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A contribution to the theory of rods (**)

1 - Introduction

In the mechanics of rods many problems arise which are interesting for the theorist and also significant for the engineer. Some of these problems provide a testbed for new analytical techniques and still others give hints for the creation of sharper mathematical tools.

In recent papers of mathematical flavour (see, e.g., [**1**]) the rod is represented as a one-dimensional continuum with local structure described by directors (as in the treatise [**2**]).

The general model with three directors is more comprehensive than any of the models accepted in engineering research and also than the classical scheme of the elastica. On the other hand the latter schemes lead to relatively simple and illuminating examples and counterexamples, in stability theory for instance (see, e.g., [**1**], and the papers quoted there; also $[\mathbf{3}]_1$). It seems, therefore, not without interest to classify the special models in a systematic way so that the limitations sometimes implicit in special results can be more easily perceived and the concrete implications of hypotheses accepted in general theorems more simply deduced.

The procedure followed here for such classification mirrors developments introduced in $[\mathbf{3}]_2$ for three-dimensional continua with affine structure.

Constraints of increasing severity are introduced and the geometric and mechanical consequences explored. Actually, to obtain the classical special theories, simplifying dynamic hypotheses must be also accepted. This aspect

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of the matter is not pursued here, though a few hints are given; nor is any mention made of special results which apply within the limits of the linear approximation.

2 - The rod

A rod is taken here to be a one-dimensional continuum with affine microstructure; anyone of its placements is defined when the following quantities are given:

(i) The position vector $\boldsymbol{p}$ measured from a fixed point, as a smooth function of a parameter s:

$$\boldsymbol{p} = \boldsymbol{p}(s)\,, \qquad s \in [0, \bar{s}]\,, \tag{2.1}$$

s is interpreted as arc length along the curve $\mathscr{C}$ defined by (2.1), so that $|\mathrm{d}\boldsymbol{p}/\mathrm{d}s| = 1$ and $\bar{s}$ is the total length of $\mathscr{C}$.

(ii) A triad of vectors $\boldsymbol{d}_{(R)}$ $(R = 1, 2, 3)$, the *directors*, again as smooth functions of s in $[0, \bar{s}]$, with the condition

$$\boldsymbol{d}_{(1)} \cdot \boldsymbol{d}_{(2)} \times \boldsymbol{d}_{(3)} \neq 0\,. \tag{2.2}$$

If a reference triad $\boldsymbol{c}_{(H)}$ $(H = 1, 2, 3)$ of fixed orthogonal unit vectors is given, one can assign, instead of the directors, the tensor $\boldsymbol{K}$ which has the property that

$$\boldsymbol{d}_{(H)} = \boldsymbol{K}\boldsymbol{c}_{(H)} \qquad (H = 1, 2, 3) \tag{2.3}$$

and is defined by

$$\boldsymbol{K} = \sum_{1}^{3}{}_{H}\, \boldsymbol{d}_{(H)} \otimes \boldsymbol{c}_{(H)}\,. \tag{2.4}$$

A variety of physical objects can be described within the scheme. Most generally one can imagine a continuous string of separately deformable links. More particularly, after the introduction of internal constraints of increasing severity, one is lead to the schemes more widely used for the description of bars, thin filaments, strings, etc.

Further notation used in the paper is as follows:

(i) $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ are tangent, normal and binormal unit vectors for $\mathscr{C}$, so that Frenet's formulae can be written in the form

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s} = \boldsymbol{t}\,, \quad \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \varkappa\boldsymbol{n}\,, \quad \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s} = -\varkappa\boldsymbol{t} - \tau\boldsymbol{b}\,, \quad \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}s} = \tau\boldsymbol{n}\,, \tag{2.5}$$

where $\varkappa$ and τ are respectively curvature and torsion of $\mathscr{C}$.

(ii) $\boldsymbol{d}^{(R)}$ $(R = 1, 2, 3)$ is the set of reciprocal directors, such that

$$\boldsymbol{d}_{(R)}\cdot\boldsymbol{d}^{(S)} = \delta^S_R\,, \qquad \sum_1^3{}_R\, \boldsymbol{d}_{(R)}\otimes \boldsymbol{d}^{(R)} = \mathbf{1}\,; \tag{2.6}$$

in terms of the tensor $\boldsymbol{K}$ we have $\boldsymbol{d}^{(H)} = \boldsymbol{K}^{-T}\boldsymbol{c}_{(H)}$, where $\boldsymbol{K}^{-T}$ is the inverse of the transpose of $\boldsymbol{K}$.

(iii) The tensor

$$\boldsymbol{W} = \sum_1^3{}_R \frac{\mathrm{d}\boldsymbol{d}_{(R)}}{\mathrm{d}s} \otimes \boldsymbol{d}^{(R)}, \tag{2.7}$$

with the property

$$\frac{\mathrm{d}\boldsymbol{d}_{(R)}}{\mathrm{d}s} = \boldsymbol{W}\boldsymbol{d}_{(R)}\,, \tag{2.8}$$

is the *wryness* of the rod.

With the use of the Ricci tensor $\boldsymbol{\varepsilon}$ as an operator over the space of vectors (second order tensors) into the space of second order tensors (vectors), one can introduce the vector $\boldsymbol{w}$ associated with $\boldsymbol{W}$, such that

$$\boldsymbol{w} = \tfrac{1}{2}\boldsymbol{\varepsilon}\boldsymbol{W}\,, \qquad \operatorname{skw}\boldsymbol{W} = \boldsymbol{\varepsilon}\boldsymbol{w}\,. \tag{2.9}$$

Because $\boldsymbol{\varepsilon}(\boldsymbol{a}\otimes \boldsymbol{b}) = \boldsymbol{a}\times\boldsymbol{b}$, one has

$$\boldsymbol{w} = \frac{1}{2}\sum_1^3{}_R \frac{\mathrm{d}\boldsymbol{d}_{(R)}}{\mathrm{d}s} \times \boldsymbol{d}^{(R)}. \tag{2.10}$$

Remarks. Using Frenet's formulae one obtains that

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}\otimes\boldsymbol{t} + \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s}\otimes\boldsymbol{n} + \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}s}\otimes\boldsymbol{b} = 2\operatorname{skw}\big(\boldsymbol{n}\otimes(\varkappa\boldsymbol{t} + \tau\boldsymbol{b})\big) \tag{2.11}$$

and the vector associated with this tensor is $\tau\boldsymbol{t} - \varkappa\boldsymbol{b}$.

