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Maximal outerplane graphs and their associated trees (**)

1 - Introduction

Two recent papers by Antonucci [1]$_{1,2}$ have treated certain enumeration problems for labelled and unlabelled maximal outerplane graphs, denoted OM-graphs in his work. In particular, using their automorphism groups, he enumerates the number of triangulations of polygons with n-vertices having no interior triangles and uses this to count the OM-graphs with three vertices of degree 2. A related enumeration problem appears in Cohen [4] where a rooted tree is associated with an OM-graph. Antonucci's enumerations correspond to associating unrooted trees with these graphs. By taking this approach we are able to obtain enumerations of OM-graphs with an arbitrary number of vertices of degree 2. We use these enumerations to give bounds on the number of graphs which correspond to a given tree, and to define a new class of trivalent trees.

Given any OM-graph G the natural way to associate a tree T with this graph is: insert one vertex into each interior face of G, and make two vertices

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adjacent if and only if the corresponding faces in G have an edge in common. Fig. 1a shows an OM-graph and its associated tree. A tree T is the associated tree of some OM-graph G if and only if the degree of every vertex of T is less than or equal to three [**2**]. Let $\mathscr{T}$ denote the collection of trees with maximum degree less than or equal to three. In this paper we investigate the set of OM-graphs associated with a given $T \in \mathscr{T}$.

We adopt the following notation: $G(T)$ denotes an OM-graph with associated T; $\mathscr{G}(T)$ denotes the set of all such graphs whose associated trees are isomorphic to T; and $|\mathscr{G}(T)|$ denotes the cardinality of this set. We emphasize here that even the same embedding of a tree T may be the associated plane tree of two or more nonisomorphic members of $\mathscr{G}(T)$. In section **2** we will show how to compute $|\mathscr{G}(T)|$ for any $T \in \mathscr{T}$. This computation is based upon theorem 3 from [**4**], the statement of which is given here for convenience:

Let G_1 and G_2 be OM-graphs with corresponding associated trees T_1 and T_2. Then T_1 and T_2 are isomorphic if and only if G_1 and G_2 are 2-isomorphic. (For a definition of 2-isomorphism, see [**5**], p. 82.)

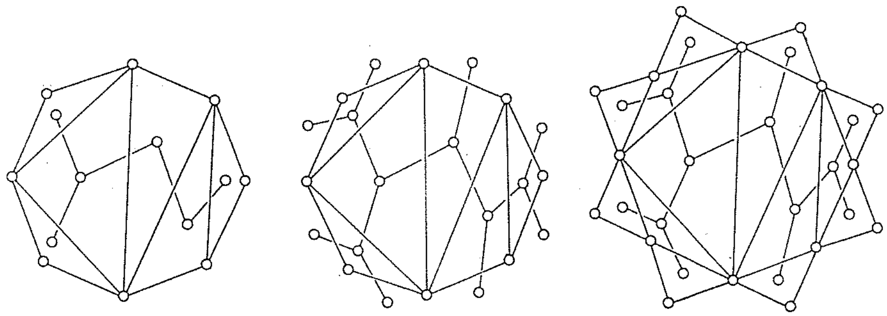
It is also shown in [**2**] that any two labelled OM-graphs with isomorphic associated trees have the same number of labelled spanning trees.

Let T_n denote a tree in $\mathscr{T}$ with n vertices. It is apparent that the relation of 2-isomorphism establishes a natural partitioning of the set of OM-graphs with $n+2$ vertices into equivalence classes $\mathscr{G}(T_n)$. Since OM-graphs can be viewed as triangulated convex polygons, counting the elements of the equivalence class $\mathscr{G}(T_n)$ is a new aspect of the classic problem of triangulating a convex polygon with $n+2$ vertices. In [**6**] Guy gives a formula for D_{n+2}, « the number of dissections of a convex polygon of $n+2$ sides into n triangles by drawing various sets of $n-1$ nonintersecting diagonals », and for E_n, « the total number of essentially different dissections ». Let $(1/2)(\overline{n+3}) = (1/2)(n+3)$ if n is odd and $(1/2)(n+2)$ if n is even. Note $D_k = 0$ if k is not on interger.

$$D_{n+2} = \frac{(2n)!}{(n+1)!\,n!}\,; \qquad E_{n+2} = \frac{1}{2n+4} D_{n+2} + \frac{1}{4} D_{\frac{1}{2}n+2} + \frac{1}{2} D_{\overline{\frac{1}{2}n+3}} + \frac{1}{3} D_{1/3(n+2)+1}\,.$$

In [**3**] Brown gives a history and bibliography of triangulations and related combinatorial problems. In particular, Brown, and Harary and Palmer ([**7**], p. 68) point out that enumerating triangulations of rooted $(n+2)$-gons (see [**7**], p. 6, for the definition) and enumerating planted plane trees with $2n+2$ vertices, (see [**7**], p. 60 and p. 66 for definitions), each of degree one or three, are equivalent, and that this correspondence has been rediscovered many times going back to Euler. Harary and Palmer then use this correspondence to enumerate planar 2-trees ([**7**], p. 76).

