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Generalized multiplication modules (**)

Introduction

All rings considered in this paper are commutative and have unity. A submodule N of an R-module M is said to be prime if whenever ak belongs to N, a in R and k in $M-N$, then $aM \subseteq N$. Equivalently whenever $AK \subseteq N$, A an ideal of R and K a submodule of M, $K \nsubseteq N$ implies that $AM \subseteq N$ [**5**]. N is said to be a multiplication submodule of M if whenever a submodule K is contained in N, there is an ideal A of R such that $K = AN$. M is said to be a multiplication module if every submodule of M is a multiplication submodule [**5**]. In case every proper submodule of M is a multiplication submodule, we call M to be a generalized multiplication module. M is said to be an almost multiplication module if M_P is a multiplication R_P-module for each prime ideal P of R. Dimension of M is defined to be r if there exist proper prime submodules $P_0, P_1, ..., P_r$ such that $P_0 < P_1 < P_2 < ... < P_r$ but there is no such chain of $r+2$ prime submodules. We prove here that the dimension of a multiplication module, even of an almost multiplication module, cannot exceed one. In section **4**, we define a (PC)-module and establish the structure of a (PC)-generalized multiplication module over a quasi-local ring.

1 – The following results can be easily derived from the corresponding results of D. D. Anderson [**1**].

Lemma 1.1. *If M is a module over a quasi-local ring R then every multiplication submodule of M is cyclic.*

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Lemma 1.2. *If N is a multiplication submodule of an R-module M and S is any multiplicatively closed subset of R then N_S is a multiplication submodule of M_S over R_S.*

Lemma 1.3. *Let N be a submodule of an R-module M such that $(O:N)$ is contained in only finitely many maximal ideals $P_1, P_2, \ldots, P_r$ of R. If N_{P_i} is a cyclic submodule of M_{P_i} over R_{P_i} for $i = 1, 2, \ldots, r$, then N is a cyclic submodule of M.*

Lemma 1.4. *If N is a multiplication submodule of an R-module M such that $(O:N)$ is contained in only finitely many maximal ideals of R then N is cyclic.*

Lemma 1.5. *Let N be a submodule of a module M over a semi-quasilocal ring R. The following statements are equivalent.*

(1) *N is a multiplication submodule.*
(2) *N is locally cyclic.*
(3) *N is cyclic.*

Lemma 1.6. *If M is a multiplication (almost multiplication) module over a semi–quasi-local ring R then every submodule of M is cyclic.*

Definition 1.7. Let N be a submodule of an R-module M. If for any two ideals A and B of R with $AN \subseteq BN$, we have $A \subseteq B + (O:N)$, then we say that N is *a weak-cancellation submodule* and if we have $A \subseteq B$, then we say that N is *a cancellation submodule.*

Lemma 1.8. *Let N be a submodule of an R-module M. The following statements are equivalent.*

(1) *N is weak cancellation and multiplication submodule.*
(2) *N is finitely generated and multiplication submodule.*
(3) *N is finitely generated and locally cyclic.*

The proof for (3) implies (1) is as follows. Let K be a submodule of M with $K \subseteq N$. Let P be any prime ideal of R. Since N is finitely generated and N_P is cyclic, $K_P = (K_P:N_P)N_P = ((K:N)N)_P$ which implies that $K = (K:N)N$ and thus N is a multiplication submodule. N_P being cyclic is weak cancellation and hence N is weak cancellation since it is finitely generated.

Corollary 1.9. *A noetherian almost multiplication module is a multiplication module.*

2 – In this section we prove that a multiplication module and an almost multiplication module has dimension at most one. The proof is based on the following lemmas.

Lemma 2.1. *Let N be a multiplication submodule of an R-module M. If a submodule K is contained in N then $K = (K:N)N$.*

Proof. Trivial.

Lemma 2.2. *If every prime submodule of M is finitely generated then every submodule of M is finitely generated.*

Proof. The proof can be easily derived from the corresponding proof for ideals.