As we shall see, in most cases of practical interest one of the directors, say $\boldsymbol{d}_{(3)}$, is constrained to be parallel to $\boldsymbol{t}$: $\boldsymbol{d}_{(3)} = \nu \boldsymbol{t}$. Then some formulae take a more special form, e.g.

$$(2.12)\qquad \boldsymbol{W} = \sum_{1}^{2}{}_{R} \frac{\mathrm{d}\boldsymbol{d}_{(R)}}{\mathrm{d}s} \otimes \boldsymbol{d}^{(R)} + \nu\left(\nu\varkappa \boldsymbol{n} + \frac{\mathrm{d}\nu}{\mathrm{d}s}\boldsymbol{t}\right) \otimes (\boldsymbol{K}\boldsymbol{K}^{T})^{-1}\boldsymbol{t}\,.$$

3 - Displacement and strain

A particular placement of the rod is fixed from now on as the reference placement and displacements are measured from it. Quantities read on the reference placement are marked with an asterisk, in particular s^* is the arc length along $\mathscr{C}^*$.

A vector $\boldsymbol{u}$ and an invertible tensor $\boldsymbol{G}$ with positive determinant (both smooth functions of s^*) are used to specify a displacement

$$(3.1)\qquad \boldsymbol{p} = \boldsymbol{p}^* + \boldsymbol{u}\,, \qquad \boldsymbol{d}_{(R)} = \boldsymbol{G}\boldsymbol{d}^*_{(R)}\,, \qquad R = 1, 2, 3\,.$$

In the formulae the independent variable is not shown explicitly; it could be either s or s^* because the correspondence between the variables s and s^* is assumed to be one-to-one: more particularly the *stretch* λ, $\lambda = \mathrm{d}s/\mathrm{d}s^*$, will be taken to be strictly positive and hence the *extension* $\delta = \lambda - 1$ always larger than -1. Notice the relations

$$(3.2)\qquad \lambda \boldsymbol{t} = \boldsymbol{t}^* + \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}s^*}\,, \qquad\qquad (3.3)\qquad \boldsymbol{G} = \sum_{1}^{3}{}_{R}\, \boldsymbol{d}_{(R)} \otimes \boldsymbol{d}^{*(R)},$$

$$(3.4)\qquad \boldsymbol{d}^{(R)} = \boldsymbol{G}^{-T}\boldsymbol{d}^{*(R)}, \qquad\qquad (3.5)\qquad \boldsymbol{W} = \frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}s}\boldsymbol{G}^{-1} + \lambda^{-1}\boldsymbol{G}\boldsymbol{W}^*\boldsymbol{G}^{-1}.$$

To characterize strain many choices are possible; later developments accord a special preference to the following choice of one vector $\boldsymbol{e}$ and two tensors $\boldsymbol{E}$ and $\boldsymbol{F}$:

$$(3.6)\qquad \begin{aligned} \boldsymbol{e} &= \lambda \boldsymbol{G}^{-1}\boldsymbol{t} - \boldsymbol{t}^*, \qquad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{G}^T\boldsymbol{G} - 1)\,, \\ \boldsymbol{F} &= \lambda \boldsymbol{G}^{-1}\boldsymbol{W}\boldsymbol{G} - \boldsymbol{W}^* = \boldsymbol{G}^{-1}\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}s^*}\,. \end{aligned}$$

The displacement is rigid if and only if $\boldsymbol{e}$, $\boldsymbol{E}$, *and* $\boldsymbol{F}$ *all vanish identically* (see [**5**], Sect. 63).

Proof. In a rigid displacement there exist a constant vector $\boldsymbol{k}$ and a constant orthogonal tensor $\boldsymbol{Q}$ (with $\det \boldsymbol{Q} = 1$) such that $\boldsymbol{p} = \boldsymbol{Q}(\boldsymbol{p}^* + \boldsymbol{k})$, $\boldsymbol{G} = \boldsymbol{Q}$; it follows that $\boldsymbol{F}$, $\boldsymbol{E}$ and $\boldsymbol{e}$ vanish identically.

Vice versa, if $\boldsymbol{F}$ vanishes for all s^*, $\boldsymbol{G}$ is independent of s^*; if also $\boldsymbol{E}$ is zero, $\boldsymbol{G}$ is necessarily orthogonal and $\det \boldsymbol{G} = +1$ (because, by hypothesis, $\det \boldsymbol{G}$ is always positive); finally, if $\boldsymbol{e}$ vanishes, then: $(\mathrm{d}/\mathrm{d}s^*)(\boldsymbol{p} - \boldsymbol{G}\boldsymbol{p}^*) = \boldsymbol{0}$, $\boldsymbol{G} \in \mathrm{Orth}^+$; hence the displacement is rigid.

Remark 1. Notice the relations

$$\boldsymbol{G} = \boldsymbol{K}(\boldsymbol{K}^*)^{-1}, \qquad \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}s^*} = \mathrm{sym}\,((\boldsymbol{1} + 2\boldsymbol{E})\boldsymbol{F})\,. \tag{3.7}$$

Remark 2. In the special case when $\boldsymbol{d}_{(3)}$ is constrained to be parallel to $\boldsymbol{t}$ (see Remark in Sect. **2**) we have $\boldsymbol{e} = (\lambda\nu^*/\nu - 1)\,\boldsymbol{t}^*$. If the further constraint $\nu = \lambda\nu^*$ is introduced, i.e. it is assumed that $\boldsymbol{G}\boldsymbol{t}^* = \lambda\boldsymbol{t}$, then $\boldsymbol{e}$ vanishes identically.

Remark 3. The definitions of this and the previous Section are given so as to allow a direct comparison with formulae of [**4**]. Other Authors prefer different strain characteristics; sometimes (see [**6**], Sect. 61) the following anholonomic components of strain of orientation are used because within certain contexts they are more convenient, as we shall mention

$$J_{RS} = (\boldsymbol{W}\boldsymbol{d}_{(R)})\cdot\boldsymbol{d}_{(S)} - (\boldsymbol{W}^*\boldsymbol{d}^*_{(R)})\cdot\boldsymbol{d}^*_{(S)}\,. \tag{3.8}$$

Actually these components J_{RS} can be interpreted also as components on the fixed system $\boldsymbol{d}^*_{(R)}$ of a new tensor $\boldsymbol{J}$

$$\boldsymbol{J} = \boldsymbol{G}^T\boldsymbol{W}\boldsymbol{G} - \boldsymbol{W}^*.$$

In fact

$$J_{RS} = (\boldsymbol{W}\boldsymbol{G}\boldsymbol{d}^*_{(R)})\boldsymbol{G}\boldsymbol{d}^*_{(S)} - (\boldsymbol{W}\boldsymbol{d}^*_{(R)})\cdot\boldsymbol{d}^*_{(S)} = ((\boldsymbol{G}^T\boldsymbol{W}\boldsymbol{G} - \boldsymbol{W}^*)\,\boldsymbol{d}^*_R)\cdot\boldsymbol{d}^*_S\,;$$

notice that

$$\boldsymbol{J} = \lambda^{-1}\boldsymbol{G}^T\Big(\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}s^*} + (\boldsymbol{G} - \lambda\boldsymbol{G}^{-T})\,\boldsymbol{W}^*\Big) = \lambda^{-1}\Big(\boldsymbol{G}^T\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}s^*} + (2\boldsymbol{E} + (1-\lambda)\boldsymbol{1})\,\boldsymbol{W}^*\Big)$$

$$= \lambda^{-1}(\boldsymbol{1} + 2\boldsymbol{E})(\boldsymbol{F} + \boldsymbol{W}^*) - \boldsymbol{W}^*.$$

These formulae can be used to prove that a displacement is rigid if and only $\boldsymbol{e} = \boldsymbol{0}$, $\boldsymbol{E} = \boldsymbol{0}$, $\boldsymbol{J} = \boldsymbol{0}$.