Let $\mathscr{T}^*$ denote the collection of trees in which each vertex has degree one or three. We have a direct correspondence between a tree $T_n \in \mathscr{T}$ and a tree $T^*_{2n+2} \in \mathscr{T}^*$. To obtain T^*_{2n+2} from T_n we adjoin the $n+2$ vertices of degree one corresponding to the $n+2$ exterior edges of $G(T_n)$, so that each of the vertices of T_n which is not of degree 3 become a vertex of degree three in T^*_{2n+2}. This construction is shown in Fig. 1b. It follows that finding $|\mathscr{G}(T_n)|$ for a given $T_n \in \mathscr{T}$ is equivalent to finding the number of plane representations of the tree T^*_{2n+2} which are distinct under all rigid motions of T^*_{2n+2}. Since $T^*_{2n+2} \in \mathscr{T}$, it can be associated with OM-graphs $G(T^*_{2n+2})$ having $2n+4$ vertices, as shown in Fig. 1c. Since this construction demonstrates the one-to-one correspondence between the OM-graphs associated with T_n and those associated with T^*_{2n+2}, we have shown that $|\mathscr{G}(T_n)| = |\mathscr{G}(T^*_{2n+2})|$.



1a 1b 1c

Figure 1. An example of T, $G(T)$, T^* and $G(T^*)$.

2 - Computation of $|\mathscr{G}(T)|$

Let $T \in \mathscr{T}$ and let u be a designated pendant vertex of T (making T a « planted » tree). Let T^* in $\mathscr{T}^*$ (all vertices having degree 1 or 3) denote the planted tree with root u which will become T if its vertices of degree one (other than u) are removed. Let v be a vertex of degree 3 in T^*. Consider the two subtrees emanating from v which do not contain the root u. If these subtrees are not isomorphic, then two distinct plane representations of T^* can be formed by interchanging the positions of the subtrees relative to u. We call v a *critical* vertex and we see that the number of plane representations of T^* will be 2^k when T^* has k critical vertices. (An algorithm to determine if two trees are isomorphic in time complexity $O(n)$ has been described by Hopcraft and Tarjan [**8**], pp. 140-142]). We define the corresponding vertex v in T to be a critical vertex of T, and note that a vertex of degree 2 is always critical in T. Recalling the above comments, it follows that if T has k critical vertices, there are 2^k OM-graphs rooted at an orientable edge having T as its associated tree.

To compute $|\mathcal{G}(T)|$ for any free cubic tree, we use the above observation and straightforward applications of Polya's Counting Theorem. The computation is necessarily done by cases:

Case 1. Some vertex of degree three in T is fixed by every automorphism of T (for example a single center of degree 3).

Case 2. Some vertex of degree two in T is fixed by every automorphism.

Case 3. T has two centers of degree two, interchanged by some automorphism.

Case 4. T has two centers of degree three, interchanged by some automorphism.

In each case we focus our attention on the fixed vertex or the centers of T and the corresponding face(s) of an OM-graph $G(T)$ with T as its associated tree. (In some instances, we consider other faces also.) The subtrees S_1, S_2, and S_3 of T emanating from these faces are viewed as planted and Polya's theorem is applied to the automorphisms of the faces which interchange isomorphic subtrees. Since most applications of Polya's theorem are straightforward, proofs for most cases are omitted.

Case 1. Suppose the tree T has a vertex c of degree three which is fixed by every automorphism of T, and let S_1, S_2 and S_3 denote the subtrees rooted at c.

(a) If S_1, S_2, and S_3 are non-isomorphic, with k_1, k_2, and k_3 critical vertices, respectively, then

$$|\mathcal{G}(T)| = 2^{k_1+k_2+k_3}, \qquad k_1, k_2, k_3 \geqslant 0 .$$

(b) If S_1 and S_2 are isomorphic, each with $k \geqslant 0$ critical vertices, and S_3 has not critical vertex, then

$$|\mathcal{G}(T)| = \tfrac{1}{2}(2^{2k} + 2^k), \qquad k \geqslant 0 .$$

(c) If S_1 and S_2 are isomorphic, each with $k_1 \geqslant 0$ critical vertices and S_3 has $k_2 \geqslant 1$ critical vertices, then

$$|\mathcal{G}(T)| = 2^{2k_1+k_2-1}, \qquad k_1 \geqslant 0 , \ k_2 \geqslant 1 .$$

(d) If S_1, S_2, and S_3 are isomorphic, each with $k \geqslant 0$ critical vertices, then

$$|\mathscr{G}(T)| = \begin{cases} \frac{1}{2} \cdot \frac{1}{6}(2^{3k} + 3 \cdot 2^{2k} + 2 \cdot 2^k), & k \geqslant 1 \\ 1, & k = 0 . \end{cases}$$

Proof of (d). The case $k = 0$ is trivial, so assume $k \geqslant 1$. The group of automorphisms on the face has order six. Consequently, the Polya theorem will give us a count of $1/6(2^{3k} + 3 \cdot 2^{2k} + 2 \cdot 2^k)$.

