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Some generalizations of Laguerre polynomials (IV) (**)

This is the last paper of a series of papers studying the properties of some generalizations of the classical Laguerre polynomials. In 1956 the author [**1**]$_1$ took a completely different approach from the previous three papers of this series [**1**]$_{2,3,4}$ and studied a generalization of the Laguerre polynomials. In this paper a few results not given in the earlier paper [**1**]$_1$ will be derived; also, a class of polynomials related to those studied by Srivastava [**4**] will be defined and some of its properties found.

Srivastava [**4**] took as his starting point a remarkable idea of Erdélyi [**2**] and obtained a generalization of Laguerre polynomials given by

$$\frac{1}{(1-u)^{\nu+1}}\exp\left[w-\frac{w}{(1-u)^{\lambda}}\right]=\sum_{m=0}^{\infty}\frac{u^m}{m!}\,\mathscr{L}^{(\nu)}_{m,\lambda}(w)\,. \tag{1}$$

He showed that $(\mathrm{D}\equiv \mathrm{d}/\mathrm{d}x)$

$$\mathscr{L}^{(\nu)}_{n,\lambda}(x)=\lambda^n x^{-\nu+n+1}\exp[x](x^{(1+1/\lambda)}\mathrm{D})^n\big(\exp[-x]\,x^{(\nu+1/\lambda)}\big)\,. \tag{2}$$

It is easy to see that

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^n}\,\mathscr{L}^{(\nu\lambda)}_{n,\lambda}(x)=G^{(\nu)}_n(x)=x^{-\nu}\exp[x](x\mathrm{D})^n\exp[-x]x^{\nu}, \tag{3}$$

where $G^{(\nu)}_n(x)$ are the polynomials studied extensively by Toscano [**7**]. Steffenson [**5**]$_{1,2}$ considers them as given by the generating function

$$\exp[\alpha t+x(1-\exp[t])]=\sum_{i=0}^{\infty}\frac{t^i}{i!}\,G^{(\alpha)}_i(x)\,. \tag{4}$$

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The author [1]₁ studied the class of polynomials $G_{n,k}^{(\alpha)}(x)$ given by

$$G_{n,k}^{(\alpha)}(x) \equiv (k-1)^n \mathscr{L}_{n,(1/k-1)}^{(\alpha-k+1)/(k-1)}(x) = x^{-\alpha-kn+n} \exp[x](x^k \mathrm{D})^n(\exp[-x]x^\alpha) . \tag{5}$$

Putting $k=2$ and $\alpha = 2-a-n$ we have

$$G_{n,2}^{(2-a-n)}(x) = x^{a-2-n} \exp[x](x^2 \mathrm{D})^n (\exp[-x]x^{2-a-2n}) , \tag{6}$$

so that

$$(-1)^n \exp[-1/x] x^n \, G_{n,2}^{(2-a-2n)}\left(\frac{1}{x}\right) = y_n(x, a, 1) , \tag{7}$$

where the polynomials $y_n(x, a, b)$ are the well-known Bessel polynomials studied by many authors and are given by

$$y_n(x, a, b) = b^{-n} x^{2-a} \mathrm{D}^n (x^{2n+a-2} \exp[-b/x]) . \tag{8}$$

1 - An integral representation of the polynomials $G_{n,k}^{(\alpha)}(x)$.

One of the results obtained by the author in his previous paper [1]₁ is

$$\lambda_n \exp[-x] \, G_{n,1+1/\lambda}^{(\alpha)}(x) = \int_0^\infty \exp[-u] \, u^{\alpha\lambda-1+n} J_{\alpha\lambda-1}^{\lambda}(xu^\lambda) \, \mathrm{d}u , \tag{1.1}$$

where $J_\nu^\lambda(x)$ is the Bessel-Maitland function defined by

$$J_\nu^\lambda(x) = \sum_{n=0}^\infty \frac{(-1)^n x^n}{n! \, \Gamma(\nu + n\lambda + 1)} . \tag{1.2}$$