4 - Kinematics

Consider a *motion* of the rod, i.e. a family of placements depending smoothly on the parameter t, the time.

Then the *speed*

$$v = \dot{p} = \dot{u}\,, \tag{4.1}$$

and the *wrenching*

$$U = \dot{K}K^{-1} = \dot{G}G^{-1} \tag{4.2}$$

(see Remark 1 of Sect. **3**) become relevant, together with a measure of the rate of change of wryness.

For this measure, use will be made of the time-derivative of the tensor F; for $\dot{F}$ the following relation applies

$$\dot{F} = G^{-1}\frac{\partial U}{\partial s^*}G\,; \tag{4.3}$$

in fact, for the inverse of a tensor one has

$$(G^{-1})^{\cdot} = -G^{-1}\dot{G}G^{-1}, \qquad \text{and} \qquad \frac{\partial G^{-1}}{\partial s^*} = -G^{-1}\frac{\partial G}{\partial s^*}dG^{-1}.$$

Notice also the expressions of the time derivatives of e and E

$$\dot{e} = -\lambda G^{-1}Ut + G^{-1}\frac{\partial v}{\partial s^*}\,, \qquad \dot{E} = \operatorname{sym}(G^T UG) = G^T(\operatorname{sym} U)G\,. \tag{4.4}$$

In all these formulae (and also later) a dot indicates, of course, a partial derivative with respect to time of the relevant function expressed in terms of s^* and t.

If $\varrho(\varrho^*)$ is the density of the rod, i.e. the mass per unit length in the placement at the instant t (in the reference placement) a standard argument leads to the *equation of conservation of mass*

$$\varrho\lambda - \varrho^* = 0\,. \tag{4.5}$$

Because $\partial v/\partial s^* = \partial^2 p/\partial s^*\partial t = (\lambda t)^{\cdot} = \dot{\lambda}t + \lambda\dot{t}$, eqn. (4.5) implies

$$\dot{\varrho} + \varrho\frac{\partial v}{\partial s}\cdot t = 0\,. \tag{4.6}$$

Remark. Local, rather then referential, developments are rarely of interest in the theory of rods. Nevertheless we notice the relation

$$\frac{\partial \varrho(s,t)}{\partial t} + \frac{\partial}{\partial s}\Big(\varrho \int_0^s \frac{\partial \boldsymbol{v}(s,t)}{\partial s} \cdot \boldsymbol{t}(s,t)\,\mathrm{d}s\Big) = 0\,, \tag{4.7}$$

which follows from (6) and from the equalities

$$s(s^*, t) = \int_0^{s^*} \lambda(s^*, t)\,\mathrm{d}s^*,$$

$$\dot{s} = \int_0^{s^*} \dot{\lambda}\,\mathrm{d}s^* = \int_0^{s^*} \frac{\partial \boldsymbol{v}}{\partial s^*} \cdot \boldsymbol{t}\,\mathrm{d}s^* = \int_0^{s} \frac{\partial \boldsymbol{v}}{\partial s} \cdot \boldsymbol{t}\,\mathrm{d}s\,,$$

$$\dot{\varrho} = \frac{\partial \varrho(s,t)}{\partial t} + \frac{\partial \varrho(s,t)}{\partial s}\,\dot{s}\,.$$

With the help of ϱ the classical measures of momentum and of inertia force per unit length are defined, respectively $\varrho \boldsymbol{v}$ and $-\varrho \dot{\boldsymbol{v}}$.

In a rod with structure there is the need to account also for a distributed moment of inertia. On this matter different choices seem possible; here we adopt the hypotheses which form the basis of the theory of three-dimensional continua with affine structure (see, e.g., [**6**]).

A symmetric positive-definite tensor $\boldsymbol{I}$, function of s, defines the Euler inertia tensor per unit mass; $\boldsymbol{I}(s(s^*, t), t)$ is supposed to be related to the inertia tensor $\boldsymbol{I}^*(s^*)$ in the reference placement through the *equation of conservation of inertia*

$$\boldsymbol{I} = \boldsymbol{G}\boldsymbol{I}^*\boldsymbol{G}^T. \tag{4.8}$$

Derivation of this equation with respect to time gives

$$\dot{\boldsymbol{I}} = 2\,\mathrm{sym}\,(\boldsymbol{I}\boldsymbol{U}^T)\,. \tag{4.9}$$

Correspondingly it is assumed that:

(i) The generalized moment of momentum per unit mass (with respect to the origin) be given by the formula

$$\boldsymbol{p} \otimes \boldsymbol{v} + \boldsymbol{S}\,, \qquad \text{with} \qquad \boldsymbol{S} = \boldsymbol{I}\boldsymbol{U}^T = \boldsymbol{G}\boldsymbol{I}^*\dot{\boldsymbol{G}}^T \tag{4.10}$$

(thus the term $\boldsymbol{S}$ is added to the usual measure).

(ii) The total kinetic energy per unit mass be

$$\frac{1}{2}(\boldsymbol{v}^2 + (\boldsymbol{I}\boldsymbol{U}^T)\cdot\boldsymbol{U}^T) = \frac{1}{2}(\boldsymbol{v}^2 + (\dot{\boldsymbol{G}}\boldsymbol{I}^*)\cdot\dot{\boldsymbol{G}}) \tag{4.11}$$

(again with the addition of an appropriate term to the classical measure).

(iii) The generalized moment of inertia forces per unit mass be

$$-(\boldsymbol{p}\otimes\dot{\boldsymbol{v}} + \boldsymbol{G}\boldsymbol{I}^*\ddot{\boldsymbol{G}}^T) = -(\boldsymbol{p}\otimes\dot{\boldsymbol{v}} + \dot{\boldsymbol{S}} - \dot{\boldsymbol{G}}\boldsymbol{I}^*\dot{\boldsymbol{G}}) = -(\boldsymbol{p}\otimes\boldsymbol{v} + \dot{\boldsymbol{S}} - \boldsymbol{U}\boldsymbol{S}). \tag{4.12}$$

Multiplying each of the quantities (4.10), (4.11), (4.12) by ϱ one obtains here the densities per unit length of the rod, densities which can be integrated over $\mathscr{C}$ to obtain the corresponding totals.

