boldsymbol{R}$. Henceforth we will identify $\bar{\boldsymbol{T}}$ and $\boldsymbol{R}$.

We get a canonical volume form on $\bar{E}$, namely $\underset{\sim}{o} \equiv \mathrm{D}t \wedge \underset{\sim}{\check{o}}'$, where $\underset{\sim}{\check{o}}'$ is any extension of $\underset{\sim}{\check{o}}$.

We denote by U the affine subspace $U \equiv \mathrm{D}t^{-1}(1)$ of $\bar{E}$, whose vector space is S.

A *special chart* is a chart $x \equiv (x^\alpha) \equiv (x^0, x^i)\colon V \subset E \to \boldsymbol{R} \times \boldsymbol{R}^3$, such that $\mathrm{D}x^0 = \mathrm{D}t$. Henceforth x will denote a special chart.

As we are concerned with the affine space E, we could utilize only the « free » vector space $\bar{E}$, for the most purposes. But, in order to get a treatment easily extendible to the constrained mechanics, we can use equivalently the « applied » vector space $TE = E\times\bar{E}$. We shall denote by $\tilde{u}$ and $\bar{u}$ (or $\underset{\sim}{u}$ and $\underline{u}$) the « free » and « applied » form of a tensor, related by the affine structure of E.

We shall be concerned with the following spaces:

the *phase-space* $\check{T}E \equiv E\times S$, the *cophase-space* $\check{T}^*E \equiv E\times S^*$,
the *unitary space* $\dot{T}E \equiv E\times U$; $\mathring{T}^*E \equiv \big((e, r\mathrm{D}t)\,|\,e\in E,\ r\in\boldsymbol{R}\big)$,
$\check{T}^2E \equiv E\times S\times S\times S$, $\check{T}\check{T}^*E \equiv E\times S^*\times S\times S^*$, $T\dot{T}E \equiv E\times U\times E\times S$,
$\check{T}\dot{T}E = E\times U\times S\times S$, $\dot{T}^2E \equiv E\times U\times U\times S$.

We have the natural inclusions

$\check{T}E$; $\dot{T}E \hookrightarrow TE$; $\check{T}^2E$, $T\dot{T}E$, $\check{T}\dot{T}E$, $\dot{T}^2E \hookrightarrow T^2E$; $\check{T}\check{T}^*E \hookrightarrow T\check{T}^*E$; $\mathring{T}^*E \hookrightarrow T^*E$ and the natural projection $T^*E \to \check{T}^*E$, denoted by $u \mapsto \check{u}$.

Moreover, let us remark that $\dot{T}E$ is the jet space of sections of the bundle η.

1.2 – *Kinematical absolute structures.* We shall be concerned with some structures related to the bundle η.

The space-like metrical structure of $\check{T}E$ is characterized by each one of the following quantities.

Definition. The *space-like*

metric tensor is $g\colon \check{T}E\otimes\check{T}E \to \boldsymbol{R}$, $u\otimes v \mapsto u\cdot v$,
metric map is $g\colon \check{T}E \to \check{T}^*E$, $\bar{u} \mapsto u$,
metric function is $g\colon \check{T}E \to \boldsymbol{R}$, $u \mapsto \frac{1}{2}u\cdot u$,
metric form is $\check{\mathrm{d}}_v g\colon \check{T}^2E \to \boldsymbol{R}$, $(e, u, v, w) \mapsto u\cdot v$.

Proposition. One has

$$g = g_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j, \qquad \dot{x}_i\circ g = g_{ij}\dot{x}^j = \partial x_i\cdot g\,,$$

$$g = \tfrac{1}{2}g_{ij}\dot{x}^i\dot{x}^j, \qquad \check{\mathrm{d}}_v g = g_{ij}\dot{x}^i\,\mathrm{d}x^j, \qquad \check{o} = \pm\,(\det(g_{ij}))^{\frac{1}{2}}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3.$$

The metrical map determines analogous quantities on $\check{T}^*E$, which will be denoted by *. In particular, it associates the metric form with a canonical form, namely, with the

space-like Liouville's form $\check{\lambda}\colon \check{T}\check{T}^*E \to \boldsymbol{R}$, $(e, \underset{\sim}{u}, \tilde{v}, \underset{\sim}{w}) \mapsto \langle \underset{\sim}{u}, \tilde{v}\rangle$.

Let $B \to V$ be an affine bundle, whose vector bundle is $\bar{B} \to V$. A pseudo-connection (a connection) is a map $C: TB \to \bar{B}$, which is an affine morphism over $B \times_V TV \to V$ and whose fiber derivatives on each fiber are 1 (and moreover which is a linear morphism over $B \to V$).

The affine connection $\Gamma: T^2E \to TE$, $(e, u, v, w) \mapsto (e, w)$, induces, by restriction, the following maps.

Definition. The *space-like connection* is $\check{\Gamma}: \check{T}^2E \to \check{T}E$, $(e, u, v, w) \mapsto (e, w)$; the *inertial connection* is $\dot{\Gamma}: T\dot{T}E \to \check{T}E$, $(e, u, v, w) \mapsto (e, w)$.

If $u, v: E \to TE$ and $a, b: E \to \check{T}E$ are vector fields, then the *covariant derivative* is $\nabla_u v \equiv \Gamma \circ Tv \circ u: E \to TE$ and the *space-like covariant derivative* is $\check{\nabla}_a b \equiv \check{\Gamma} \circ \check{T}b \circ a: E \to \check{T}E$, which can be in a natural way extended to tensors.

Proposition. One has $\check{\nabla} g = 0$. Moreover, one has $\Gamma^0_{\alpha\beta} = 0$,

$$\dot{x}^i \circ \check{\Gamma} = \ddot{x}^i + \Gamma^i_{hk} \hat{\dot{x}}^h \dot{x}^k,$$

$$\dot{x}^i \circ \dot{\Gamma} = \ddot{x}^i + \Gamma^i_{hk} \hat{\dot{x}}^h \dot{x}^k + \Gamma^i_{h0} \hat{\dot{x}}^h \dot{x}^0 + \Gamma^i_{0k} \dot{x}^k + \Gamma^i_{00} \dot{x}^0,$$

$$\gamma_{i,hk} = \tfrac{1}{2}(\partial_h g_{ik} + \partial_k g_{ih} - \partial_i g_{hk}), \qquad \gamma_{j,0i} + \gamma_{i,0j} = \partial_0 g_{ij},$$

where $\gamma_{i,\alpha\beta} \equiv g_{ih} \Gamma^h_{\alpha\beta}$. The metrical map determines a connection $\check{\Gamma}^*$ on $\check{T}^*E$.

Let us remark that we could consider the further map $\urcorner: T^2E \to \check{T}E$, $(e, u, v, w) \mapsto (e, u^0 w - w^0 u)$, which reduces to $\dot{\Gamma}$ on $T\dot{T}E$.

Let V be a $2n$ manifold. A symplectic form is an exterior 2-form s, such that $ds = 0$ and s^n is a volume form.

Let V be a $2n+1$ manifold (and let θ be a closed 1-form). A contact (pseudo-contact) form is an exterior 2-form c, such that $dc = 0$ $(\theta \wedge dc = 0)$ and $\gamma \wedge c^n$ $(\theta \wedge c^n)$, where $d\gamma = c$, is a volume form.

We define the map $\nu: T^2E \to \check{T}E$, $(e, u, v, w) \mapsto (e, u_0 v - v_0 u)$, whose restriction to $T\dot{T}E$ is denoted by $\dot{\nu}$.

Definition. The *space-like form* is $\check{\psi} \equiv g \circ (\check{\Gamma} \wedge \varkappa): \check{T}^2E \times_{\check{T}E} \check{T}^2E \to \boldsymbol{R}$, the *inertial form* is $\dot{\psi} \equiv g \circ (\dot{\Gamma} \wedge \dot{\nu}): T\dot{T}E \times_{\dot{T}E} T\dot{T}E \to \boldsymbol{R}$; and moreover $\varphi \equiv g \circ (\check{\Gamma} \wedge (\varkappa - \pi)): \check{T}^2E \times_{\check{T}E} \check{T}^2E \to \boldsymbol{R}$.

Proposition. $\check{\psi}$ is the space-like symplectic form induced by the metric, namely $\check{\psi} = \check{d}\check{d}_v g$.