However, because of the perfect symmetry of T it is necessary to provide the face with an orientation in order to propertly root the subtrees. To find $|\mathscr{G}(T)|$ we must therefore divide by 2. This properly adjusts the count since each automorphism has some orbit with an odd number of orientable subtrees.

Case 2. Suppose the tree T has a vertex c of degree two which is fixed by every automorphism, and let S_1 and S_2 denote the subtrees rooted at c.

(a) If the rooted subtrees S_1 and S_2 are not isomorphic and have k_1 and k_2 critical vertices, respectively, then

$$|\mathscr{G}(T)| = 2^{k_1+k_2}, \qquad k_1, k_2 \geqslant 0 .$$

(b) If the rooted subtrees S_1 and S_2 are isomorphic, each with $k \geqslant 0$ critical vertices, then

$$|\mathscr{G}(T)| = \tfrac{1}{2}[2^{2k} + 2^k], \qquad k \geqslant 0 .$$

Case 3. Suppose the tree T has two centers c_1 and c_2 of degree two which are interchanged by some automorphism of T. In this case

$$|G(T)| = 2^{2k} + 2^k, \qquad k \geqslant 0 ,$$

where k is the number of critical vertices of the subtrees S_1 and S_2 rooted at c_1 and c_2, respectively.

Proof. The subtrees S_1 and S_2 are necessarily isomorphic. There are two possible ways to arrange the two faces containing c_1 and c_2, as shown by Fgs. 2a and 2b. Polya's theorem gives $\frac{1}{2}(2^{2k} + 2^k)$ for each structure.

Figure 2. Arrangements of the faces at the two central vertices of degree two.

Case 4. Suppose the tree T has two centers c_1 and c_2 of degree three which are interchanged by some automorphism. In this case edge (c_1, c_2) is a line of symmetry for T, and each of c_1 and c_2 subtend a pair of rooted subtrees S_1 and S_2.

(a) If the rooted subtrees S_1 and S_2 are non-isomorphic with k_1 and k_2 critical vertices, respectively, then

$$|\mathscr{G}(T)| = 2^{2(k_1+k_2)} + 2^{k_1+k_2}, \qquad k_1, k_2 \geqslant 0 .$$

(b) If the rooted subtrees S_1 and S_2 are isomorphic each with k critical vertices, then

$$|\mathscr{G}(T)| = \tfrac{1}{4}\,[2^{4k} + 3\cdot 2^{2k}], \qquad k \geqslant 0 .$$

3 - Upper and lower bounds for $|\mathscr{G}(T_n)|$

In this section we will use the results of section **2** to find bounds for $|\mathscr{G}(T_n)|$ in terms of n and the number of vertices of degree one, two, and three. We note that for any tree in $\mathscr{T}$ the number of vertices of degree one is always two more than the number of vertices of degree three. If $T_n \in \mathscr{T}$ has exactly two vertices of degree one, then it is path P_n, for if it has more then T_n will contain a vertex of degree 3.

Theorem 1. *If P_n is a path with n vertices*

$$|\mathscr{G}(P_n)| = \begin{cases} 2^{n-4} + 2^{(n-5)/2} & n \text{ odd}, \\ 2^{n-4} + 2^{(n-4)/2} & n \text{ even}. \end{cases}$$

Proof. If n is odd, case 2 (b) applies; case 3 applies if n is even.

In general, if T_n has k vertices of degree one, we have the following upper bound.

Theorem 2. $|\mathscr{G}(T_n)| \leqslant 2^{n-k-1}$ *where k is the number of vertices of degree one.*

Proof. There are at most $n - k - 1$ vertices of T_n for which the orientation of the associated triangle in $G(T_n)$ makes a difference since none of the vertices of degree one is critical and one other vertex is used to root T_n.

For $k \geqslant 3$ this bound is achieved by an identity tree (see [7], p. 64, for the definition).

Theorem 3. $|\mathcal{G}(T_n)| \leqslant |\mathcal{G}(P_n)|$.

Proof. If T_n is not a path, it must have at least 3 vertices of degree one, hence $|G(T_n)| \leqslant 2^{n-4}$, and the conclusion follows from Theorem 1.