5 - Equations of balance

Consider now any arc c of $\mathscr{C}$ with end-points A, B; the balance of momentum for c is expressed by the classical relation

$$\int_c \varrho\boldsymbol{f}\,\mathrm{d}s + \boldsymbol{s}_B - \boldsymbol{s}_A = \int_c \varrho\dot{\boldsymbol{v}}\,\mathrm{d}s, \tag{5.1}$$

here: $\boldsymbol{f}$ is external force per unit mass and $\boldsymbol{s}$ is the resultant of internal forces on any cross-section of the rod; in particular $\boldsymbol{s}_A$, $\boldsymbol{s}_B$ are the values of $\boldsymbol{s}$ on the cross-sections through A and B.

The validity of (5.1) for any choice of c implies the local relation

$$\varrho(\boldsymbol{f} - \dot{\boldsymbol{v}}) + \frac{\partial \boldsymbol{s}}{\partial s} = \boldsymbol{0}. \tag{5.2}$$

The balance equation of generalized moment of momentum is modelled on the corresponding equation for a three-dimensional body with affine structure; precisely it is expressed as follows

$$\int_c \varrho(\boldsymbol{L} + \boldsymbol{p}\otimes\boldsymbol{f})\,\mathrm{d}s + \boldsymbol{p}_B\otimes\boldsymbol{s}_B - \boldsymbol{p}_A\otimes\boldsymbol{s}_A + \boldsymbol{H}_B - \boldsymbol{H}_A + \int_c \boldsymbol{Z}\,\mathrm{d}s$$
$$= \int_c \varrho(\dot{\boldsymbol{S}} - \boldsymbol{U}\boldsymbol{S} + \boldsymbol{p}\otimes\dot{\boldsymbol{v}})\,\mathrm{d}s, \tag{5.3}$$

here account is taken of (4.12) and the following additional notation is used: $\boldsymbol{L}$, generalized moment of external forces per unit mass; $\boldsymbol{H}$, generalized resultant moment of internal forces on any cross-section of the rod, in particular $\boldsymbol{H}_A$, $\boldsymbol{H}_B$, values of $\boldsymbol{H}$ on the cross-sections through A and B; $\boldsymbol{Z}$, generalized moment of internal forces per unit length of c.

As is known, $\boldsymbol{Z}$ need not be zero, though the mutual character of internal forces implies that

$$\operatorname{skw} \boldsymbol{Z} = \boldsymbol{0} . \tag{5.4}$$

Taking into account (5.2), eq. (5.3) can be reduced to

$$\int_c \varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S}) \,\mathrm{d}s + \int_c (\boldsymbol{t} \otimes \boldsymbol{s} + \boldsymbol{Z}) \,\mathrm{d}s + \boldsymbol{H}_B - \boldsymbol{H}_A = \boldsymbol{0} . \tag{5.5}$$

Local consequence of (5.5) is the relation

$$\varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S}) + \boldsymbol{t} \otimes \boldsymbol{s} + \boldsymbol{Z} + \frac{\partial \boldsymbol{H}}{\partial s} = \boldsymbol{0} . \tag{5.6}$$

The power of inertia forces acting on c is given by

$$\pi^{(i)} = -\int_c \varrho(\dot{\boldsymbol{v}} \cdot \boldsymbol{v} + (\dot{\boldsymbol{S}} - \boldsymbol{U}\boldsymbol{S}) \cdot \boldsymbol{U}^T) \,\mathrm{d}s = -\left(\tfrac{1}{2} \int_c \varrho(\boldsymbol{v}^2 + \boldsymbol{S} \cdot \boldsymbol{U}^T) \,\mathrm{d}s\right)^{\boldsymbol{\cdot}}, \tag{5.7}$$

i.e., it is the opposite of the time-derivative of the kinetic energy of c (see (4.11)).

The power of external forces is given by

$$\pi^{(e)} = \int_c \varrho(\boldsymbol{f} \cdot \boldsymbol{v} + \boldsymbol{L} \cdot \boldsymbol{U}^T) \,\mathrm{d}s + \boldsymbol{s}_B \cdot \boldsymbol{v}_B - \boldsymbol{s}_A \cdot \boldsymbol{v}_A + \boldsymbol{H}_B \cdot \boldsymbol{U}_B^T - \boldsymbol{H}_A \cdot \boldsymbol{U}_A^T . \tag{5.8}$$

In view of (5.2), (5.6) the sum of $\pi^{(i)}$ and $\pi^{(e)}$ is the opposite of the power $\pi^{(m)}$ of mutual internal forces

$$\pi^{(m)} = \int_c \left(\boldsymbol{s} \cdot \left(\frac{\partial \boldsymbol{v}}{\partial s} - \boldsymbol{U}\boldsymbol{t}\right) + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{U}^T}{\partial s} - \boldsymbol{Z} \cdot \operatorname{sym} \boldsymbol{U}\right) \mathrm{d}s . \tag{5.9}$$

The latter expression and the consequent specification of the power density per unit length of $\mathscr{C}$

$$\pi^{(m)} = \boldsymbol{s} \cdot \left(\frac{\partial \boldsymbol{v}}{\partial s} - \boldsymbol{U}\boldsymbol{t}\right) + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{U}^T}{\partial s} - \boldsymbol{Z} \cdot \operatorname{sym} \boldsymbol{U} \tag{5.10}$$

are of particular relevance for the development of the theory. Even more significant is the expression of the power density $\pi^{*(m)}$ per unit length of $\mathscr{C}^*$, when account is taken of formulae (4.3), (4.4)

$$\pi^{*(m)} = (G^T s)\cdot\dot{e} + (G^{-1}HG)^T\cdot\dot{F} - \lambda(G^{-1}ZG^{-T})\cdot\dot{E}\,. \tag{5.11}$$

This expression indicates already the convenience of the introduction of Lagrangian components of forces and moments

$$s^* = G^T s\,, \qquad H^* = G^{-1}HG\,, \qquad Z^* = \lambda G^{-1}ZG^{-T}, \tag{5.12}$$

in terms of which $\pi^{*(m)}$ becomes

$$\pi^{*(m)} = s^*\cdot\dot{e} + H^{*T}\cdot\dot{F} - Z^*\dot{E}\,. \tag{5.13}$$

s^*, H^*, Z^* can be introduced also in the balance equations, which then take the form

$$\varrho^* G^T(b - \dot{v}) - F^T s^* + \frac{\partial s^*}{\partial s} = 0\,, \tag{5.14}$$

$$\varrho^* G^{-1}(L - \dot{S} + US)G + (t^* + e)\otimes s^* + Z^*(1 + 2E) + FH^* - H^*F + \frac{\partial H^*}{\partial s^*} = 0\,.$$

Formula (5.13) suggests also the hypothesis, analytically attractive and physically significant, of the existence of a potential function φ of the variables e, F and E such that

$$\pi^{*(m)} = \frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial e}\cdot\dot{e} + \frac{\partial\varphi}{\partial F}\cdot\dot{F} + \frac{\partial\varphi}{\partial E}\cdot\dot{E}\,, \tag{5.15}$$

which implies the *constitutive equations*

$$s^* = \frac{\partial\varphi}{\partial e}, \qquad H^* = \left(\frac{\partial\varphi}{\partial F}\right)^T, \qquad Z^* = -\frac{\partial\varphi}{\partial E}, \qquad \text{or} \tag{5.16}$$

$$s = G^{-T}\frac{\partial\varphi}{\partial e}, \qquad H = G\left(\frac{\partial\varphi}{\partial F}\right)^T G^{-1}, \qquad Z = -\frac{1}{\lambda}\, G\,\frac{\partial\varphi}{\partial E}\,G^T. \tag{5.17}$$

Remark 1. The validity of the consequences (5.16), (5.17) of the existence of a potential is restricted by the proviso that no condition is imposed upon the choice of e, F, E.