To give another integral representation we follow Szegö [6]₁ and easily get

$$\begin{aligned} &\int_{+\infty}^{(0^+)} \exp[-u] \, u^{\alpha\lambda-1+n} J_{\alpha\lambda-1}^{\lambda}(-xu^\lambda) \, \mathrm{d}u \\ &= \sum_{r=0}^\infty \frac{(-1)^r x^r}{r! \, \Gamma(\alpha\lambda + \eta r)} \int_{+\infty}^{(0^+)} \exp[-u] \, u^{\alpha\lambda-1+n+\lambda r} \, \mathrm{d}u \\ &= 2 \sum_{r=0}^\infty \frac{(-1)^r x^r}{r! \, \Gamma(\alpha\lambda + \lambda r)} \sin(\alpha\lambda - 1 + \lambda r)\pi \exp[-i\pi(\alpha\lambda - 1 + \lambda r)] \, \Gamma(\alpha\lambda + \lambda r + n) \\ &= 2 \sin(\alpha\lambda - 1)\pi \exp[i\pi(1-\alpha\lambda)] \sum_{r=0}^\infty \frac{(-1)^r x^r}{r!} (\alpha\lambda + \lambda r) \dots (\alpha\lambda - 1 + \lambda r + n) \\ &= 2\lambda^n \sin(\alpha\lambda - 1)\pi \exp[i\pi(1-\alpha\lambda)] \exp[-x] \, G_{n,1/\lambda+1}^{(\alpha)}(x) , \end{aligned} \tag{1.3}$$

for $\alpha\lambda > 0$ and $\alpha\lambda \neq 1, 2, \dots$.

2 - Turán's inequality

In a very interesting paper, Szegö [6]$_2$ shows that Turán's inequality

$$\Delta_n(x) = (U_n(x))^2 - U_{n-1}(x)\,U_{n+1}(x) > 0\,, \tag{2.1}$$

holds good, provided $U_n \equiv U_n(x)$ have a generating function $F(z)$ such that

$$\sum_{n=0}^{\infty} U_n \frac{z^n}{n!} = F(z) \quad \text{and} \quad F(z) = C \exp[-\alpha z^2 + \beta z] \prod \Big(1 - \frac{z}{z_m}\Big) \exp[z/z_m]\,, \tag{2.2}$$

also, Skowgaard [3] has made the observation that if $F(z, \alpha)$ satisfies the functional equation $\partial F/\partial z = F(z, \alpha + 1)$, exploited fully by Truesdell [8], and is of the form given in (2.2) then it satisfies Turán's inequality.

From the generating function of $G^{(\alpha)}_{n,k}(x)$ we see that

$$\big(G^{(\alpha)}_{n,k}(x)\big)^2 - G^{(\alpha)}_{n-1,k}(x)\, G^{(\alpha)}_{n+1,k}(x) \geqslant 0 \qquad \text{for } n \geqslant 1\,. \tag{2.3}$$

3 - Relation of the polynomials $G^{(\alpha)}_{n,k}(x)$ with the hypergeometric series

To find the relation of the polynomials $G^{(\alpha)}_{n,k}(x)$ with the hypergeometric series we proceed as follows.

We know that [1]$_1$

$$n!\, P^{(\alpha)}_{n,r}(x) = \mathscr{L}^{(\alpha)}_{n,r}(x^r)\,, \tag{3.1}$$

where

$$P^{(\alpha)}_{n,r}(x) = \frac{\exp[x^r]\, x^{-\alpha}}{n!} \frac{d^n}{dx^n} \big(x^{n+\alpha} \exp[-x^r]\big)\,. \tag{3.2}$$

If r is a positive integer then it is easy to express $P^{(\alpha)}_{n,r}(x)$ through the confluent hypergeometric function pF_q of Kummer by the formula

$$P^{(\alpha)}_{n,r}(x) = \exp[x^r] \frac{x^n}{n!} \prod_{m=1}^{r} \Big\{ \frac{\Gamma((n+\alpha+m)/r)}{\Gamma((\alpha+m)/r)} \Big\}\, rF_r\,, \tag{3.3}$$

where

$$rF_r\Big(\frac{n+\alpha+1}{r}, \ldots, \frac{n+\alpha+r}{r}\,;\; \frac{\alpha+1}{r}, \ldots, \frac{\alpha+r}{r}\,; -x^r\Big) \equiv rF_r\,. \tag{3.4}$$

From this formula we can easily deduce a general form for $P^{(\alpha)}_{n,r}(x)$.

Using the fact that the product of two hypergeometric functions of any order $rF_s\ (a_1, \ldots, a_r; b_1, \ldots, b_s; x)$ and $_\varrho F_\sigma\ (\alpha_1, \ldots, \alpha_\varrho; \beta_1, \ldots, \beta_\sigma; -x)$ can be put in the form

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{(a_1, n) \ldots (a_r, n)}{(b_1, n) \ldots (b_s, n)} \cdot s + \varrho + 1\, F_{r+\sigma}\left\{ \begin{matrix} -n, 1-b_1-n, \ldots, \alpha_1, \ldots \\ 1-a_1-n, \ldots, \beta_1, \ldots \end{matrix} ; \pm 1 \right\},$$

where the positive or negative in the argument is to be taken according as $(r-s)$ is even or odd, we get (since $\exp[x^r] = {}_0F_0(x^r)$)

$$P_{n,r}^{(\alpha)}(x) = \frac{r^n}{n!} \frac{\Gamma((n+\alpha+1)/r) \ldots \Gamma((n+\alpha+r)/r)}{\Gamma((\alpha+1)/r) \ldots \Gamma((\alpha+r)/r)} \tag{3.5}$$

$$\cdot \sum_{m=0}^{\infty} \frac{x^{rm}}{m!}\, {}_{r+1}F_r\left(-m, \frac{n+\alpha+1}{r}, \ldots, \frac{n+\alpha+r}{r}; \frac{\alpha+1}{r}, \ldots, \frac{\alpha+1}{r}; 1\right).$$