One has $\mathrm{d}\dot{\psi} = 0$ and $\mathrm{d}t\wedge\dot{\psi}\wedge\dot{\psi}\wedge\dot{\psi}$ is a volume form (connected with $\check{o}$). We get

$$\check{\psi}(e, u; v, w; a, b) = w\cdot a - b\cdot v\,,$$

$$\varphi(e, u; v, w; a, b) = w\cdot(a - u) - b\cdot(v - u)\,,$$

$$\dot{\psi}(e, u; v, w; a, b) = w\cdot(a - a^0 u) - b\cdot(v - v^0 u)\,,$$

$$\check{\psi} = g_{ij}(\check{\mathrm{d}}\dot{x}^i + \Gamma^i_{hk}\dot{x}^h\check{\mathrm{d}}x^k)\wedge\mathrm{d}x^j = g_{ij}\check{\mathrm{d}}\dot{x}^i\wedge\check{\mathrm{d}}x^j + \partial_i g_{jh}\dot{x}^h\check{\mathrm{d}}x^i\wedge\check{\mathrm{d}}x^j,$$

$$\varphi = g_{ij}(\check{\mathrm{d}}\dot{x}^i + \Gamma^i_{hk}\dot{x}^h\check{\mathrm{d}}x^k)\wedge\check{\mathrm{d}}x^j + g_{ij}\dot{x}^i(\check{\mathrm{d}}\dot{x}^i + \Gamma^i_{hj}\dot{x}^h\check{\mathrm{d}}x^j)\circ(\pi^2 - \pi^1)$$

$$= g_{ij}\check{\mathrm{d}}\dot{x}^i\wedge\check{\mathrm{d}}x^j + \partial_i g_{hk}\dot{x}^h\check{\mathrm{d}}x^i\wedge\check{\mathrm{d}}x^k + (g_{ij}\dot{x}^i\check{\mathrm{d}}x^j + \tfrac{1}{2}\partial_j g_{hk}\dot{x}^h\dot{x}^k\check{\mathrm{d}}x^j)\circ(\pi^2 - \pi^1)\,,$$

$$\dot{\psi} = g_{ij}(\mathrm{d}\dot{x}^i + \Gamma^i_{hk}\dot{x}^h\mathrm{d}x^k + \Gamma^i_{0k}\mathrm{d}x^k + \Gamma^i_{h0}\dot{x}^h\mathrm{d}x^0 + \Gamma^i_{00}\mathrm{d}x^0)\wedge(\mathrm{d}x^j - \dot{x}^j\mathrm{d}x^0)$$

$$= g_{ij}\,\mathrm{d}\dot{x}^i\wedge\mathrm{d}x^j + g_{ih}\dot{x}^h\mathrm{d}x^0\wedge\mathrm{d}x^i + (\partial_i g_{jh}\dot{x}^h + \gamma_{j,0i})\,\mathrm{d}x^i\wedge\mathrm{d}x^j$$

$$+ (\tfrac{1}{2}\partial_i g_{hk}\dot{x}^h\dot{x}^k + \partial_0 g_{hi}\dot{x}^h + \gamma_{i,00})\,\mathrm{d}x^0\wedge\mathrm{d}x^i,$$

$$\check{\psi}\wedge\check{\psi}\wedge\check{\psi} = 3\,!\det(g_{ij})\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}\dot{x}^1\wedge\mathrm{d}\dot{x}^2\wedge\mathrm{d}\dot{x}^3,$$

$$\mathrm{d}t\wedge\dot{\psi}\wedge\dot{\psi}\wedge\dot{\psi} = 3\,!\det(g_{ij})\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}\dot{x}^1\wedge\mathrm{d}\dot{x}^2\wedge\mathrm{d}\dot{x}^3.$$

The metric map associates with $\check{\psi}$ the canonical space-like symplectic form $\check{\mathrm{d}}\check{\lambda}$ of $\check{T}^*E$.

Let us remark that we could consider the further map $\psi \equiv g\circ(\neg\wedge\nu)$: $T^2E\times_{TE}T^2E \to \boldsymbol{R}$, which reduces to Γ on $T\dot{T}E\times_{\dot{T}E}T\dot{T}E$.

Let X: $\dot{T}E \to T\dot{T}E$ (or X: $\check{T}E \to \check{T}^2E$) be a section. Then the following conditions are equivalent:

(a) if c: $T \to \dot{T}E$ (or c: $\boldsymbol{R} \to \check{T}E$) is a section (a map such that $t\circ c$ is constant), such that $X\circ c = \mathrm{d}c$, then $c = \mathrm{d}(\pi\circ c)$,

(b) $s\circ X = X$.

Definition. The *space-like equation* is $\check{X}$: $\check{T}E \to \check{T}^2E$, $(e, u) \mapsto (e, u, u, 0)$, the *inertial equation* is $\dot{X}$: $\dot{T}E \to T\dot{T}E$, $(e, u) \mapsto (e, u, u, 0)$.

Proposition. One has

$$\check{X} = \dot{x}^i\partial x_i - \Gamma^i_{hk}\dot{x}^h\dot{x}^k\partial\dot{x}_i\,, \qquad \dot{X} = \partial x_0 + \dot{x}^i\partial x_i - (\Gamma^i_{hk}\dot{x}^h\dot{x}^k + 2\Gamma^i_{0h}\dot{x}^h + \Gamma^i_{00})\partial\dot{x}_i.$$

Moreover, $\check{X}$ and $\dot{X}$ are the sections characterized, respectively, by each one of the following conditions:

(a) $s\circ\check{X}=\check{X}\,,\quad \check{\Gamma}\circ X=0\,,$ (a′) $s\circ\dot{X}=\dot{X}\,,\quad \dot{\Gamma}\circ\dot{X}=0\,,$

(b) $i_{\check{X}}\check{\psi}=-\check{\mathrm{d}}g\,,$ (c′) $i_{\dot{X}}\mathrm{d}t=1\,,\quad i_{\dot{X}}\dot{\psi}=0\,.$

(c) $i_{\check{X}}\varphi=0\,,$

1.3 – *The one-body absolute kinematics.*

Definition. A *motion* is a section $M\colon \boldsymbol{T}\to E$ and its *world-line* is $M(\boldsymbol{T})\hookrightarrow E$.

The *velocity* of M is $\mathrm{d}M\equiv TM_1=(M,\mathrm{D}M)\colon \boldsymbol{T}\to \dot{T}E$.

The *acceleration* of M is $\dot{\nabla}\,\mathrm{d}M\equiv\dot{\Gamma}\circ\mathrm{d}^2M=(M,\mathrm{D}^2M)\colon \boldsymbol{T}\to\check{T}E$.

M is characterized by its world-line, which is a one dimensional submanifold of E, meeting each S_τ just in one point.

Proposition. One has

$$\mathrm{d}M=\partial x^0\circ M+\mathrm{D}M^i(\partial x_i\circ M)\,,$$

$$\dot{\nabla}\,\mathrm{d}M=(\mathrm{D}^2M^i+\Gamma^i_{hk}\circ M\,\mathrm{D}M^h\,\mathrm{D}M^k+2\Gamma^i_{0k}\circ M\,\mathrm{D}M^k+\Gamma^i_{00}\circ M)(\partial x_i\circ M)\,.$$

A motion M is *inertial* if it is an affine map, i.e. if $\dot{\nabla}\,\mathrm{d}M=0$.

If $0\colon \boldsymbol{T}\to E$ is a motion, then $\delta 0$ will denote the map

$$\delta 0\colon E\to\check{T}E\,,\qquad e\mapsto\big(e-O(t(e))\big)\,.$$

1.4 – *The absolute kinematics of a continuum.*

Definition. *A continuum of world-lines* is a family $\mathscr{P}\equiv(\boldsymbol{T}_q)_{q\in\boldsymbol{P}}$ of world lines, such that

(a) $E=\bigcup\limits_{q\in\boldsymbol{P}}\boldsymbol{T}_q$;

(b) if $q\neq q'$, then $\boldsymbol{T}_q\cap\boldsymbol{T}_{q'}=\emptyset$;

(c) there exists an atlas A of C^∞ charts $x\equiv(x_0,x^i)\colon V\subset E\to\boldsymbol{R}\times\boldsymbol{R}^3$ adapted to each world-line of $\mathscr{P}$ in V.

The previous definition is easily extendible to local continua of world-lines. A special chart determines (at least locally) a continuum of world-lines.

Henceforth we assume a continuum of world-lines to be given.

We consider the map $p: E \to \boldsymbol{P}$, $e \mapsto$ (the unique $q \in \boldsymbol{P}$, such that $e \in \boldsymbol{T}_q$).

Proposition. There is a unique C^∞ structure on $\boldsymbol{P}$, such that p is C^∞. More precisely, it is induced by the maps $(x^i): V \to \boldsymbol{R}^3$, associated with the atlas A, which factorize through p.

Moreover, the atlas A can be reduced to special charts.

Henceforth, when we are concerned with $\mathscr{P}$, will be a special chart, adapted to $\mathscr{P}$, and we shall denote the chart induced on $\boldsymbol{P}$, by the same notation $(x^i): \boldsymbol{P} \to \boldsymbol{R}^3$.

Definition. The *motion* of $\mathscr{P}$ is the map $P: \boldsymbol{T} \times \boldsymbol{P} \to E$, $(\tau, q) \mapsto$ (the unique $e \in S_\tau \cap \boldsymbol{T}_q$), or the map $\mathring{P} = P \circ (\mathrm{id}_{\boldsymbol{T}} \times p): \boldsymbol{T} \times E \to E$, or the family of maps $\mathring{P}_{(\tau,\sigma)} = \mathring{P}_{\tau \restriction S_\sigma}: S_\sigma \to S_\tau$.