If, on the contrary, constraints are present, some of those relations are invalid, as we shall see in detail later; for instance, if the constraint quoted in Remark 2 of Sect. **3** is present, then the first relation in both (5.16) and (5.17) must be left out and $\boldsymbol{s}^*$ becomes a reaction to the constraint.

Remark 2. Particularly important but very different in character is the special case (which is considered here only marginally) when the material properties and the data assure the inclusions

$$\boldsymbol{L} \in \mathrm{Skw}\,, \qquad \dot{\boldsymbol{S}} - \boldsymbol{U}\boldsymbol{S} \in \mathrm{Skw}\,, \qquad \boldsymbol{H} \in \mathrm{Skw}\,. \tag{5.18}$$

Then (5.6) can be split into the two balance equations

$$\varrho\boldsymbol{\varepsilon}(\boldsymbol{L} - \dot{\boldsymbol{S}} - \boldsymbol{U}\boldsymbol{S}) + \boldsymbol{t}\times\boldsymbol{s} + \frac{\partial \boldsymbol{h}}{\partial s} = 0\,, \qquad \boldsymbol{h} = \boldsymbol{\varepsilon}\boldsymbol{H}\,, \tag{5.19}$$

$$\boldsymbol{Z} = -\,\mathrm{sym}\,\boldsymbol{t}\otimes \boldsymbol{s}\,, \tag{5.20}$$

and the expressions (5.10), (5.11) of the power densities can be reduced to

$$\begin{aligned} \pi^{(m)} &= \boldsymbol{s}\cdot\Big(\frac{\partial \boldsymbol{v}}{\partial s} - (\mathrm{skw}\,\boldsymbol{U})\,\boldsymbol{t}\Big) + \boldsymbol{H}\cdot\frac{\partial \boldsymbol{U}^T}{\partial s}\,, \\ \pi^{*(m)} &= \boldsymbol{s}^*\cdot\big(\dot{\boldsymbol{e}} + \lambda\boldsymbol{G}^{-1}(\mathrm{sym}\,\dot{\boldsymbol{G}}\boldsymbol{G}^{-1})\,\boldsymbol{t}\big) + \boldsymbol{H}^*\cdot\dot{\boldsymbol{F}}^T. \end{aligned} \tag{5.21}$$

The expressions show that a different definition of characteristics of strain would be appropriate.

The case

$$\boldsymbol{L} = \boldsymbol{0}\,, \qquad \boldsymbol{S} = \boldsymbol{0}\,, \qquad \boldsymbol{H} \in \mathrm{Skw} \tag{5.22}$$

is also frequently studied; then (5.19) becomes even simpler

$$\boldsymbol{t}\times\boldsymbol{s} + \frac{\partial \boldsymbol{h}}{\partial s} = \boldsymbol{0}\,. \tag{5.23}$$

6 - Constraints

For most studies the results and formulae of the previous Sections are too general; constraints are normally introduced to reflect certain physical assumptions and also to simplify the analysis.

We examine here a few constraints, most commonly accepted, in order of increasing severity and deduce first consequences of a geometric character.

The « real » rod. As declared at the outset, the word « rod » is used in the previous Sections as an abbreviation for « one-dimensional continuum with affine structure ». The usage is not wholly appropriate; normally more specific properties are attributed to a rod. Actually, from now on, it will be assumed that, in a rod, $\boldsymbol{G}$ has the property of transforming the unit vector $\boldsymbol{t}^*$ into a vector parallel to $\boldsymbol{t}$, more precisely that the following constraint applies

$$\boldsymbol{G}\boldsymbol{t}^* = \lambda \boldsymbol{t} . \tag{6.1}$$

As a consequence (see (3.6) and Remark 2 in Sect. **3**) $\boldsymbol{e}$ vanishes identically and the strain is characterized by $\boldsymbol{E}$ and $\boldsymbol{F}$ alone; furthermore, whereas the constraint does not restrict the choice of $\dot{\boldsymbol{E}}$ and $\dot{\boldsymbol{F}}$, it requires $\dot{\boldsymbol{e}} = \boldsymbol{0}$. Condition (6.1) suggests the convenience of adopting $\lambda\boldsymbol{t}$ as one of the directors; from now on we accept the identification

$$\boldsymbol{d}_{(3)} = \lambda \boldsymbol{t} , \tag{6.2}$$

which obviously implies (6.1).

The bar of engineering theories. In addition to (6.2) one assumes that $\boldsymbol{G}$ transforms all vectors in the plane $\boldsymbol{d}^*_{(1)}$, $\boldsymbol{d}^*_{(2)}$ rigidly. In other words, there exists an orthogonal tensor $\boldsymbol{Q}$ (with $\det \boldsymbol{Q} = 1$) such that (see (3.3))

$$\boldsymbol{G} = \sum_{1}^{2}{}_{R}\, \boldsymbol{Q}\boldsymbol{d}^*_{(R)} \otimes \boldsymbol{d}^{*(R)} + \lambda \boldsymbol{t} \otimes \boldsymbol{d}^{*(3)}, \tag{6.3}$$

or

$$\boldsymbol{G} = \boldsymbol{Q}(\boldsymbol{1} + \boldsymbol{g} \otimes \boldsymbol{d}^{*(3)}) , \tag{6.4}$$

with

$$\boldsymbol{g} = \lambda \boldsymbol{Q}^T \boldsymbol{t} - \boldsymbol{t}^* \qquad \text{and} \qquad \det(\boldsymbol{1} + \boldsymbol{g} \otimes \boldsymbol{d}^{*(3)}) = \det \boldsymbol{G} > 0 . \tag{6.5}$$

As a consequence

$$\boldsymbol{E} = \operatorname{sym}\left((\boldsymbol{g} + \tfrac{1}{2}(\boldsymbol{g}^2)\, \boldsymbol{d}^{*(3)}) \otimes \boldsymbol{d}^{*(3)}\right) \tag{6.6}$$

and, of course,

$$\dot{\boldsymbol{E}} = \operatorname{sym}\left((\dot{\boldsymbol{g}} + (\boldsymbol{g} \cdot \dot{\boldsymbol{g}})\, \boldsymbol{d}^{*(3)}) \otimes \boldsymbol{d}^{*(3)}\right) . \tag{6.7}$$

Remark 1. The expression (6.6) specifies for $\boldsymbol{E}$ a form of the type

$$\boldsymbol{E} = \operatorname{sym}(\boldsymbol{c} \otimes \boldsymbol{d}^{*(3)}) \tag{6.8}$$

with $\boldsymbol{c} = \boldsymbol{g} + \frac{1}{2}(\boldsymbol{g})^2 \boldsymbol{d}^{*(3)}$.