Lemma 2.3. *If N is a prime submodule of M then $(N:M)$ is a prime ideal of R.*

Proof. Let $(N:M) = P$. Let $ab \in P$. Thus $abM \subseteq N$ but $aM \nsubseteq N$. N being prime, we get that $bM \subseteq N$ which implies that $b \in P$.

Lemma 2.4. *Let M be an R-module and N a proper prime submodule of M. Let $(N:M) = P$ (it is prime ideal of R). Then there is an injective map f from the set of prime submodules of M which are contained in N to the set of proper prime submodules of M_P. Moreover if M is a multiplication submodule of M then f is bijective.*

Proof. Let $S = R - P$ which is a multiplicatively closed subset of R containing the unity 1 of R. Let N^e denotes the extension of N to M_P and N^{ec} the contraction of N^e to M. If K is a prime submodule of M contained in N then it is easy to see that K^e is a prime submodule of M_P. We show that $K^e < M_P$. Suppose $K^e = M_P$. If $m \in M - N$ then $(m/1) \in M_P = K^e \subseteq N^e$. Let $m/1 = n/s$ for some $n \in N$ and $s \in S$. Thus we can find $s' \in S$ such that $s'(sm - n) = 0$ which implies that $s'sm = s'n \in N$. But $m \notin N$ and so $s'sM \subseteq N$. Therefore $s's \in (N:M) = P$ which is impossible.

Now suppose that there is a prime submodule H contained in N such that $K^c = H^c$. We shall prove that $K = H$. Suppose $K \neq H$. There exists $k \in K - H$ (or $h \in H - K$). Then $k/1 \in K^c = H^c$. Let $k/1 = h/s$ for some $h \in H$ and $s \in S$. Let $s' \in S$ such that $s'(sk - h) = 0$. Thus $s'sk = s'h \in H$ and $k \notin H$ which implies that $s'sM \subseteq H$. Hence $s's \in (H:M) \subseteq (N:M) = P$ which is impossible.

Now we assume that M is a multiplication submodule of M. Thus for each submodule K, $K = (K:M)M$. Let T be any proper prime submodule of M_P. It is easy to see that T^c is a prime submodule of M. We shall prove that $T^c \subseteq N$. If $T^c \nsubseteq N$ then $(T^c:M) \nsubseteq (N:M) = P$. Let $s \in (T^c:M) \cap S$ which is non-empty. Thus $sM \subseteq T^c$ which implies that $(s)^e M_P \subseteq T^{ce} = T$. But $(s)^e = R_P$. Hence $M_P \subseteq T$ which is impossible. This completes the proof.

Lemma 2.5. *Let M be an R-module such that every submodule of M is cyclic. If for a non-zero element x of M, (x) and (O) are proper prime submodule of M then (x) is a maximal submodule of M.*

Proof. Let $m \in M$ such that $M = (m)$. Let A be the annihilator ideal of M. Then we know that M and R/A, as modules, are isomorphic. Since (O) is prime submodule of M, it is easy to see that A is a prime ideal of R. Thus R/A is a domain. As each submodule of M is cyclic, each ideal of R/A is principal and hence every non-zero proper prime ideal of R/A is maximal which implies that every non-zero proper prime submodule of M is maximal.

Lemma 2.6. *If $f: M \to M'$ is an epimorphism of R-modules with kernel K then there is one-to-one correspondence between the set of prime submodules of M which contain K and the set of proper prime submodules of M'.*

Proof. Let N be a prime submodule of M containing K. Let $r \in R$ and $m \in M$ such that $rf(m) \in f(M)$ but $f(m) \notin f(N)$. We show that $rM' \subseteq f(N)$. As $f(rm) = rf(m) \in f(N)$, $f(rm) = f(n)$ for some $n \in N$. Thus $rm - n \in K$ and so $rm \in K + N = N$ and $m \notin N$ which implies that $rM \subseteq N$. Hence $rm' = rf(M) = f(rM) \subseteq f(N)$. Similarly it can be proved that if N' is any prime submodule of M' then $f^{-1}(M')$ is a prime submodule of M containing K. That the correspondence is one-to-one is clear.