Proposition. Each one of P, $\mathring{P}$ and $\mathring{P}_{(\tau,\sigma)}$ characterizes $\mathscr{P}$. Moreover, one has

$$t \circ P = \pi^1, \qquad p \circ P = \pi^2,$$

$$t \circ \mathring{P} = \pi^1, \qquad \mathring{P} \circ j = \mathrm{id}_E, \qquad \mathring{P} \circ (\mathrm{id}_{\boldsymbol{T}} \times \mathring{P}) = \mathring{P} \circ (\mathrm{id}_{\boldsymbol{T}} \times \pi^2),$$

$$\mathring{P}_{(\tau'',\tau')} \circ \mathring{P}_{(\tau',\tau)} = \mathring{P}_{(\tau'',\tau)}, \qquad \mathring{P}_{(\tau,\tau)} = \mathrm{id}_{S_\tau},$$

$$P_\sigma = \mathring{P}_{(\sigma,\tau)} \circ P_\tau, \qquad p_\tau \circ P_\tau = \mathrm{id}_{\boldsymbol{P}}, \qquad P_\tau \circ p_\tau = \mathrm{id}_{S_\tau}.$$

Definition. We define the following quantities relative to $\mathscr{P}$.

The *velocity*	is	$\overline{P} \equiv \mathrm{d}_1 \mathring{P} \circ j: E \to \dot{T}E$,
the *acceleration*	is	$\overline{\overline{P}} \equiv \mathrm{d}_1^2 P \circ j: E \to \check{T}E$,
the *strain*	is	$\varepsilon_{\mathscr{P}} \equiv \mathrm{sym}\, \dot{\nabla} \overline{P}: E \to \check{T}^*E \otimes \check{T}E$,
the *spin*	is	$\omega_{\mathscr{P}} \equiv \mathrm{antisym}\, \check{\nabla} \overline{P}: E \to \check{T}^*E \otimes \check{T}E$,
the *angular velocity*	is	$\Omega_{\mathscr{P}} \equiv \frac{1}{2} * \omega_{\mathscr{P}}: E \to \check{T}E$.

Proposition. $\overline{P}$ is a complete field, which characterizes $\mathscr{P}$. Moreover, one has

$$T_2 \mathring{P} \circ \hat{j}: TE \to \check{T}E, \qquad (e, u) \mapsto (e, \check{u}_{\mathscr{P}}(e)); \qquad T_2^2 \mathring{P} \circ \hat{j}: T^2E \to \check{T}^2E,$$

$$(e, u, v, w) \mapsto \big(e, \check{u}_{\mathscr{P}}(e), \check{v}_{\mathscr{P}}(e), \check{w}_{\mathscr{P}}(e) - u^0 \check{\mathrm{D}}\widetilde{P}(e)(\check{v}_{\mathscr{P}}(e))$$
$$- v^0 \check{\mathrm{D}}\widetilde{P}(e)(u_{\mathscr{P}}(e)) - \check{u}^0 v^0 \widetilde{\widetilde{P}}(e)\big),$$

where $\check{u}_{\mathscr{P}}(e) \equiv u - u \circ \tilde{P}(e)$;

$$\overline{P} = \nabla_{\overline{P}} \overline{P}; \qquad \varepsilon_{\mathscr{P}} = L_{\overline{P}}\, g; \qquad \overline{P} = \partial x_0;$$

$$\overline{P} = \Gamma^i_{00} \partial x_i; \qquad \check{\nabla} \overline{P} = \Gamma^j_{i0}\, \mathrm{d}x^i \otimes \partial x_j; \qquad \varepsilon_{\mathscr{P}} = \partial_0 g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j;$$

$$\omega_{\mathscr{P}} = (\gamma_{j,0i} - \gamma_{i,0j})\, \mathrm{d}x^i \otimes \mathrm{d}x^j; \qquad \Omega_{\mathscr{P}} = \tfrac{1}{2} (\det(g_{ij}))^{\frac{1}{2}} \varepsilon^{kij} \gamma_{j,0i} \partial x_k .$$

Definition. $\mathscr{P}$ is *affine* if $\check{\nabla}^2 \overline{P} = 0$. $\mathscr{P}$ is *rigid* if $\check{\nabla}^2 \overline{P} = 0$ and $\varepsilon_{\mathscr{P}} = 0$. $\mathscr{P}$ is *translating* if $\nabla \overline{P} = 0$. $\mathscr{P}$ is *inertial* if $\nabla \overline{P} = 0$.

Proposition. The following conditions are equivalent:

(a) $\check{\nabla}^2 \overline{P} = 0$, (a′) $\check{\nabla} \overline{P}$ is factorizable through $E \to \boldsymbol{T}$,

(b) $\check{\mathrm{D}}_2^2 \mathring{P} = 0$, (b′) $\check{\mathrm{D}}_2 \mathring{P}$ is factorizable through $\boldsymbol{T} \times E \to \boldsymbol{T} \times \boldsymbol{T}$,

(c) the map $\mathring{P}_{(\sigma,\tau)}\colon S_\tau \to S_\sigma$ is affine, i.e.

$$\mathring{P}_\sigma(e') = \mathring{P}_\sigma(e) + \mathrm{D}\mathring{P}_{(\sigma,\tau)}(e' - e),$$

(d) the map $\tilde{P}_{|S_\tau}$ $S_\tau \to U$ is affine, i.e.

$$\tilde{P}(e') = \tilde{P}(e) + \tfrac{1}{2} \varepsilon_{\mathscr{P}}(\tau)(e' - e) + \Omega_{\mathscr{P}}(\tau) \times (e' - e) .$$

The following conditions are equivalent:

(a) $\check{\nabla}^2 \overline{P} = 0$ and $\varepsilon_{\mathscr{P}} = 0$,

(a′) $\check{\nabla} \overline{P}$ is factorizable through $E \to \boldsymbol{T}$ and it is unitary,

(b) $\check{\mathrm{D}}_2^2 \mathring{P} = 0$ and $\check{\mathrm{D}}_2 \mathring{P}$ is unitary,

(b′) $\check{\mathrm{D}}_2 \mathring{P}$ is factorizable through $\boldsymbol{T} \times E \to \boldsymbol{T} \times \boldsymbol{T}$ and it is unitary,

(c) the map $\mathring{P}_{(\sigma,\tau)}\colon S_\tau \to S_\sigma$ is rigid, i.e.

$$\|\mathring{P}_{(\sigma,\tau)}(e') - \mathring{P}_{(\sigma,\tau)}(e)\| = \|e' - e\|,$$

(d) the map $\tilde{P}_{|S_\tau}\colon S_\tau \to U$ is affine and $\tilde{P}(e') = \tilde{P}(e) + \Omega_{\mathscr{P}}(\tau) \times (e' - e)$.

The following conditions are equivalent:

(a) $\check{\nabla} P = 0$, (a′) $\overline{P}$ is factorizable through $E \to \boldsymbol{T}$,

(b) $\check{\mathrm{D}}_2 \mathring{P} = \mathrm{id}_S$,

(c) the map $\mathring{P}_{(\sigma,\tau)}\colon S_\tau \to S_\sigma$ is a translation, i.e.

$$\mathring{P}_{(\sigma,\tau)}(e') = \mathring{P}_{(\sigma,\tau)}(e) + (e' - e) ,$$

(d) the map $\tilde{P}_{|S_\tau}\colon S_\tau \to U$ is constant.

The following conditions are equivalent:

(a) $\nabla\bar{P} = 0$,

(b) $\check{D}_2\tilde{P} = \mathrm{id}_S$ and $D_1\tilde{P}$ is constant,

(c) the map $\mathring{P}\colon \boldsymbol{T}\times E \to E$ is affine, i.e.

$$\mathring{P}(\tau, e) = e + \tilde{P}(\tau - t(e)) , \qquad \text{with } \tilde{P} \in U,$$

(d) the map $\tilde{P}\colon E \to U$ is constant.

2 - Observed kinematics

2.1 – *The position space.* We get useful representations of $\boldsymbol{P}$.

The diffeomorphism $P\colon \boldsymbol{T}\times\boldsymbol{P} \to E$ induces the diffeomorphisms $P_\tau\colon \boldsymbol{P} \to S_\tau$, whose inverse ones are $p_\tau\colon S_\tau \to \boldsymbol{P}$ and which are related together by $\mathring{P}_{(\tau,\sigma)}$. Hence, in this way, we get a time depending representation of $\boldsymbol{P}$.

Moreover, $\mathscr{P}$ determines the equivalence relation in E

$$e' \sim e \Leftrightarrow p(e') = p(e) \Leftrightarrow e' = \mathring{P}(t(e'), e) .$$

The equivalence classes are just the world-lines of $\mathscr{P}$; each one of these meets each fiber S_τ in one point and its representatives at the times τ and σ are related by $\mathring{P}_{(\tau,\sigma)}$.

Definition. The *position space* is the quotient space $E_{/\mathscr{P}}$.

Henceforth we shall make the natural identification $\boldsymbol{P} \equiv E_{/\mathscr{P}}$, which unifies the time dependent representations of $\boldsymbol{P}$ by a time independent one.