It is relevant to remark that, given $\boldsymbol{c}$, $\boldsymbol{g}$ can be determined uniquely; in fact the components of $\boldsymbol{c}$ and $\boldsymbol{g}$ orthogonal to $\boldsymbol{d}^{*(3)}$ coincide, whereas for the components (say, c_3, g_3) along $\boldsymbol{d}^{*(3)}$ one has

$$|\boldsymbol{d}^{*(3)}| g_3^2 + 2g_3 + |\boldsymbol{d}^{*(3)}|(\boldsymbol{c}^2 - c_3^2) - 2c_3 = 0 ;$$

but the discriminant of this equation is equal to $(\det \boldsymbol{G})^2$, and the alternative choice for the root is again decided by the condition $\det \boldsymbol{G} > 0$.

A necessary and sufficient condition for $\boldsymbol{E}$ to be of the form (6.8) is that the covariant components $E_{RS} = \boldsymbol{d}^*_{(R)} \cdot \boldsymbol{E} \boldsymbol{d}^*_{(S)}$ vanish for $R, S = 1, 2$

$$E_{RS} = 0 \qquad (R, S = 1, 2) . \tag{6.9}$$

The first part of the statement is an immediate corollary of the properties of reciprocal directors. To prove that (6.9) implies (6.8) with an appropriate choice of $\boldsymbol{c}$ it is sufficient to observe that (6.8) follows from the general formula

$$\boldsymbol{E} = \sum_{1}^{3}{}_{R,S} E_{RS} \boldsymbol{d}^{*(R)} \otimes \boldsymbol{d}^{*(S)}$$

when one puts $\boldsymbol{c} = 2(E_{13} \boldsymbol{d}^{*(1)} + E_{23} \boldsymbol{d}^{*(2)})$.

Remark 2. As the deformation transforms $\boldsymbol{d}_{(1)}$ and $\boldsymbol{d}_{(2)}$ (actually their whole plane) rigidly, no restriction would be implied by *the assumption that* $\boldsymbol{d}_{(1)}$ *and* $\boldsymbol{d}_{(2)}$ *are two orthogonal unit vectors or equivalently that* $\boldsymbol{d}_{(1)}$, $\boldsymbol{d}_{(2)}$, $\lambda^{-1} \boldsymbol{d}^{(3)}$ *is a triad of orthogonal unit vectors. This assumption leads to a significant simplification of formulae in concrete problems and also helps in some of our developments.*

The other strain characteristic has now the expression

$$\boldsymbol{F} = \left(\boldsymbol{1} - \frac{\boldsymbol{g} \otimes \boldsymbol{d}^{*(3)}}{1 + \boldsymbol{g} \cdot \boldsymbol{d}^{*(3)}}\right)\Big(\boldsymbol{Q}^T \frac{\partial \boldsymbol{Q}}{\partial s^*}(\boldsymbol{1} + \boldsymbol{g} \otimes \boldsymbol{d}^{*(3)}) + \frac{\partial \boldsymbol{g}}{\partial s^*} \otimes \boldsymbol{d}^{*(3)} + \boldsymbol{g} \otimes \frac{\partial \boldsymbol{d}^{*(3)}}{\partial s^*}\Big) . \tag{6.10}$$

Notice that $1 + \boldsymbol{g}\cdot\boldsymbol{d}^{*(3)}$ is different from zero because its vanishing would imply that

$$\boldsymbol{t}\cdot\boldsymbol{Q}\boldsymbol{d}^{*(3)} = 0\,,$$

i.e. that $\boldsymbol{t}$ is in the plane of $\boldsymbol{d}_{(1)}$ and $\boldsymbol{d}_{(2)}$ which is explicitly forbidden by the condition $\det\boldsymbol{G} \neq 0$ ($\boldsymbol{Q}\boldsymbol{d}^{*(3)}$ is in the direction of the normal to the plane of $\boldsymbol{d}_{(1)}$ and $\boldsymbol{d}_{(2)}$).

The expression of $\dot{\boldsymbol{F}}$ is more complex and we need not write it out here. Suffice it to say that $\dot{\boldsymbol{F}}$ is linear in $\dot{\boldsymbol{g}}$, $(\partial\boldsymbol{g}/\partial s^*)^{\cdot}$, $(\boldsymbol{Q}^T(\partial\boldsymbol{Q}/\partial s^*))^{\cdot}$ whereas, as appears in (6.7), $\dot{\boldsymbol{E}}$ is linear in $\dot{\boldsymbol{g}}$ only. On the whole 9 parameters are involved rather than 15 in the specification of $\dot{\boldsymbol{E}}$ and $\dot{\boldsymbol{F}}$; actually only the choice of $\dot{\boldsymbol{E}}$ is restricted, whereas $\dot{\boldsymbol{F}}$ can assume locally any value through an appropriate local choice of $\dot{\boldsymbol{g}}$, $(\partial\boldsymbol{g}/\partial s^*)^{\cdot}$, $(\boldsymbol{Q}^T(\partial\boldsymbol{Q}/\partial s^*))^{\cdot}$.

The elastica. In addition to (6.2) and (6.3) one assumes that

$$\boldsymbol{t} = \boldsymbol{Q}\boldsymbol{t}^*. \tag{6.11}$$

Thus the triad of vectors $\boldsymbol{d}_{(1)}$, $\boldsymbol{d}_{(2)}$, $\boldsymbol{t} = \lambda^{-1}\boldsymbol{d}_{(3)}$ is subject to rigid displacements only during the deformation; no restriction ensues if we assume that they form an orthogonal triad. Such assumption, together with the assumption within Remark 2 above, is accepted here to simplify formulae; in particular $\boldsymbol{d}^{(3)}$ is brought to coincide with $\lambda^{-1}\boldsymbol{t}$ and $\boldsymbol{d}^{*(3)}$ with $\boldsymbol{t}^*$; $\boldsymbol{G}$, $\boldsymbol{E}$, $\dot{\boldsymbol{E}}$, $\boldsymbol{F}$ become respectively

$$\boldsymbol{G} = \boldsymbol{Q}(\boldsymbol{1} + \delta\boldsymbol{t}^*\otimes\boldsymbol{t}^*)\,, \qquad \boldsymbol{E} = \tfrac{1}{2}(\lambda^2 - 1)\,\boldsymbol{t}^*\otimes\boldsymbol{t}^*, \tag{6.12}$$

$$\dot{\boldsymbol{E}} = \lambda\dot{\lambda}\boldsymbol{t}^*\otimes\boldsymbol{t}^*, \tag{6.13}$$

$$\boldsymbol{F} = \Big(\boldsymbol{1} - \frac{\delta}{\lambda}\,\boldsymbol{t}^*\otimes\boldsymbol{t}^*\Big)\Big(\boldsymbol{Q}^T\frac{\partial\boldsymbol{Q}}{\partial s^*}(\boldsymbol{1} + \delta\boldsymbol{t}^*\otimes\boldsymbol{t}^*) + \frac{\partial\delta}{\partial s^*}\,\boldsymbol{t}^*\otimes\boldsymbol{t}^* + 2\delta\varkappa^*\,\mathrm{sym}\,(\boldsymbol{t}^*\otimes\boldsymbol{n}^*)\Big)\,,$$