Lemma 2.7. *If M is an R-module such that every submodule of M is cyclic then* $\dim(M) \leqslant 1$.

Proof. Suppose that there exist proper prime submodules P_1, P_2, P_3 of M such that $P_1 < P_2 < P_3$. By Lemma 2.6 we get that $(O) < P_2/P_1 < P_3/P_1$ are proper prime submodules of M/P_1 which is a contradiction to Lemma 2.5 since every submodule of M/P_1 is cyclic.

Theorem 2.8. *If M is a multiplication module (almost multiplication module) over a ring R then* $\dim(M) \leqslant 1$.

Proof. If P is any prime ideal of R then M_P is a multiplication module over the quasi-local ring R_P. By Lemma 1.1 we get that every submodule of M_P is cyclic. Let N be any proper prime submodule of M. Then $P = (N:M)$ is a prime ideal of R. Thus $\dim(M_P) \leqslant 1$ by Lemma 2.7. Using Lemma 2.4 we deduce that $\dim(M) \leqslant 1$.

Lemma 2.9. *Let M be a cyclic module over a ring R. Let N be a non-cyclic submodule of M such that every submodule properly containing N is cyclic. Then N is a prime submodule of M.*

Proof. Let $M = (x)$. Let A be the annihilator ideal of M. We know that M and R/A, as modules, are isomorphic. Let $f: M \to R/A$ be this isomorphism. Thus $f(N)$ is a non-principal ideal of R/A and every ideal properly containing $f(N)$ is principal. Hence $f(N)$ is a prime ideal of R/A ([**2**]$_2$, p. 33) and consequently N is a prime submodule of M.

Theorem 2.10. *If every prime submodule of an R-module M is cyclic then every submodule of M is cyclic.*

Proof. Let $M = (x)$. Let A and f be as in Lemma 2.9. Since every prime submodule of M is cyclic, every prime ideal of R/A is principal and hence every ideal of R/A is principal ([**2**], Theorem 2.1). Hence every submodule of M is cyclic.

Theorem 2.11. *A finitely generated almost multiplication module M over a multiplication ring R is a multiplication module.*

Proof. Let K and N be submodules of M with $K \subseteq N$. As $(K:M) \subseteq (N:M)$, there is an ideal A of R such that $(K:M) = A(N:M)$. For any prime ideal P of R, $(K_P:M_P) = A_P(N_P:M_P)$ since M is finitely generated. By Lemma 1.1, M_P is cyclic. Thus $K_P = (K_P:M_P)M_P = A_P(N_P:M_P)M_P = A_PN_P = (AN)_P$. Hence $K = AN$.

Theorem 2.12. *Let M be a module over an almost multiplication ring R. If M is a multiplication submodule of M, then M is an almost multiplication module.*

Proof. Let P be any prime ideal of R and K, N be any two submodules of M_P with $K \subseteq N$. As M_P is cyclic by Lemma 1.1 and R_P is a multiplication ring, there is an ideal A of R_P such that $(K:M_P) = A(N:M_P)$ so that $K = (K:M_P)M_P = A(N:M_P)M_P = AN$. Thus M_P is a multiplication R_P-module or each prime ideal P of R. This completes the proof.

Theorem 2.13. *Let M be a module over a multiplication ring R. If M is a multiplication submodule of M then M is a multiplication module.*

Proof. Let K and N be any two submodules of M with $K \subseteq N$. Since M is a multiplication submodule of M, $K = (K:M)M$ and $N = (N:M)M$. Now $(K:M) \subseteq (N:M)$ and R is a multiplication ring. There is an ideal A of R such that $(K:N) = A(N:M)$. Hence $K = (K:M)M = A(N:M)M = AN$. This completes the proof.