Analogous statements can be obtained for $T\boldsymbol{P}$ and $T^2\boldsymbol{P}$, by taking into account the diffeomorphisms

$$T_2 P\colon \boldsymbol{T}\times T\boldsymbol{P} \to \check{T}E \qquad \text{and} \qquad T_2^2 P\colon \boldsymbol{T}\times T^2\boldsymbol{P} \to \check{T}^2 E ,$$

and the equivalence relations in $\check{T}E$ and in $\check{T}^2E$

$$a' \sim a \Leftrightarrow Tp(a') = Tp(a) \Leftrightarrow a' = \check{T}_2\mathring{P}(t(a'), a) ,$$

$$a' \sim a \Leftrightarrow T^2p(a') = T^2p(a) \Leftrightarrow a' = \check{T}_2^2\mathring{P}(t(a'), a) .$$

In this way we get a time dependent representation of $\boldsymbol{TP}$ and $T^2\boldsymbol{P}$ and an identification $\boldsymbol{TP} \equiv \check{T}E_{/\mathscr{P}}$ and $T^2\boldsymbol{P} = \check{T}^2E_{/\mathscr{P}}$.

Moreover the diffeomorphisms

$$\mathrm{d}P \equiv TP_1 \colon \boldsymbol{T}\times T\boldsymbol{P} \to \dot{T}E \qquad \text{and} \qquad \mathrm{d}^2P \equiv T\,\mathrm{d}P_1 \colon \boldsymbol{T}\times T^2\boldsymbol{P} \to \dot{T}^2E$$

give further useful time dependent representations of $\boldsymbol{TP}$ and $T^2\boldsymbol{P}$.

Proposition. One as

$$p(e) = [e]\,, \qquad Tp(e, u) = [e, \check{u}_{\mathscr{P}}(e)]\,,$$

$$T^2p(e, u, v, w)$$

$$= [e, \check{u}_{\mathscr{P}}(e), \check{v}_{\mathscr{P}}(e), \check{w}_{\mathscr{P}}(e) - u^0\check{\mathrm{D}}\tilde{P}(e)(\check{v}_{\mathscr{P}}(e)) - v^0\check{\mathrm{D}}\tilde{P}(e)(\check{u}_{\mathscr{P}}(e)) - u^0v^0\tilde{\tilde{P}}(e)]\,,$$

$$P(t(e), [e]) = e\,, \qquad T_2P(t(e), [e, u]) = (e, u)\,,$$

$$T_2^2P(t(e), [e, u, v, w]) = (e, u, v, w)\,,$$

$$\mathrm{d}P(t(e), [e, u]) = (e, u + \tilde{P}(e))\,,$$

$$\mathrm{d}^2P(t(e), [e, u, v, w]) = (e, u + \tilde{P}(e), v + \tilde{P}(e), w + \check{\mathrm{D}}\tilde{P}(e)(u + v) + \tilde{P}(e))\,.$$

Henceforth we shall be concerned with the *observed phase-space* $\boldsymbol{T}\times T\boldsymbol{P}$, the *observed cophase-space* $\boldsymbol{T}\times T^*\boldsymbol{P}$ and moreover with the spaces $\boldsymbol{T}\times T^2\boldsymbol{P}$ and $\boldsymbol{T}\times TT^*\boldsymbol{P}$.

If $f\colon \boldsymbol{T}\times A \to B$ is a map, we shall denote by $\overset{T}{f}$ the lift $\overset{T}{f} \equiv (\mathrm{id}_{\boldsymbol{T}}, f)\colon \boldsymbol{T}\times A \to \boldsymbol{T}\times B$.

Moreover, we shall denote by the sign ˇ the vertical derivatives with respect to the bundle $\boldsymbol{T}\times \boldsymbol{P} \to \boldsymbol{P}$.

2.2 – *Observed kinematical structures.* The time dependent representation of $\boldsymbol{P}$ induces interesting structures on it.

Definition. The *observed time-dependent*

metric tensor is $g_{\mathscr{P}} \equiv g\circ(T_2P\times T_2P)\colon \boldsymbol{T}\times T\boldsymbol{P}\times_{\mathscr{P}} T\boldsymbol{P} \to \boldsymbol{R}\,,$

metric map is $g_{\mathscr{P}} \equiv T^*p\circ g\circ T_2P\colon \boldsymbol{T}\times T\boldsymbol{P} \to T^*\boldsymbol{P}\,,$

metric function is $g_{\mathscr{P}} \equiv g\circ T_2P\colon \boldsymbol{T}\times T\boldsymbol{P} \to \boldsymbol{R}\,,$

metric form is $g_{\mathscr{P}} \equiv g\circ T_2^2P\colon \boldsymbol{T}\times T^2\boldsymbol{P} \to \boldsymbol{R}\,.$

Proposition. One has

$$g_{\mathscr{P}}(t(e), [e, u], [e, v]) = u \cdot v\,, \qquad g_{\mathscr{P}}(t(e), [e, \tilde{u}]) = [e, \underset{\sim}{u}]\,,$$

$$g_{\mathscr{P}}(t(e), [e, u]) = \tfrac{1}{2} u \cdot u\,, \qquad g_{\mathscr{P}}(t(e), [e, u, v, w]) = u \cdot v\,,$$

$$g_{\mathscr{P}} = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad \dot{x}_i \circ g_{\mathscr{P}} = g_{ij} \dot{x}^j,$$

$$g_{\mathscr{P}} = \tfrac{1}{2} g_{ij} \dot{x}^i \dot{x}^j, \qquad g_{\mathscr{P}} = \check{\mathrm{d}}_o g_{\mathscr{P}} = g_{ij} \dot{x}^i\, \mathrm{d}x^j.$$

The metrical map determines analogous quantities on $\boldsymbol{T} \times T^*P$.

Definition. The *observed time-dependent*

space-like connection is $\check{\Gamma}_{\mathscr{P}} \equiv Tp \circ \check{\Gamma} \circ T^2_2 P\colon \boldsymbol{T} \times T^2\boldsymbol{P} \to T\boldsymbol{P}$,
inertial pseudo-connection is $\dot{\Gamma}_{\mathscr{P}} \equiv Tp \circ \dot{\Gamma} \circ \mathrm{d}^2 P\colon \boldsymbol{T} \times T^2\boldsymbol{P} \to T\boldsymbol{P}$.

The *Coriolis map* is $C_{\mathscr{P}} \equiv Tp \circ \check{\nabla} \vec{P} \circ T_2 P\colon \boldsymbol{T} \times T\boldsymbol{P} \to T\boldsymbol{P}$, the *dragging map* is $\mathrm{D}_{\mathscr{P}} \equiv Tp \circ \overline{\overline{P}} \circ P\colon \boldsymbol{T} \times \boldsymbol{P} \to T\boldsymbol{P}$.

If $u, v\colon \boldsymbol{T} \times \boldsymbol{P} \to T\boldsymbol{P}$ are vector fields, we define the *covariant derivative* $\check{\nabla}_{\mathscr{P}_u} v \equiv \check{\Gamma}_{\mathscr{P}} \circ \check{T} v \circ u\colon \boldsymbol{T} \times T\boldsymbol{P} \to T\boldsymbol{P}$, which can be naturally extended to tensors.

Proposition. One has

$$\check{\Gamma}_{\mathscr{P}}(t(e), [e, u, v, w]) = [e, w]\,,$$

$$\dot{\Gamma}_{\mathscr{P}}(t(e), [e, u, v, w]) = [e, w + \check{\mathrm{D}}\vec{P}(e)(u + v) + \tilde{\tilde{P}}(e)]\,,$$

hence

$$\dot{\Gamma}_{\mathscr{P}} = \check{\Gamma}_{\mathscr{P}} + \hat{C}_{\mathscr{P}} + \hat{C}_{\mathscr{P}} \circ s + \hat{\hat{\mathrm{D}}}_{\mathscr{P}}\,,$$

$$\ddot{x}^i \circ \check{\Gamma}_{\mathscr{P}} = \ddot{x}^i + \Gamma^i_{hk} \hat{\dot{x}}^h \hat{\dot{x}}^k, \qquad \ddot{x}^i \circ \dot{\Gamma}_{\mathscr{P}} = \ddot{x}^i + \Gamma^i_{hk} \hat{\dot{x}}^h \hat{\dot{x}}^k + 2\Gamma^i_{h0} \hat{\dot{x}}^h + \Gamma^i_{00} \hat{\dot{x}}^0.$$

Moreover, one has $\check{\nabla}_{\mathscr{P}} g_{\mathscr{P}} = 0$.

The metrical map induces analogous quantities on $T^*\boldsymbol{P}$ and $TT^*\boldsymbol{P}$.

We put $\dot{\nu}_{\mathscr{P}} = Tp \circ \dot{\nu} \circ \mathrm{d}^2 P\colon \boldsymbol{T} \times T^2\boldsymbol{P} \to T\boldsymbol{P}$ getting $\dot{\nu}_{\mathscr{P}} = \varkappa - \pi$, i.e. $\dot{\nu}_{\mathscr{P}}(t(e), [e, u, v, w]) = [e, v - u]$.