(6.14)

$$\boldsymbol{F} = \Big(1 - \frac{\delta}{\lambda}\,\boldsymbol{t}^*\otimes\boldsymbol{t}^*\Big)\,\boldsymbol{Q}^T\frac{\partial\boldsymbol{Q}}{\partial s^*}(\boldsymbol{1} + \delta\boldsymbol{t}^*\otimes\boldsymbol{t}^*) + \frac{1}{\lambda}\frac{\partial\delta}{\partial s^*}\,\boldsymbol{t}^*\otimes\boldsymbol{t}^* + \delta\varkappa^*\,\boldsymbol{n}^*\otimes\boldsymbol{t}^* + \frac{\delta\varkappa^*}{\lambda}\,\boldsymbol{t}^*\otimes\boldsymbol{n}^*.$$

When $\boldsymbol{n}$ and $\boldsymbol{b}$ are uniquely defined, one can introduce an angle γ such that

$$\boldsymbol{d}_{(1)} = \boldsymbol{n}\cos\gamma - \boldsymbol{b}\sin\gamma\,, \qquad \boldsymbol{d}_{(2)} = \boldsymbol{b}\cos\gamma + \boldsymbol{n}\sin\gamma\,, \tag{6.15}$$

and make use of the formulae

$$(6.16)_1 \qquad \frac{\mathrm{d}\boldsymbol{d}_{(1)}}{\mathrm{d}s} = -\left(\tau + \frac{\mathrm{d}\gamma}{\mathrm{d}s}\right)\boldsymbol{d}_{(2)} - \varkappa \cos\gamma \boldsymbol{t}\,,$$

$$(6.16)_2 \qquad \frac{\mathrm{d}\boldsymbol{d}_{(2)}}{\mathrm{d}s} = \left(\tau + \frac{\mathrm{d}k}{\mathrm{d}s}\right)\boldsymbol{d}_{(1)} - \varkappa \sin k\boldsymbol{t}\,,$$

$$(6.16)_3 \qquad \frac{\mathrm{d}\boldsymbol{d}_{(3)}}{\mathrm{d}s} = \frac{\mathrm{d}\lambda}{\mathrm{d}s}\boldsymbol{t} + \lambda\varkappa\boldsymbol{n}\,, \qquad \text{and}$$

$$(6.17) \qquad \boldsymbol{w} = -\left(\tau + \frac{\mathrm{d}\gamma}{\mathrm{d}s}\right)\boldsymbol{t} + \varkappa\boldsymbol{b}\,.$$

With the additional notation

$$(6.18) \qquad \boldsymbol{d} = \boldsymbol{n}^* \cos(\gamma - \gamma^*) + \boldsymbol{b}^* \sin(\gamma - \gamma^*)\,,$$

$$(6.18) \qquad \boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{t}\times\dot{\boldsymbol{t}} + \boldsymbol{n}\times\dot{\boldsymbol{n}} + \boldsymbol{b}\times\dot{\boldsymbol{b}})\,,$$

one can give an alternative expression of $\boldsymbol{F}$

$$(6.19) \qquad \boldsymbol{F} = \frac{1}{\lambda}\frac{\mathrm{d}\delta}{\mathrm{d}s^*}\boldsymbol{t}^*\otimes\boldsymbol{t}^* - \varkappa(\boldsymbol{t}^*\otimes\boldsymbol{d} - \lambda^2\boldsymbol{d}\otimes\boldsymbol{t}^*)$$
$$- \boldsymbol{\varepsilon}\left(\varkappa^*\boldsymbol{b}^* + \left(\lambda\tau - \tau^* + \frac{\mathrm{d}(\gamma-\gamma^*)}{\mathrm{d}s^*}\right)\boldsymbol{t}^*\right),$$

and a compact expression of $\boldsymbol{U}$

$$(6.20) \qquad \boldsymbol{U} = \frac{\delta}{\lambda}\boldsymbol{t}\otimes\boldsymbol{t} - \boldsymbol{\varepsilon}(\boldsymbol{q} + \dot{\gamma}\boldsymbol{t})\,.$$

Even with the new notation, (6.19) is not a convenient expression to work with. In fact, in the study of elastica, the use of the characteristics of strain mentioned in Remark 3 of Section **3** is preferred; they are null except for

$$(6.21)_1 \qquad J_{12} = -J_{21} = \tau^* + \frac{\partial\gamma^*}{\partial s^*} - \left(\tau + \frac{\partial\gamma}{\partial s}\right),$$

$$(6.21)_2 \qquad J_{13} = -J_{31} = \varkappa^*\cos\gamma^* - \gamma\varkappa\cos\gamma\,,$$

$$(6.21)_3 \qquad J_{23} = -J_{32} = \varkappa^*\sin\gamma^* - \gamma\varkappa\sin\gamma\,, \qquad J_{33} = -\lambda\frac{\partial\lambda}{\partial s}\,.$$

Under the present circumstances $\boldsymbol{F}$ and $\boldsymbol{J}$ are related as follows

$$\boldsymbol{J} = \lambda^{-1}(\mathbf{1} + (\lambda^2 - 1)\,\boldsymbol{t}^* \otimes \boldsymbol{t}^*)(\boldsymbol{F} + \boldsymbol{W}^*) - \boldsymbol{W}^*.$$

The inextensible elastica. It is characterized by the further constraint

$$\lambda = 1 \qquad \text{or} \qquad \delta = 0\,. \tag{6.22}$$

Then $\boldsymbol{E}$ vanishes identically and $\boldsymbol{F}$ is a skew tensor. Obvious simplifications ensue in the formulae; in particular $\boldsymbol{F}$ coincides with $\boldsymbol{J}$.

7 - Pure balance equations for the constrained cases

We accept here a dynamic notion of frictionless constraint in the form introduced in a general context by Gurtin and Podio-Guidugli [**5**] and specified for affine bodies in $[\mathbf{3}]_2$. In brief, the absence of friction is bound to the property of vanishing power of reaction stresses $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{H}})$ for all virtual kinematic fields; in addition the active portions of each one the stress field $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{Z}}, \hat{\boldsymbol{H}})$ are supposed to belong to the subspaces orthogonal to those spanned respectively in the space of vectors, of second order symmetric tensor, of second order tensor by $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{H}})$. Pure dynamic equations are those which involve $\hat{\boldsymbol{s}}$, $\hat{\boldsymbol{Z}}$, $\hat{\boldsymbol{H}}$ only (and not at all $\bar{\boldsymbol{s}}$, $\bar{\boldsymbol{Z}}$, $\bar{\boldsymbol{H}}$).