Whence an immediate conclusion is that

$${}_{r+1}F_r\left(-m, \frac{n+\alpha+1}{r}, \ldots, \frac{n+\alpha+r}{r}; \frac{\alpha+1}{r}, \ldots, \frac{\alpha+r}{r}; 1\right) = 0 \tag{3.6}$$

for $m > n$.

4 - A class of polynomials related to $\mathscr{L}_{m,\lambda}^{(\nu)}(x)$

Let us now define another class of polynomials $L_{m,\lambda}^{(\nu)}(x)$ which are closely related to the ones studied by Srivastava [**4**] as defined in (1).

We define our polynomials $L_{m,\lambda}^{(\nu)}(w)$ by

$$\frac{\exp[-wu/(1-u)^\lambda]}{(1-u)^{\nu+1}} = \sum_{m=0}^{\infty} \frac{u^m}{m!} L_{m,\lambda}^{(\nu)}(w). \tag{4.1}$$

Differentiating with respect to w, we have

$$\frac{-u \exp[-wu/(1-u)^\lambda]}{(1-u)^{\nu+\lambda+1}} = \sum_{m=0}^{\infty} \frac{u^m}{m!} \frac{\mathrm{d}}{\mathrm{d}w} L_{m,\lambda}^{(\nu)}(w),$$

which easily gives the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}w} L_{m,\lambda}^{(\nu)}(w) = -mL_{m-1,\lambda}^{(\nu+\lambda)}(w). \tag{4.2}$$

Also

$$\frac{\exp[w - w/(1-u)^\lambda]}{(1-u)^{2\nu+1}} = \frac{\exp[w - w/(1-u)^{\lambda-1}]}{(1-u)^{\nu+\alpha+1}} \cdot \frac{\exp[-wu/(1-u)^\lambda]}{(1-u)^{\nu-\alpha}},$$

which shows that

$$\sum_{m=0}^{\infty} \frac{u^m}{m!} \mathscr{L}_{m,\lambda}^{(2\nu)}(w) = \sum_{m=0}^{\infty} \frac{u^m}{m!} L_{m,\lambda}^{(\nu-\alpha-1)}(w) \cdot \sum_{m=0}^{\infty} \frac{u^m}{m!} \mathscr{L}_{m,\lambda-1}^{(\nu+\alpha)}(w).$$

Hence

(4.3) $$\mathscr{L}_{m,\lambda}^{(2\nu)}(w) = \sum_{r=0}^{m} \binom{m}{r} L_{r,\lambda}^{(\nu-\alpha-1)}(w) \cdot \mathscr{L}_{m-r,\lambda-1}^{(\nu+\alpha)}(w).$$

Now, let us multiply (4.1) by $\exp[-w]$ and differentiate with respect to w. We have on using $[\mathbf{1}]_1$

$$-\frac{d}{dx}\{\exp[-x]\, G_{m,k}^{(\alpha)}(x)\} = \exp[-x]\, G_{m,k}^{(\alpha+1)}(x),$$

the interesting relation

$$\mathscr{L}_{m,\lambda}^{(2\nu+\lambda)}(w)$$

$$= \sum_{r=0}^{m} \binom{m}{r} L_{r,\lambda}^{(\nu-\alpha+1)}(w) \cdot \mathscr{L}_{m-r,\lambda-1}^{(\nu+\alpha+\lambda-1)}(w) + m \sum_{r=0}^{m} \binom{m-1}{r} L_{r,\lambda}^{(\nu-\alpha+\lambda-1)}(w) \cdot \mathscr{L}_{m-r-1,\lambda-1}^{(\nu+\alpha)}(w).$$

Again,

$$\frac{\exp[-wu/(1-u)^\lambda]}{(1-u)^\nu} = \sum_{m=0}^{\infty} \frac{u^m}{m!} L_{m,\lambda}^{(\nu)}(w) \cdot (1-w).$$

Hence, we have

(4.5) $$L_{m,\lambda}^{(\lambda-1)}(w) = L_{m,\lambda}^{(\nu)}(w) - m L_{m-1,\lambda}^{(\nu)}(w).$$

Putting $\nu = 0$ gives

(4.6) $$L_{m,\lambda}^{(-1)}(w) = L_{m,\lambda}(w) - m L_{m-1,\lambda}(w).$$

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