Definition. The *observed time-dependent*

space-like form is $\check{\psi}_{\mathscr{P}} \equiv \check{\psi}\circ(T^2_2 P\times T^2_2 P)\colon\ \boldsymbol{T}\times T^2\boldsymbol{P}\times_{TP} T^2\boldsymbol{P}\to\boldsymbol{R}$,
inertial form is $\dot{\psi}_{\mathscr{P}} \equiv \dot{\psi}\circ(\mathrm{d}^2 P\times \mathrm{d}^2 P)\colon\ \boldsymbol{T}\times T^2\boldsymbol{P}\times_{TP} T^2\boldsymbol{P}\to\boldsymbol{R}$,

Moreover, we define $\varphi_{\mathscr{P}} \equiv \varphi\circ(T^2_2 P\times T^2_2 P)\colon\ \boldsymbol{T}\times T^2\boldsymbol{P}\times_{TP} T^2\boldsymbol{P}\to\boldsymbol{R}$.

Proposition. One has

$$\check{\psi}_{\mathscr{P}}(t(e), [e, u; v, w; a, b]) = w\cdot a - b\cdot v\,,$$
$$\varphi_{\mathscr{P}}(t(e), [e, u; v, w; a, b]) = w\cdot(a-u) - b\cdot(v-u)\,,$$
$$\dot{\psi}_{\mathscr{P}}(t(e), [e, u; v, w; a, b]) = w\cdot(a-u) - b\cdot(v-u) + \underset{\sim}{\varepsilon}_{\mathscr{P}}(e)(u, a-v) + 2\tilde{\Omega}_{\mathscr{P}}(e)\times v\cdot a + \tilde{\tilde{P}}(e)\cdot(a-v)\,,$$

$$\check{\psi}_{\mathscr{P}} = g_{\mathscr{P}}\circ(\check{\Gamma}_{\mathscr{P}}\wedge\varkappa) = g_{ij}\,\mathrm{d}\dot{x}^i\wedge\mathrm{d}x^j + \partial_i g_{jh}\dot{x}^h\,\mathrm{d}x^i\wedge\mathrm{d}x^j,$$
$$\varphi_{\mathscr{P}} = g_{\mathscr{P}}\circ(\check{\Gamma}_{\mathscr{P}}\wedge(\varkappa-\pi)) = g_{ij}\,\mathrm{d}\dot{x}^i\wedge\mathrm{d}x^j + \partial_i g_{jh}\dot{x}^h\,\mathrm{d}x^i\wedge\mathrm{d}x^j + (\mathrm{d}\dot{x}^i + \gamma_{k,hi}\dot{x}^h x^k\,\mathrm{d}x^i)(\pi^2-\pi^1)\,,$$
$$\dot{\psi}_{\mathscr{P}} = g_{\mathscr{P}}\circ(\dot{\Gamma}_{\mathscr{P}}\wedge\dot{v}_{\mathscr{P}}) = g_{ij}\,\mathrm{d}\dot{x}^i\wedge\mathrm{d}x^j + (\partial_j g_{jh}\dot{x}^h + \gamma_{j,0i})\,\mathrm{d}x^i\wedge\mathrm{d}x^j + g_{ih}\dot{x}^h\,\mathrm{d}\dot{x}^i\circ(\pi^2-\pi') + (\tfrac{1}{2}\partial_i g_{hk}\dot{x}^h\dot{x}^k + \partial_0 g_{ih}\dot{x}^h + \gamma_{i,00})\,\mathrm{d}x^i\circ(\pi^2-\pi')\,.$$

Definition. The *observed time-dependent*

space-like equation is $\check{X}_{\mathscr{P}} \equiv T^2 p\circ\check{X}\circ T_2 P\colon\ \boldsymbol{T}\times T\boldsymbol{P}\to T^2\boldsymbol{P}$,
inertial equation is $\dot{X}_{\mathscr{P}} \equiv T^2 p\circ\dot{X}\circ\mathrm{d}P\colon\ \boldsymbol{T}\times T\boldsymbol{P}\to T^2\boldsymbol{P}$.

The metrical map determines analogous quantities on $\boldsymbol{T}\times T^*\boldsymbol{P}$.

Proposition. One has

$$\check{X}_{\mathscr{P}}(t(e), [e, u]) = [e, u, u, 0]\,,\qquad \dot{X}_{\mathscr{P}}(t(e), [e, u]) = [e, u, u, -2\check{\mathrm{D}}\tilde{P}(e)(u) - \tilde{\tilde{P}}(e)],$$

$$\check{X}_{\mathscr{P}} = \dot{x}^i\partial x_i - \Gamma^i_{hk}\dot{x}^h\dot{x}^k\partial\dot{x}_i\,,\qquad \dot{X}_{\mathscr{P}} = \dot{x}^i\partial x_i - (\Gamma^i_{hk}\dot{x}^h\dot{x}^k + 2\Gamma^i_{h0}\dot{x}^h + \Gamma^i_{00})\partial\dot{x}_i\,.$$

Moreover, $\check{X}_{\mathscr{P}}$ and $\dot{X}_{\mathscr{P}}$ are the sections characterized, respectively, by each one of the following conditions:

(a) $s\circ\check{X}_{\mathscr{P}} = \check{X}_{\mathscr{P}}$, $\check{\Gamma}_{\mathscr{P}}\circ\check{X}_{\mathscr{P}} = 0$, (a′) $s\circ\dot{X}_{\mathscr{P}} = \dot{X}_{\mathscr{P}}$, $\dot{\Gamma}_{\mathscr{P}}\circ\dot{X}_{\mathscr{P}} = 0$,
(b) $i_{\check{X}}\check{\psi}_{\mathscr{P}} = -\check{\mathrm{d}}g_{\mathscr{P}}$, (b′) $i_{\dot{X}_{\mathscr{P}}}\dot{\psi}_{\mathscr{P}} = 0$,
(c) $i_{\check{X}_{\mathscr{P}}}\varphi_{\mathscr{P}} = 0$.

Hence $\boldsymbol{P}$ has a time-dependent affine structure, characterized by $\check{\Gamma}_{\mathscr{P}}$, and a euclidean structure, characterized by $g_{\mathscr{P}}$.

If $\mathscr{P}$ is affine, then $\boldsymbol{P}$ is an affine space, whose vector space is $\boldsymbol{P} \equiv (\boldsymbol{T}\times S)_{/\mathscr{P}}$, with respect to the equivalence relation

$$(\tau', u') \sim (\tau, u) \Leftrightarrow u' = \mathrm{D}\mathring{P}_{(\tau',\tau)}(u) .$$

In this case, $\check{\Gamma}_{\mathscr{P}}$ results into the (time independent) affine connection of $\boldsymbol{P}$.

If $\mathscr{P}$ is rigid, then $g_{\mathscr{P}}$ is time independent and $\boldsymbol{P}$ turn out to be an affine euclidean space.

If $\mathscr{P}$ is translating, then we get $\overline{\boldsymbol{P}} = S$ and $C_{\mathscr{P}} = 0$.

If $\mathscr{P}$ is inertial, then we get $\check{\Gamma}_{\mathscr{P}} = \dot{\Gamma}_{\mathscr{P}}$, $\varphi_{\mathscr{P}} = \dot{\psi}_{\mathscr{P}}$, $\check{X}_{\mathscr{P}} = \dot{X}_{\mathscr{P}}$.

2.3 – *The representation of E.* We can take the continuum of world lines $\mathscr{P}$ as an *observer*, which splits E into space and time, associating with the absolute quantities the physically observed ones. The pair $(\mathscr{P}, x)$ is a *frame of reference*, which, moreover, gives the numerical representation of the observed quantities.

The representation of E is given by the inverse diffeomorphisms $P\colon \boldsymbol{T}\times\boldsymbol{P} \to E$ and $(t, p)\colon E \to \boldsymbol{T}\times\boldsymbol{P}$.

Hence $\mathscr{P}$ determines the splitting of the absolute bundle $E \to \boldsymbol{T}$, by means of the fiber $\boldsymbol{P}$, and it determines the bundle $E \to \boldsymbol{P}$, whose absolute fiber is $\boldsymbol{T}$. Analogous representations hold for the tangent spaces.

We define the following spaces

$$T_{\mathscr{P}}E \equiv \big(e, r\tilde{P}(e)\big)_{e\in E,\, r\in \boldsymbol{R}} \hookrightarrow TE\,, \qquad T^*_{\mathscr{P}}E \equiv \big((e, v) \in T^*E \,|\, \langle v, \tilde{P}(e)\rangle = 0\big) \hookrightarrow T^*E,$$

which give the useful representations

$$TE = T_{\mathscr{P}}E \oplus \check{T}E\,, \qquad (e, u) = \big(e, u^0\tilde{P}(e) + \check{u}_{\mathscr{P}}(e)\big)\,,$$

$$T^*E = \mathring{T}^*E \oplus \check{T}^*_{\mathscr{P}}E\,, \qquad (e, v) = \big(e, \langle v, \tilde{P}(e)\rangle\,\mathrm{D}t + \check{v}_{\mathscr{P}}(e)\big)\,.$$

2.4 – *Observed kinematics.* Let M be a motion.