The «real» rod. $\boldsymbol{s}$ is a reaction due to the constraint (i.e., it has no active component) and is given by the formula

$$\bar{\boldsymbol{s}} = -\Big(\varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S})^T + \boldsymbol{Z} + \frac{\partial \boldsymbol{H}^T}{\partial s}\Big)\,\boldsymbol{t}\,, \tag{7.1}$$

or alternatively by

$$\bar{\boldsymbol{s}} = -2(\mathbf{1} - \tfrac{1}{2}\boldsymbol{t} \otimes \boldsymbol{t})\Big(\varrho \,\mathrm{sym}\,(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S}) + \boldsymbol{Z} + \mathrm{sym}\,\frac{\partial \boldsymbol{H}}{\partial s}\Big)\,\boldsymbol{t}\,, \tag{7.2}$$

while $\boldsymbol{H}$ and $\boldsymbol{Z}$ must be specified by constitutive equations (such as $(5.17)_2$, $(5.17)_3$). The pure (i.e., reaction-free) equations of balance are

$$\Big(\varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S})^T + \boldsymbol{Z} + \frac{\partial \boldsymbol{H}^T}{\partial s}\Big)\,\boldsymbol{n} = \mathbf{0}\,,$$

$$\Big(\varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S})^T + \boldsymbol{Z} + \frac{\partial \boldsymbol{H}^T}{\partial s}\Big)\,\boldsymbol{b} = \mathbf{0}\,, \tag{7.3}$$

$$\varrho(\boldsymbol{f} - \boldsymbol{a}) + \frac{\partial}{\partial s}\Big(\Big(\varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S})^T + \boldsymbol{Z} + \frac{\partial \boldsymbol{H}^T}{\partial s}\Big)\boldsymbol{t}\Big) = \mathbf{0}\,. \tag{7.4}$$

The bar of engineering theories. The tensor

$$\overline{\boldsymbol{Z}}^* = \sum_{1}^{2}{}_{HK} Z^{*HK} \boldsymbol{d}^*_{(H)} \otimes \boldsymbol{d}^*_{(K)} \tag{7.5}$$

formed with the contravariant components of $\boldsymbol{Z}^*$ has reactive character; $\boldsymbol{Z}^*$ can be split into the sum

$$\boldsymbol{Z}^* = \overline{\boldsymbol{Z}}^* + 2\,\mathrm{sym}\left((Z^{*13}\boldsymbol{d}^*_{(1)} + Z^{*23}\boldsymbol{d}^*_{(2)} + \tfrac{1}{2}Z^{*33}\boldsymbol{d}^*_{(3)}) \otimes \boldsymbol{d}^*_{(3)}\right), \qquad \text{or} \tag{7.6}$$

$$\boldsymbol{Z} = \overline{\boldsymbol{Z}} + 2\,\mathrm{sym}\left((Z^{13}\boldsymbol{d}_{(1)} + Z^{23}\boldsymbol{d}_{(2)} + \tfrac{1}{2}Z^{33}\boldsymbol{d}_{(3)}) \otimes \boldsymbol{d}_{(3)}\right), \tag{7.7}$$

which implies

$$\boldsymbol{Z}\boldsymbol{d}_{(3)} = \boldsymbol{z} = \frac{1}{\lambda}\sum_{1}^{3}{}_{R} Z^{R3}\boldsymbol{d}_{(R)}\,.$$

The reactive components of $\boldsymbol{Z}$ can be determined through the formulae

$$\left(\varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S})^T + \overline{\boldsymbol{Z}} + \frac{\partial \boldsymbol{H}^T}{\partial s}\right)\boldsymbol{d}^{(R)} = \boldsymbol{0} \qquad (R = 1, 2)\,,$$

whereas the only pure relation is

$$\varrho(\boldsymbol{f} - \boldsymbol{a}) - \frac{\partial}{\partial s}\left(\frac{1}{\boldsymbol{t}\cdot\boldsymbol{d}^{(3)}}\left(\varrho(\boldsymbol{L} - \dot{\boldsymbol{S}} + \boldsymbol{U}\boldsymbol{S})^T + \frac{\partial \boldsymbol{H}^T}{\partial s}\right)\boldsymbol{d}^{(3)} + \boldsymbol{z}\right)\Big) = \boldsymbol{0}\,.$$

Remark. These formulae are still cumbersome. Easier developments follow from the acceptance (which is general and usually tacit) of the additional hypotheses (5.18) or even (5.22). We do not intend to pursue the matter here; to the Remark of Section **5** we add only the observation that, if hypotheses (6.2), (6.4) are also assumed, (5.21) changes into

$$\pi^{(m)} = \tfrac{1}{2}(1 + \boldsymbol{d}^{*(3)}\cdot\boldsymbol{g})^{-1}\left((\boldsymbol{d}^{*(3)}\cdot\boldsymbol{t})\,\boldsymbol{Q}^T + \boldsymbol{Q}^T\boldsymbol{t}\otimes\boldsymbol{d}^{*(3)}\right)\boldsymbol{s}\cdot\boldsymbol{g} + \boldsymbol{H}\cdot\frac{\partial \boldsymbol{U}^T}{\partial s}\,,$$

which may be taken as a new point of departure for a discussion of the special case.

The inextensible elastica. Because $\dot{\boldsymbol{E}}$ vanishes, $\boldsymbol{Z}$ becomes a reaction due to the constraint and $\hat{\boldsymbol{Z}}$ vanishes. $\dot{\boldsymbol{F}}$ is skew; hence $\mathrm{sym}\,\boldsymbol{H}^*$ has reactive character, $\hat{\boldsymbol{H}}^*$ is skew and so is $\hat{\boldsymbol{H}}$.

Together with $\hat{\boldsymbol{h}}^* = \boldsymbol{\varepsilon}\hat{\boldsymbol{H}}^*$, put $\boldsymbol{j} = \frac{1}{2}\boldsymbol{\varepsilon}\boldsymbol{J}$, then $\hat{\boldsymbol{H}}^* \cdot \boldsymbol{J}^* = \hat{\boldsymbol{h}}^*(\partial \boldsymbol{j}/\partial t)$, so that, when a potential exists $\hat{\boldsymbol{h}}^* = \partial\varphi/\partial \boldsymbol{j}$.

In classical analyses the pure equation is not used generally; a mixed equation of the form (5.19) is preferred, where $\hat{\boldsymbol{h}} = \boldsymbol{\varepsilon}\hat{\boldsymbol{H}}$ takes the place of $\boldsymbol{h}$.

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