Theorem 2.14. *Let M be a faithfull module over a domain R such that M is a multiplication submodule of M. Then*

(i) *M is an almost multiplication module if and only if R is an almost multiplication ring;*

(ii) *M is a multiplication module if and only if R is a multiplication ring.*

Proof. Let $0 \neq m \in M$ and $r \in R$ such that $rm = 0$. Let $(m) = AM$ where A is an ideal of R. As $rAM = r(m) = (0)$, $rA = (0)$. Thus $r = 0$ since $A \neq (0)$. It shows that M is torsion free. It is easy to see that M is locally cyclic and torsion free and hence locally cancellation module and consequently M itself is a cancellation module. The desired results are now immediate.

3 – In this section we study weak multiplication modules which are defined to be the modules in which every prime submodule is a multiplication submodule.

Theorem 3.1. *A weak multiplication module is an almost multiplication module and hence its dimension is $\leqslant 1$.*

Proof. Let M be a weak multiplication module over a ring R. Let P be any proper prime ideal of R. It is clear that M_P is also a weak multiplication module over the quasi-local ring R_P. Thus every prime submodule of M_P, being a multiplication submodule, is cyclic by Lemma 1.1. Theorem 2.10 implies that every submodule of M_P is cyclic. Thus M_P is a multiplication module and hence M is an almost multiplication module. By Theorem 2.8 we deduce that $\dim(M) \leqslant 1$.

Corollary 3.2. *A weak multiplication module over a quasi-local (semi–quasi-local) ring is a multiplication module.*

Lemma 3.3. *A maximal submodule is prime.*

Proof. Trivial.

Theorem 3.4. *If M is an R-module such that M is a multiplication submodule of M then M possesses a maximal (and hence prime) submodule.*

Proof. Let $0 \neq x \in M$. Let P be any maximal ideal of R containing A, the annihilator ideal of x. For some ideal I of R, $(x) = IM$. Observe that $PM < M$. In fact if $PM = M$ then $P(x) = PIM = IM = (x)$. Nakayama's Lemma implis that $P + A = R$ which is impossible. Let N be any submodule of M such that $PM \subseteq N$. Thus $P \subseteq (N:M)$. Therefore $(N:M) = P$ or R and consequently $N = (N:M)M = PM$ or RM which proves that N is a maximal submodule.

Lemma 3.5. *Let M be a cancellation module over a ring R. If M is a weak multiplication module then M is a multiplication module and R is a multiplication ring.*

Proof. Any submodule N of M is of the type $N = AM$ for some ideal A of R. It can be checked that the mapping $A \to AM$, A an ideal of R, is an isomorphism from the lattice of ideals of R to the lattice of submodules of M in which prime ideals correspond to prime submodules. Since M is a weak multiplication module, R is a weak multiplication ring and hence a multiplication ring ([6], p. 429) which implies that M is a multiplication module.

Theorem 3.6. *A finitely generated, faithfull and weak multiplication module over a ring R is a multiplication module and R is a multiplication ring.*

Proof. Let A and B be any two ideals of R such that $AM \subseteq BM$. For any proper prime ideal P of R, M_P is cyclic by Lemma 1.1 and Theorem 3.1. Thus M_P is a weak cancellation module. $AM \subseteq BM$ implies that $A_P M_P \subseteq B_P M_P$ and so $A_P \subseteq B_P + (0:M)_P$. Therefore $A \subseteq B + (0:M) = B$ since M is faithfull. The result now follows from Theorem 3.5.

4 – In this section we study generalized multiplication modules which are defined to be the modules in which every proper submodule is a multiplication submodule.

Definition 4.1. A module M over a ring R is said to be *a pseudo cancellation module* (*(PC)-module*) if for each proper ideal A of R, $AM < M$.

Lemma 4.2. *Let M be a (PC)-module over a ring R. If M is a multi-*

plication submodule of M then every proper submodule of M is contained in a maximal submodule.

Proof. Let N be a proper submodule of M. As $(N:M)M = N < M$, $(N:M)$ is a proper ideal of R. Let P be any maximal ideal of R containing $(N:M)$. So $N \subseteq PM < M$. It is easy to check that PM is a maximal submodule of M.

Lemma 4.3. *A generalized multiplication module has dimension* $\leqslant 2$.