Definition. The *observed motion* is $M_{\mathscr{P}} \equiv p\circ M\colon \boldsymbol{T} \to \boldsymbol{P}$. The *velocity* and the *acceleration* of $M_{\mathscr{P}}$ are

$$\mathrm{d}M_{\mathscr{P}}\colon \boldsymbol{T} \to T\boldsymbol{P} \qquad \text{and} \qquad \dot{\nabla}_{\mathscr{P}}\overset{T}{\mathrm{d}}M_{\mathscr{P}} \equiv \dot{\Gamma}_{\mathscr{P}}\circ\overset{T}{\mathrm{d}}{}^2M_{\mathscr{P}}\colon \boldsymbol{T} \to T\boldsymbol{P}\,.$$

Proposition. One has

$$M^i_{\mathscr{P}} = M^i, \qquad \mathrm{d}M_{\mathscr{P}} = \mathrm{d}M_{\mathscr{P}} - \overline{P}\circ\overset{T}{M}_{\mathscr{P}} = \mathrm{D}M^i(\partial x_i\circ\overset{T}{M}_{\mathscr{P}})\,,$$

$$\check{\nabla}_{\mathscr{P}}\overset{T}{\mathrm{d}}M_{\mathscr{P}} = \dot{\nabla}\,\mathrm{d}M - 2C_{\mathscr{P}}\circ\overset{T}{\mathrm{d}}M_{\mathscr{P}} - \mathrm{D}_{\mathscr{P}}\circ\overset{T}{M}_{\mathscr{P}} = \big(\mathrm{D}^2M^i + \Gamma^i_{hk}\circ\overset{T}{M}_{\mathscr{P}}\mathrm{D}M^h\mathrm{D}M^k\big)(\partial x_i\circ\overset{T}{M}_{\mathscr{P}}),$$

where we have represented the values of $\mathrm{d}M_{\mathscr{P}}$ and of $\overset{T}{\dot{\nabla}}_{\mathscr{P}}\mathrm{d}M_{\mathscr{P}}$ in $T\boldsymbol{P}$, by means of their absolute representatives at each time.

Hence the observed quantities determine the absolute ones, taking into account the kinematical structures of $\boldsymbol{P}$.

3 - Absolute dynamics

3.1 – *Dynamical metric.*

Definition. A *mass* is a conformal transformation of $\{g\}$, characterized by a number $m \in \boldsymbol{R}$. We call *dynamical* the new metric mg and all the new quantities induced by the transformation $g \mapsto mg$.

Henceforth we assume such an m to be given.

In this way, we get

$$G \equiv mg\,, \qquad \bar{G} \equiv \frac{1}{m}\,\bar{g}\,, \qquad \check{\Psi} \equiv m\check{\psi}\,, \qquad \dot{\Psi} \equiv m\dot{\psi}\,, \qquad \Phi \equiv m\varphi\,, \qquad \Gamma_{i,\alpha\beta} \equiv m\gamma_{i,\alpha\beta}\,.$$

Henceforth (except when it is not explicitly pointed out) we shall be concerned with such dynamical quantities.

3.2 – *Forces.*

Definition. A *force* is a morphism over E, $F\colon \dot{T}E \to T^*E$, such that $\langle \underset{\sim}{F}(e, u), u\rangle = 0$. A force F is characterized by the morphisms over E, $\check{F}\colon \dot{T}E \to \check{T}^*E$ and $\check{\check{F}}\colon \dot{T}E \to \check{T}E$ or by the sections obtained, by lift, from F, $\check{F}$ and $\check{\check{F}}$, which will be denoted by the same notations.

Proposition. One has

$$\langle \underset{\sim}{F}(e, u), v\rangle = \langle \underset{\sim}{\check{F}}(e, u), v - v^0 u\rangle = \underset{\sim}{G}(\check{\check{F}}(e, u), v - v^0 u)\,,$$

$$F = -F_i \dot{x}^i\,\mathrm{d}x^0 + F_i\,\mathrm{d}x^i, \qquad \check{F} = F_i\,\check{\mathrm{d}}x^i, \qquad \check{\check{F}} = F^i\,\partial x_i\,, \qquad F^i = \frac{1}{m}\,g^{ij}F_j\,.$$

A force F is *event*, or *time*, or *velocity dependent*, respectively, if it is factorizable through $\dot{T}E \to E$, or $\dot{T}E \to \boldsymbol{T}$, or $\dot{T}E \to U$.

A force $\check{F}$ is *space-like closed* if $\check{\mathrm{d}}\check{F} = 0$ and it is *space-like exact* if there is a function $f\colon E \to \boldsymbol{R}$, such that $\check{F} = \check{\mathrm{d}}f$. We get $\check{\mathrm{d}}f' = \check{F} = \check{\mathrm{d}}f \Leftrightarrow f' = f$

$+\,\varphi\circ t$, where $\varphi\colon \boldsymbol{T} \to \boldsymbol{R}$. Moreover, $\check{F}$ is space-like closed if and only if it is space-like exact. If $\check{F}$ is time dependent, then it is space-like exact and $f_{|S_\tau}\colon S_\tau \to \boldsymbol{R}$ is an affine function.

The following conditions are equivalent:

(a) $\mathrm{d}t\wedge \mathrm{d}F = 0$,
(b) $\check{F}$ is event-dependent and space-like closed.

There are not *closed* forces, namely, if $\mathrm{d}F = 0$, then $F = 0$.

Let us consider an interesting example of force.

The *force of Lorentz* generated by the *classical electromagnetic field* (see [**8**]) $\mathscr{F}\colon E \to \wedge^2 E$ is $F(u) = i_u \mathscr{F}$.

3.3 – *Absolute dynamical structures.* A force modifies the inertial structures of $\dot{T}E$.

Definition. A *dynamical connection* is a connection $\Gamma\colon T\dot{T}E \to \check{T}E$, which reduces to the inertial connection on $\check{T}\dot{T}E$.

A *dynamical form* is an exterior 2-form $\psi\colon T\dot{T}E \times_{\dot{T}E} T\dot{T}E \to \boldsymbol{R}$, which differs from the inertial form for a pull-back 2-form and which reduces to it on $\check{T}\dot{T}E \times_{\dot{T}E} \check{T}\dot{T}E$.

A *dynamical equation* is a section $X\colon TE \to T\dot{T}E$, such that $s\circ X = X$.

Proposition. The maps

$$\check{\check{F}} \mapsto \overset{F}{\Gamma} \equiv \dot{\Gamma} - \mathrm{d}t \otimes \check{\check{F}}\,, \qquad F \mapsto \overset{F}{\Psi} \equiv \dot{\Psi} - \mathrm{d}t\wedge F\,,$$

$$\check{\check{F}} \mapsto \overset{F}{X} \equiv \dot{X} + \check{\check{F}}\,, \qquad \Gamma \mapsto \Psi = G\circ(\Gamma\wedge\dot{\nu})\,,$$

$\Psi \mapsto X$, where X is the section $X\colon \dot{T}E \to T\dot{T}E$ characterized by $i_X \mathrm{d}t = 1$ and $i_X\Psi = 0$; $X \mapsto \Gamma$, where Γ is the dynamical connection such that $\Gamma\circ X = 0$, induce bijections among forces, dynamical connections, forms and equations, they relate the null force with the inertial quantities and their compositions commute.

More precisely, one has

$$\overset{F}{\Gamma}(e, u, v, w) = \big(e, w - v^0 \check{\check{F}}(e, u)\big)\,,$$

$$\overset{F}{\Psi}(e, u; v, w; a, b) = mw\cdot(a - a^0 u) - mb\cdot(v - v^0 u) - \langle \underset{\sim}{\check{F}}(e, u), v^0 a - a^0 v\rangle\,,$$

$$\overset{F}{X}(e, u) = \big(e, u, u, \check{\check{F}}(e, u)\big)\,.$$

The coordinate expressions of $\overset{F}{\Gamma}$, $\overset{F}{\Psi}$ and $\overset{F}{X}$ are obtained from those of $\dot{\Gamma}$, $\dot{\Psi}$ and $\dot{X}$ replacing g_{ij} with G_{ij}, $\gamma_{i,\alpha\beta}$ with $\Gamma_{i,\alpha\beta}$ and $\gamma_{i,00}$ with $\gamma_{i,00} - F_i$.

We get $dt \wedge d\overset{F}{\Psi} = dt \wedge d\dot{\Psi}$ and $dt \wedge \overset{F}{\Psi} \wedge \overset{F}{\Psi} \wedge \overset{F}{\Psi} = dt \wedge \dot{\Psi} \wedge \dot{\Psi} \wedge \dot{\Psi}$.

Definition (*Absolute Newton's law of motion*). A *dynamical system* is a pair $\mathscr{D} \equiv (G, F)$. A *dynamical solution* of $\mathscr{D}$ is a motion (at least defined locally on $\boldsymbol{T}$) $M: \boldsymbol{T} \to E$, such that $\overset{F}{\nabla} dM = 0$.