Corollary. *If* $T_n^* \in \mathcal{T}^*$ *then* $|\mathcal{G}(T_n^*)| \leqslant |\mathcal{G}(P_{(n-2)/2})|$.

Proof. Note that T_n^* has $(n+2)/2$ vertices of degree 1 and $(n-2)/2$ vertices of degree 3. The inequality is obtained by applying Theorem 3 to the tree obtained by removing the vertices of degree 1.

Since $|\mathcal{G}(T_n)|$ is determined by the number of critical vertices and the symmetry of T_n, we can give lower bounds for $|\mathcal{G}(T_n)|$ in terms of the number of vertices of degree two.

Theorem 4. *If r is the number of vertices of degree two in T_n and* $r \equiv 0 \pmod 3$, *then*

$$|\mathcal{G}(T_n)| \geqslant \tfrac{1}{3}(2^{r-2} + 3 \cdot 2^{(2r/3)-2} + 2^{(r/3)-1}).$$

Proof. This is case 1 (d) with the assumption that the only critical vertices are of degree two and k replaced by $r/3$.

If we do not have the special symmetry present in case 1 (d) this bound can be improved.

Theorem 5. *If r is the number of vertices of degree two in T_n and* $r \not\equiv 0 \pmod 3$, *then*

$$|\mathcal{G}(T_n)| \geqslant |\mathcal{G}(P_{r+2})|.$$

Proof. For any T_n in $\mathcal{T}$, the path P_n has the greatest number of vertices of degree 2, namely $n-2$. Thus $r+2 \leqslant n$ for any T_n. Since in any case the number of critical vertices (either k, k_1+k_2, or $k_1+k_2+k_3$) is greater than or equal to r, the inequality is seen by comparing $|\mathcal{G}(P_{r+2})|$ from Theorem 1 to $|\mathcal{G}(T_n)|$ in each case.

We will call a tree $T \in \mathcal{T}$ a *single tree* if $|\mathcal{G}(T)| = 1$. A planted tree from $\mathcal{T}$ will have one and only one associated OM-graph if and only if it has no critical vertices. Such trees will be called *symmetric* trees of height r (see Fig. 3), and are a subclass of the single trees.

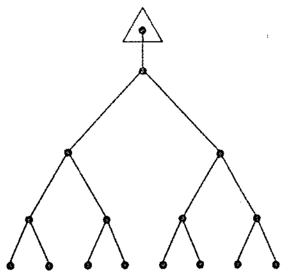


Figure 3. A symmetric tree of height four.

A symmetric tree of height r has $2^r - 1$ vertices (excluding the root), and we note that two of these trees are isomorphic if and only if they have the same height.

It is easily seen that single trees can arise in all of the above cases except for Case 3 and Case 4 (a). Moreover, with the exception of trees in Case 1 (c), all the single trees are formed by joining symmetric trees of perhaps different heights at the rooted vertices. In the exceptional case subtrees S_1 and S_2 are symmetric trees of equal height while S_3 contains exactly one critical vertex v adjacent to the fixed vertex c. There are two possibilities here:

(i) vertex v is of degree two and acts as the root of a symmetric tree;

(ii) vertex v is of degree three and acts as the root for two symmetric trees of different heights.

In view of the limited structure of the single trees in the various cases we have the following interesting result.

Theorem 6. *If T_n is a single tree, then the number of vertices is*

$$n = 2^r + 2^s + 2^t - 2 \qquad r, s, t \geq 0 .$$

Proof. Suppose first that the single tree T_n is not in Case 1 (c). Then T_n is formed by joining three symmetric trees of heights r, s, and t (where a single vertex can be thought of as a symmetric tree of height 0) at a common root, then T_n has $(2^r - 1) + (2^s - 1) + (2^t - 1) + 1 = 2^r + 2^s + 2^t - 2$ vertices.

Suppose next that T_n is in Case 1 (c), with c the fixed vertex of degree 3.

Let the isomorphic subtrees S_1 and S_2 be symmetric trees both of height $r-1$, $r \geqslant 1$, with root at c. Rooted at v, adjacent to c, we have two symmetric trees of different heights s and t (again s or t may be 0). Thus T_n has $2(2^{r-1}-1)+1+(2^s-1)+(2^t-1)+1=2^r+2^s+2^t-2$ vertices.

Corollary. *The smallest integer for which no single tree* T_n *exists is* $n=13$; *the smallest even integer is* $n=28$.

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Abstract

Any trivalent tree may be regarded as the interior dual of a member of a class of non-isomorphic maximal outerplane graphs. We count the number of graphs associated with any given tree, and use these counts to give bounds on the class size, and to define a new class of trivalent trees.

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