Proof. It follows from Theorem 2.8.

Lemma 4.4. *Let M be a module over a ring R such that M is a multiplication submodule of M. If R is noetherian then M is noetherian. If M has a torsion free element and R is a domain then the converse is also true.*

Proof. For every submodule N of M, $N = (N:M)M$. The direct part now follows from the fact that if $N_1 \subseteq N_2 \subseteq \ldots$, be a chain of submodules of M then $(N_1:M) \subseteq (N_2:M) \subseteq \ldots$, is a chain of ideals of R. Let x be a torsion free element in M. If $A_1 \subseteq A_2 \subseteq \ldots$, is a chain of ideals in domain R then $A_1 x \subseteq A_2 x \subseteq \ldots$, is a chain of submodules of M. For some r, $A_r x = A_{r+1} x$ which implies that $A_r = A_{r+1}$.

Theorem 4.5. *If M is a (PC)-generalized multiplication module over a noetherian ring R then M is noetherian.*

Proof. Let N be any proper submodule of M. Then N is a multiplication submodule of M. By Lemma 4.4, N is finitely generated. Only thing remains to prove is that M is finitely generated. Consider two cases.

(i) If M has a maximal submodule K then $M = K + (x)$ for every $x \in M - K$. K being finitely generated, M is finitely generated.

(ii) If M has no maximal submodule then for each submodule A of M, there is a submodule B of M such that $A < B < M$. B being a multiplication submodule, for some ideal I of R, $A = IB \subseteq IM < M$. Again there is an ideal J of R such that $A = J(IM) = (JI)M$ which shows that M is a multiplication submodule of M. Again by Lemma 4.4, we get that M is noetherian.

Corollary 4.6. *A (PC)-generalized multiplication module over a quasi-local (semi–quasi-local) ring is noetherian.*

Definition 4.7. A module M over a ring R is said to be *quasi-local* if M has a unique maximal submodule which contains all the proper submodules.

Theorem 4.8. *Let M be a (PC)-generalized multiplication module over a quasi-local (semi–quasi-local) ring R. If M is not a multiplication module then there is a maximal submodule N of M such that N is a local module.*

Proof. If M is not a multiplication module than there is a submodule N of M such that there is no ideal A of R with $N = AM$. If N is not maximal then there is a submodule K such that $N < K < M$. For some ideal I of R, $N = IK \subseteq IM < M$. Again for some ideal J or R, $N = J(IM) = (JI)M$ which contradicts the choice of N. Thus N is a maximal submodule. Let $\{x_j : j \in W\}$ be the collection of those elements of N for which $(x_j) < N$. As in the preceeding part of the proof, we can find ideals I_j of R such that $(x_j) = I_j M$. Thus $\sum_{j \in W} (x_j) = (\sum_{j \in W} I_j)\ M$ and it follows that $\sum_{j \in W} (x_j) \neq N$. It is obvious that $\sum_{j \in W} (x_j)$ is the unique maximal submodule of N. By Lemma 1.6, every submodule of N is cyclic and hence N is a local module.

Lemma 4.9. *Let M be a (PC)-module over a ring R. If M is a multiplication submodule of M then there is a $1 \leftrightarrow 1$ correspondence between the set of maximal ideals of R and the set of maximal submodules of M. Thus R is quasi-local (semi–quasi-local) if and only if M is quasi-local (semi–quasi-local).*

Proof. The proof follows from the fact that if P is a maximal ideal of R then PM is a maximal submodule of M and if S is a maximal submodule of M then $(S:M)$ is a maximal ideal of R and $(S:M)M = S$.

Lemma 4.10. *Let M be module over a domain R having a torsion free element m. If M is a multiplication submodule of M then M is torsion free.*

Proof. Suppose $0 \neq x \in M$ and $r \in R$ such that $rx = 0$. For some ideal I of R, $(x) = IM$. Thus $rIM = (0)$. Therefore for each $i \in I$, $rim = 0$ which implies that $ri = 0$ and so $rI = (0)$. Since $I \neq (0)$, $r = 0$.