Theorem. *Let $\mathscr{D}$ be a dynamical system and let M be a motion. The following conditions are equivalent*:

(a) $\overset{F}{\nabla} dM \equiv \overset{F}{\Gamma} \circ d^2 M = 0$,

(b) $\dot{\nabla} dM \equiv \dot{\Gamma} \circ d^2 M = \check{\bar{F}} \circ dM$,

(c) $\langle \overset{F}{\Psi} \circ dM, d^2 M \rangle = 0$,

(d) $\overset{F}{X} \circ dM = d^2 M$,

(e) $D^2 M + \Gamma^i_{hk} \circ M \, DM^h DM^k + 2\Gamma^i_{0k} \circ M \, DM^k + \Gamma^i_{00} \circ M = F^i \circ dM$.

4 - Observed dynamics

4.1 – *Observed dynamical metric.* We call *dynamical* the new observed metric and the new observed quantities induced by the transformation $g_{\mathscr{P}} \mapsto m g_{\mathscr{P}}$.

In this way, we get

$$G_{\mathscr{P}} \equiv m g_{\mathscr{P}}, \quad \bar{G}_{\mathscr{P}} \equiv \frac{1}{m} \bar{G}_{\mathscr{P}}, \quad \check{\Psi}_{\mathscr{P}} \equiv m \check{\psi}_{\mathscr{P}}, \quad \dot{\Psi}_{\mathscr{P}} \equiv m \dot{\psi}_{\mathscr{P}}, \quad \Phi_{\mathscr{P}} \equiv m \varphi_{\mathscr{P}}.$$

4.2 – *Observed forces.* Let F be a force.

Definition. The *observed force* is the morphism over $\boldsymbol{P}$

$$F_{\mathscr{P}} \equiv T^* p \circ \check{F} \circ dP : \boldsymbol{T} \times T\boldsymbol{P} \to T^* \boldsymbol{P}, \quad \text{or} \quad \bar{F}_{\mathscr{P}} \equiv Tp \circ \check{\bar{F}} \circ dP : \boldsymbol{T} \times T\boldsymbol{P} \to T\boldsymbol{P}.$$

The *observed power* is the function

$$F \circ_{\mathscr{P}} : \boldsymbol{T} \times T\boldsymbol{P} \to \boldsymbol{R}, \quad (t(e), [e, u]) \mapsto \langle \underset{\sim}{\check{F}}(e, u + \tilde{P}(e)), u \rangle.$$

Moreover, let M be a motion and let $\tau_0 \in \boldsymbol{T}$, then the *observed work*, along M and with starting point τ_0, is the function $L\colon \boldsymbol{T} \to \boldsymbol{R},\ \tau \mapsto \int_{[\tau_0,\tau]} F_{0\mathscr{P}} \circ \overset{T}{\mathrm{d}} M_{\mathscr{P}}$.

Definition. F is *positional* (with respect to $\mathscr{P}$) if it is factorizable through $\boldsymbol{T} \times T\boldsymbol{P} \to \boldsymbol{P}$, by $f_{\mathscr{P}}\colon \boldsymbol{P} \to T^*\boldsymbol{P}$.

Let F be positional; it is *conservative* if there is $f_{\mathscr{P}}\colon \boldsymbol{P} \to \boldsymbol{R}$, such that $f_{\mathscr{P}} = \check{\mathrm{d}} f_{\mathscr{P}}$.

We get $F_{\mathscr{P}} = F_{\mathscr{P}i}\mathrm{d}x^i$, $F_{0\mathscr{P}} = F_{\mathscr{P}i}\dot{x}^i$, where $F_{\mathscr{P}i} = F_i \circ \mathrm{d}P$.

Let us examine the previous example of force.

We can write $\mathscr{F} = \mathrm{d}t \wedge \varepsilon_{\mathscr{P}} - * H$, where $\varepsilon_{\mathscr{P}} = i_{\bar{P}}\mathscr{F}$ and $H = - * \check{\mathscr{F}}$. Then we get $F(e, u) = -\langle \varepsilon_{\mathscr{P}}(e), u \rangle \mathrm{d}t + \varepsilon_{\mathscr{P}} + (u \times H)_{\mathscr{P}}$, where $(u \times H)_{\mathscr{P}}$ is the extension of $u \times H$ induced by $\mathscr{P}$, and $F_{\mathscr{P}}(t(e), [e, u]) = \varepsilon_{\mathscr{P}}(t(e), [e]) + u \times_{\mathscr{P}} H_{\mathscr{P}}(t(e), [e])$, $F_{0\mathscr{P}}(t(e), [e, u]) = \langle \varepsilon_{\mathscr{P}}(t(e), [e]), u \rangle$, where we have made trivial abuse of notations.

4.3 – *Observed dynamical structures.* Let F be a force.

Definition. The *observed dynamical*

pseudo-connection is $\overset{F}{\Gamma}_{\mathscr{P}} \equiv Tp \circ \overset{F}{\Gamma} \circ T^2_2 P\colon \boldsymbol{T} \times T^2\boldsymbol{P} \to T\boldsymbol{P}$,

form is $\overset{F}{\Psi}_{\mathscr{P}} \equiv \overset{F}{\Psi} \circ (\mathrm{d}^2 P \times \mathrm{d}^2 P)\colon \boldsymbol{T} \times T^2\boldsymbol{P} \times_{T\boldsymbol{P}} T^2\boldsymbol{P} \to \boldsymbol{R}$,

equation is $\overset{F}{X}_{\mathscr{P}} \equiv T^2 p \circ \overset{F}{X} \circ \mathrm{d}P\colon \boldsymbol{T} \times T\boldsymbol{P} \to T^2\boldsymbol{P}$.

Proposition. The observed dynamical quantities satisfy relations analogous to those satisfied by the inertial ones. Moreover the expressions of the dynamical quantities can be obtained from those of the inertial ones, replacing $\tilde{\tilde{P}}(e)$ by $\tilde{\tilde{P}}(e) - \check{\bar{F}}(e, u)$ and $\gamma_{i,00}$ by $\gamma_{i,00} - F_{i\mathscr{P}}$.

Theorem. *Let $\mathscr{D}$ be a dynamical system and let M be a motion. The following conditions are equivalent:*

(o) *M is a dynamical solution of $\mathscr{D}$,*

(a) $\overset{F}{\nabla}_{\mathscr{P}} \overset{T}{\mathrm{d}} M_{\mathscr{P}} \equiv \overset{F}{\Gamma}_{\mathscr{P}} \circ \mathrm{d}^2 M_{\mathscr{P}} = 0$,

(b) $\dot{\nabla}_{\mathscr{P}} \overset{T}{\mathrm{d}} M_{\mathscr{P}} \equiv \dot{\Gamma}_{\mathscr{P}} \circ \mathrm{d}^2 M_{\mathscr{P}} = \check{\bar{F}}_{\mathscr{P}} \circ \overset{T}{\mathrm{d}} M_{\mathscr{P}}$,

(c) $\check{\nabla}_{\mathscr{P}} \overset{T}{\mathrm{d}} M_{\mathscr{P}} \equiv \check{\Gamma}_{\mathscr{P}} \circ \mathrm{d}^2 M_{\mathscr{P}} = (\bar{F}_{\mathscr{P}} - \bar{C}_{\mathscr{P}} - \bar{D}_{\mathscr{P}}) \circ \overset{T}{\mathrm{d}} M_{\mathscr{P}}$,

(d) $\langle \overset{F}{\psi}_{\mathscr{P}} \circ \overset{T}{\mathrm{d}} M_{\mathscr{P}}, \overset{T}{\mathrm{d}}{}^2 M_{\mathscr{P}} \rangle = 0$,

(e) $\overset{F}{X}_{\mathscr{P}} \circ \overset{T}{\mathrm{d}} M_{\mathscr{P}} = \overset{T}{\mathrm{d}}{}^2 M_{\mathscr{P}}$, *and, if $\mathscr{P}$ is rigid, the conditions analogous to* (a), ..., (e), *obtained replacing* $\overset{T}{\mathrm{d}}M_{\mathscr{P}}$ *with* $\overset{T*}{\mathrm{d}}M_{\mathscr{P}} \equiv G_{\mathscr{P}} \circ \overset{T}{\mathrm{d}}M_{\mathscr{P}}$, *and moreover the condition* (*Lagrange equations*), *for any special chart* $y: E \to \boldsymbol{R} \times \boldsymbol{R}^3$,

$$\text{(f)} \quad \mathrm{D}(\partial \dot{y}_i \cdot G_{\mathscr{P}}) \circ \overset{T}{\mathrm{d}}M_{\mathscr{P}} + (\partial y_i \cdot G_{\mathscr{P}}) \circ \overset{T}{\mathrm{d}}M_{\mathscr{P}} = F_{i\mathscr{P}} \circ \overset{T}{\mathrm{d}}M_{\mathscr{P}} .$$

Proposition. Let M a solution of $\mathscr{D}$. One has $\mathrm{D}(G_{\mathscr{P}} \circ \overset{T}{\mathrm{d}}M_{\mathscr{P}}) = F_{0\mathscr{P}} \circ \overset{T}{\mathrm{d}}M_{\mathscr{P}}$.

Corollary. *Let $\mathscr{P}$ be inertial and let F be conservative. Then the function* $H_{\mathscr{P}} \equiv G_{\mathscr{P}} - \hat{f}_{\mathscr{P}}: \boldsymbol{T} \times T\boldsymbol{P} \to \boldsymbol{R}$ *satisfies* $\overset{F}{X}_{\mathscr{P}} \cdot H_{\mathscr{P}} = 0$, i.e. $\mathrm{D}(H_{\mathscr{P}} \circ \overset{T}{\mathrm{d}}M_{\mathscr{P}}) = 0$, *for each dynamical solution* M.

5 - Remark on the n-body absolute and observed kinematics

In order to study n not interacting particles, we could repeat n times the model induced by the classical event framework for one particle. However we can introduce the pull-back bundle $\eta^{(n)} \equiv (E^{(n)}, t^n, \boldsymbol{T})$, of the bundle $E \to \boldsymbol{T}$ by the diagonal map $\boldsymbol{T} \hookrightarrow \boldsymbol{T}^n$.

Namely $E^{(n)}$ is the set of the n-plets $(e_1, ..., e_n) \in E^n$, such that $t(e_1) = ... = t(e_n)$. Then $\dim E^{(n)} = 3n + 1$.

The canonical i-th projection $\pi_i: E^{(n)} \to E$ maps each n-event into the event space of the i-th particle.

The bundle η and its structures induce analogous structures on the bundle $\eta^{(n)}$, which will be denoted by the sign « $^{(n)}$ ».

We leave to the reader the extension to this case of all the proceeding results.

6 - The constrained mechanics

6.1 - Definition. A *configuration time bundle* is a C^∞ (not necessarily affine) subbundle of $\eta^{(n)}$ of dimension $1 + l$, $\mu \equiv (C, t, \boldsymbol{T})$. Henceforth we assume such a μ to be given.

A *constraint* is a (local) function $f: E^{(n)} \to \boldsymbol{R}$, such that $f_{|C} = 0$ and $(\mathrm{d}f)_{|C} \neq 0$.

In the following $x: E^{(n)} \to \boldsymbol{R} \times \boldsymbol{R}^{3n+1}$ will be a chart *adapted* to C. It can be choosen to be *orthogonal*, (at least) at one point $e \in C$, i.e such that $\partial x_{l+1}, ..., \partial x_{3n}$ are orthogonal to C in e, with respect to the space-like metric of $E^{(n)}$.

An observer of $E^{(n)}$ is *adapted* to C if its world-lines, which meet at on point, belong to it. Adapted charts induce adapted observers only.

We denote by $c: C \hookrightarrow E^{(n)}$ the canonical injection.

We define the following spaces and structures along the line adopted for the free mechanics.

Definition.

The *phase space* is $\check{T}C \equiv \dot{t}^{-1}(0) = TC \cap \check{T}E^{(n)}$,
the *unitary space* is $\dot{T}C \equiv \dot{t}^{-1}(1) = TC \cap \dot{T}E^{(n)}$.

The bilinear form $g_c \equiv g^n \circ (\check{T}c, \check{T}c)\colon \check{T}C \times_C \check{T}C \to \boldsymbol{R}$ is the *space-like Riemannian metric*.

The space-like metric g^n-induces the parallel and orthogonal projections $\pi''\colon \check{T}E_{|C} \to \check{T}C$ and $\pi^\perp\colon \check{T}E_{|C} \to (TC)^\perp$.

6.2 – Proposition. The map $\check{\Gamma}_c \equiv \pi'' \circ \check{\Gamma} \circ T^2c\colon \check{T}^2C \to \check{T}C$ is a space-like connection, namely the space-like Riemannian connection.

The map $\pi^\perp \circ \check{\Gamma} \circ \check{T}^2c\colon \check{T}^2C \to (\check{T}C)^\perp$ is a morphism over C, which can be factorized through the canonical projection $\check{T}^2C \to \check{T}C \times_C \check{T}C$, by a bilinear map $\check{T}C \times_C \check{T}C \to (\check{T}C)^\perp$. Hence, its restriction to the symmetric subspace of $\check{T}^2C$ can be factorized through the canonical projection sym $\check{T}^2C \to \check{T}C$, by a quadratic map $\check{N}\colon \check{T}C \to (TC)^\perp$. The map $\dot{\Gamma}_c \equiv \pi'' \circ \dot{\Gamma} \circ T^2c\colon T\dot{T}C \to \check{T}C$ is a connection.

The map $\pi^\perp \circ \dot{\Gamma} \circ T^2c\colon T\dot{T}C \to (\check{T}C)^\perp$ is a morphism over C, which can be factorized through the canonical projection $T\dot{T}C \to \dot{T}C \times_C \dot{T}C$, by a map $\dot{T}C \times_C \dot{T}C \to (TC)^\perp$. Hence, its restriction to the symmetric subspace of $T\dot{T}C$ can be factorized through the canonical projection sym $T\dot{T}C \to \dot{T}C$, by an affine quadratic map $\dot{N}\colon \dot{T}C \to (TC)^\perp$.

Let x be an orthogonal chart. Then we get the following expressions (where the orthogonality holds)

$$\dot{x}^i \circ \check{\Gamma}_c = \ddot{x}^i + \Gamma^i_{hk} \hat{\dot{x}}^h \hat{\dot{x}}^k, \qquad \dot{x}^r \circ \check{N} = \Gamma^r_{hk} \dot{x}^h \dot{x}^k,$$

$$\dot{x}^i \circ \dot{\Gamma}_c = \ddot{x}^i + \Gamma^i_{hk} \hat{\dot{x}}^h \hat{\dot{x}}^k + \Gamma^i_{0k} \hat{\dot{x}}^k + \Gamma^i_{h0} \hat{\dot{x}}^h \hat{\dot{x}}^0 + \Gamma^i_{00} \hat{\dot{x}}^0,$$

$$\dot{x}^r \circ \dot{N} = \Gamma^r_{hk} \dot{x}^h \dot{x}^k + 2\Gamma^r_{0k} \dot{x}^k + \Gamma^r_{00}\,,$$

with $1 \leqslant i, h, k \leqslant 1$ and $l + 1 \leqslant r \leqslant 3n$.

The previous expressions show the relations among $\check{\Gamma}_c$, $\dot{\Gamma}_c$, $\check{N}$, $\dot{N}$ and the tensors $\varepsilon_{\mathscr{P}}$, $\Omega_{\mathscr{P}}$ and $\overline{\overline{P}}$, restricted to C, associated with any observer $\mathscr{P}$ adapted to C. Let us remark that these quantities are space-like and that their parallel projections on C do not depend on the choice of such a $\mathscr{P}$.

In the n-body mechanics, the affine structure of $\eta^{(n)}$ does not play any essential role. The main results where obtained taking into account the C^∞ bundle structure on $\boldsymbol{T}$, the space-like metric, the space-like connection and the inertial connection. Then the theory of mechanics on $E^{(n)}$ can be easily and completely revised on C.

In particular, a *force* is a morphism over C $F_c\colon \dot{T}C \to T^*C$ such that $F_{c\alpha}\dot{x}^\alpha = 0$, a *dynamical system* is a pair $\mathscr{D}_c \equiv (G_c, F_c)$ and a *solution* of $\mathscr{D}_c$ is a motion $M\colon \boldsymbol{T} \to C$, which satisfies the *Newton law of motion* $\overset{F_c}{\nabla}_c \mathrm{d}M = 0$.

We can make a direct comparison between free and constrained mechanics.

Theorem. *Let* $\check{\bar{F}}^{(n)}\colon \dot{T}E^{(n)} \to \check{T}E^{(n)}$ *be a force. Moreover, let* $\bar{F}_c \equiv \pi'' \circ \check{\bar{F}}^{(n)}\colon \dot{T}C \to \check{T}C$, $\bar{F}^\perp = \pi^\perp \circ \check{\bar{F}}^{(n)}\colon TC \to (\check{T}C)^\perp$ *and* $\bar{R} = \dot{N} - \bar{F}^\perp\colon \dot{T}C \to (\check{T}C)^\perp$.

Let $M\colon \boldsymbol{T} \to C$ *be a motion which is a solution of* (G_c, F_c), i.e. *such that* $\dot{\nabla}_c \mathrm{d}M \equiv \bar{F}_c \circ \mathrm{d}M$.

The M *is a solution of* $(G, F^{(n)} + R)$, i.e. *we have* $\dot{\nabla}\,\mathrm{d}M = (\bar{F} + \bar{R}) \circ \mathrm{d}M$.

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Abstract

An axiomatic and geometrical approach to classical mechanics is presented, emphasising the basis role of connections.

A bundle $E \to \boldsymbol{T}$ endowed with a space-like metric g, a space-like connection $\check{\Gamma}$ and an inertial connection $\dot{\Gamma}$ leads to the absolute mechanics and to Newton's law $\dot{\nabla}\, \mathrm{d}M = 0$.

The 1-particle treatment can be easily extended to the n-particles and to the constrained cases.

The observed mechanics is obtained by taking into account a continuum $\mathscr{P}$, which determines the position space $\boldsymbol{P} \equiv E/\mathscr{P}$, endowed with a space-like metric $g_{\mathscr{P}}$, a space-like connection $\check{\Gamma}_{\mathscr{P}}$ and an inertial pseudo-connection $\dot{\Gamma}_{\mathscr{P}}$.

Several classical results and approaches are contained as particular cases.

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