Lemma 4.11. *Let M be a module over a domain R having a torsion free element. If P is any non-zero prime ideal of R then M_P has a torsion free element over the domain R_P.*

Proof. Trivial.

Theorem 4.12. *If M is a multiplication module over a domain R having a torsion free element m then R is a Dedekind domain.*

Proof. The proof follows from the fact that R and Rm, as R-modules, are isomorphic.

Theorem 4.13. *Let M be a (PC)-module over a quasi-local ring R. Then M is a generalized multiplication module if and only if M satisfies one of the following*: (a) *M is a uniserial module*; (b) *M has a unique infinite descending chain of submodules without proper refinements*; (c) *M possesses maximal submodules and through each maximal submodule, there passes a unique composition series of M*; (d) *M possesses maximal submodules and all the non-zero submodules contained in a maximal submodule form an infinite descending chain without proper refinements.*

Proof. Suppose M is a generalized multiplication module. We discuss the following two cases.

Case I. M is a multiplication submodule of M.
Since R is quasi-local, M is cyclic. Let $M = (w)$. If A denotes the annihilator ideal of M then we known that M and R/A as R-modules, are isomorphic. Now every proper submodule of M, being multiplication submodule, is cyclic. Thus every ideal of R/A is principal. Also R is a quasi-local ring implies that R/A is a local ring. If P is the unique maximal ideal of R then $\bar{P} = P/A$ is the unique maximal ideal of $\bar{R} = R/A$ and $\bigcap_{n=1}^{\infty} (\bar{P})^n = (\bar{0})$. Two cases arise

(i) $(\bar{P})^n = (\bar{P})^{n+1}$ for some least positive integer n. Then $(\bar{P})^n = (\bar{0})$. If $x \in R$ is such that $(\bar{x}) = \bar{P}$ then the only ideals of $\bar{R}$ are R, $(\bar{x}), (\bar{x})^2, \ldots, (\bar{x})^n = (\bar{0})$. Thus there exist $x_1, x_2, \ldots, x_n$ in M such that the only submodules of M are M, $(x_1), (x_2), \ldots, (x_n)$ with $M > (x_1) > (x_2) > \ldots > (x_n) = (0)$. Thus M is a uniserial module.

(ii) $(\bar{P})^n > (\bar{P})^{n+1}$ for every integer n. Then $(\bar{0}) = \bigcap_{n=1}^{\infty} (\bar{P})^n$ is a prime ideal of $\bar{R}$. Let I be any non-zero proper ideal of $\bar{R}$. Let $r \in R$ be such that $I = (\bar{r})$. Let k be the integer such that $\bar{r} \in (\bar{P})^k - (\bar{P})^{k+1}$. Let $\bar{r} = \bar{z}(\bar{x})^k$ where $z \in R$ and $\bar{z} \notin \bar{P}$. But then $\bar{z}$ is a unit and hence $(\bar{r}) = (\bar{x})^k = (\bar{P})^k$. Thus the only ideals of $\bar{R}$ are $\bar{R}$, $(\bar{0}), (\bar{x}), (\bar{x})^2, (\bar{x})^3, \ldots$ with $\bar{R} > (\bar{x}) > (\bar{x})^2 > (\bar{x})^3 > \ldots$. Therefore we can find $x_1, x_2, x_3, \ldots$ in M such that the only submodules of M are $(0), M, (x_1), (x_2), (x_3), \ldots$ with $M > (x_1) > (x_2) > (x_3) > \ldots$.

Case II. M is not a multiplication submodule of M.

In this case, M possesses a maximal submodule by Theorem 4.8. Let N be any maximal submodule. Since M is a generalized multiplication module, N is a multiplication submodule. As discussed in Case I, either N is a uniserial module or the non-zero submodules of N form an infinite descending chain without proper refinements. Thus one of the following holds:

(i) M possesses maximal submodules and through each maximal submodule, there passes a unique composition series of M;

(ii) M possesses maximal submodules and all the non-zero submodules contained in a maximal submodule form an infinite descending chain without proper refinements.

Conversely, assume that M satisfies any one of (a), (b), (c) or (d).

(i) Let M satisfies (a). Let $(0) < N_1 < N_2 < \dots < N_r = M$ be the unique composition series of M. If $0 \neq x \in N_1$ and $y \in N_2 - N_1$ then clearly $(x) = N_1$. Also the submodule (y) is one of $N_1, N_2, \dots, N_r$ and it is not difficult to see that $(y) = N_2$. Continuining in this way, we get that all submodules of M are cyclic and hence M is a multiplication module.

(ii) Let M satisfies (b). Let $M > N_1 > N_2 > \dots$ be the unique infinite descending chain of submodule without refinements. If $x \in M - N_1$ is any element then $x \notin N_1$ and hence (x) is different from $N_1, N_2, N_3, \dots$.

It implies that $(x) = M$. Similarly if we consider the chain $N_1 > N_2 > N_3 \dots$ then we find that N_1 is cyclic. Continuing we conclude that every submodule of M is cyclic and hence M is a multiplication module.

(iii) Suppose M satisfies (c). Let N_1 be a maximal submodule of M. Let $M > N_1 > N_2 > \dots > N_r = (0)$ be the unique composition series of M passing through N_1. Let S be any proper submodule of M. Then $M > S$ is a normal series of M and it can be refined to a composition series

$$M > S_1 > S_2 > \dots > S_{i-1} > S_i = S > S_{i+1} > \dots > S_n = (0) .$$

Thus S_1 is a maximal submodule and through S_1, there passes a unique composition series. Thus S_1 is uniserial and therefore each submodule of S_1 is cyclic. It implies S is cyclic. We find that each proper submodule of M is cyclic and hence M is a generalized multiplidation module.

(iv) Suppose M satisfies (d). Let N be a maximal submodule of M and all the submodules contained in N be $N_1, N_2, N_3, \dots$ such that $M > N > N_1 > N_2 > N_3 > \dots$. Thus $N > N_1 > N_2 > N_3 \dots$ is the unique descending chain of submodules of N.

Hence $N, N_1, N_2, \dots$ all are cyclic. If $N = (x)$ and $y \in M - N$ then clearly $M = (x) + (y)$. Let S be any submodule of M different from (0) and M. Then $\bar{M} = M/S$ is generated by $\bar{x} = x + S$ and $\bar{y} = y + S$. Let F be the family of all proper submodules of $\bar{M}$. Let $(\bar{0}) \subseteq \bar{B}_1 \subseteq \bar{B}_2 \subseteq \ldots$ be any infinite ascending chain of elements of F. Let $\bar{B} = \bigcap_{i=1}^{\infty} \bar{B}_i$ which is a submodule of $\bar{M}$. Two cases arise.

(1) $\bar{B} < \bar{M}$ (2) $\bar{B} = \bar{M}$. If $\bar{B} = \bar{M}$ then $\bar{x} \in \bar{B}_i$ and $y \in \bar{B}_j$ for some i and j. If $i \leqslant j$ then $\bar{x}, \bar{y} \in \bar{B}_j$ and thus $\bar{B}_j = \bar{M}$ which is not true. Thus only possibility is $\bar{B} < \bar{M}$ and hence $\bar{B} \in F$ is the least upper bound of the chain $\bar{0} \subseteq \bar{B}_1 \subseteq \bar{B}_2, \ldots$. Zorn's lemma guarantees the existence of a maximal element $\bar{P}$ in F. It immediately implies that P is a maximal submodule of M containing S. Thus there exists a unique infinite descending chain $M > P > S_1 > S_2 > \ldots > S_{i-1} > S_i = S > S_{i+1} > \ldots$ passing through M and S, since $S \subseteq P$. Thus P has a unique infinite descending chain $P > S_1 > S_2 > \ldots$ without refinements. Thus $P, S_1, S_2, \ldots$ all are cyclic and hence S is cyclic. Thus every proper submodule of M is cyclic and therefore M is a generalized multiplication module.

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