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Asymptotic behavior of nonlinear Stepanoff-bounded functional perturbation problems (**)

1 - Introduction

Consider the functional differential equation

$$x'(t) = A(t)x(t) + f(t, T(t, x)) , \tag{1}$$

where $t \in R$, $x \in R^n$, $A(t)$ is a continuous $n \times n$ matrix, $f \in C[R \times C[B, R^n], R^n]$, B a compact subset of R and $T: R \times C[R, R^n] \to C[B, R^n]$ is defined by

$$T(t, x)(\vartheta) = x(\alpha(t, \vartheta)) \qquad \vartheta \in B ,$$

for given $\alpha \in C[R \times B, R]$.

Problem (1) can be thought of as a perturbation of the linear problem

$$y'(t) = A(t)y(t) . \tag{2}$$

We are going to study the asymptotic relationship between problems (1) and (2), such that to each bounded solution $y = y(t)$ of (2) there corresponds at least one bounded solution $x = x(t)$ of (1) such that $\lim_{|t| \to \infty} |y(t) - x(t)| = 0$.

The question of this asymptotic relationship has been answered by Hallam [**3**], generalizing previous work by Coppel [**2**], Staikos [**5**], Brauer and

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Wong [**1**] and Talpalaru [**6**]. Hallam's innovation is based upon the introduction of four projections, the use of the entire real axis R, the general asymptotic growth conditions imposed on the solutions of the linear problem (2) and the allowable degree of nonlinearity of the functional perturbation f.

In the present work, it is our purpose to relax the conditions on the asymptotic estimate of the nonlinear perturbation f, in the cost of slightly strengthening the conditions on the linear problem (2). To this end, we are employing « Stepanoff-like » conditions on f for L^q-type theorems, $1 \leqslant q < \infty$. In a simplified version of the above question, Lovelady [**4**] treated L^q-type theorems ($1 < q < \infty$) for nonlinear Stepanoff-bounded perturbation problems.

2 - Preliminaries

In the sequel, we suppose that R^n can be decomposed as the direct sum $R^n = X_0 \oplus X_{-1} \oplus X_1 \oplus X_\infty$, where the subspaces X_i, $i = 0, \pm 1, \infty$, are determined as follows.

We have $y_0 \in X_0$ if and only if the solution $y(t; 0, y_0)$ is bounded on R; $y_0 \in X_{-1} \oplus X_0$ if and only if the solution $y(t; 0, y_0)$ is bounded on $[0, \infty)$; $y_0 \in X_1 \oplus X_0$ if and only if the solution $y(t; 0, y_0)$ is bounded on $(-\infty, 0]$; and X_∞ is the direct complement of $X_0 \oplus X_{-1} \oplus X_1$.

To the above complementary subspaces, we associate the corresponding projections P_i ($i = 0, \pm 1, \infty$). Then $y(t; t_0, y_0)$ is written as

$$y(t; t_0, y_0) = [\Phi_0(t; t_0) + \Phi_{-1}(t; t_0) + \Phi_1(t; t_0) + \Phi_\infty(t; t_0)]\, y_0\,,$$

where $\Phi_i(t; t_0) = Y(t) P_i Y^{-1}(t_0)$ ($i = 0, \pm 1, \infty$), and $Y(t)$ denotes the fundamental matrix of the linear problem (2).

In what follows, $\beta(t)$ and $\Gamma(t)$ are nonsingular $n \times n$ matrices that are continuous on R. In our main results, we are going to make use of the following lemma proved by Hallam [**3**].

Lemma 1. (i) *Let there exist a projection P and constants t_0, $K > 0$ and q, $1 \leqslant q < \infty$, such that*

$$\Big[\int_{t_0}^{t} |\beta(t)\, Y(t) P Y^{-1}(s) \Gamma(s)|^q \, ds\Big]^{1/q} \leqslant K \qquad t \geqslant t_0\,,$$

and suppose that $\int^\infty |\Gamma^{-1}(t)\beta^{-1}(t)|^{-q}\, dt = \infty$. *Then* $\lim_{t \to \infty} |\beta(t)\, Y(t) P| = 0$.

(ii) *Let there exist a projection* P *and constants* t_0, $K > 0$ *and* q, $1 \leqslant q < \infty$, *such that*

$$\Big[\int_{t_0}^{t} |\beta(t) P Y^{-1}(s) \Gamma(s)|^q \, ds\Big]^{1/q} \leqslant K \qquad t_0 \geqslant t ,$$

and suppose that $\int_{-\infty} |\Gamma^{-1}(t)\beta^{-1}(t)|^{-q} \, dt = \infty$. *Then* $\lim_{t\to\infty} |\beta(t)\, Y(t) P| = 0$.

3 - Main results

Theorem 1. *Suppose that equations* (1) *and* (2) *satisfy the following hypotheses.*

(i) *There exist supplementary projections* P_i $(i = 0, \pm 1, \infty)$ *and constants* $K > 0$, $a > 1$, *such that for all* $t \in R$

$$(3) \qquad \sum_{k=t}^{-\infty} \varphi(k) \int_{k-1}^{k} |\beta(t) \Phi_{-1}(t; s) \Gamma(s)| \, ds + \sum_{\substack{k=0, \text{ if } t \geqslant 0 \\ k=-1, \text{ if } t<0}}^{t} \varphi(k) \int_{k}^{k+1} |\beta(t) \Phi_0(t; s) \Gamma(s)| \, ds$$
$$+ \sum_{k=t}^{+\infty} \varphi(k) \int_{k}^{k+1} |\beta(t) \Phi_1(t; s) \Gamma(s)| \, ds \leqslant K ,$$

where

$$\varphi(k) = \begin{cases} |k|^a & k \in Z - \{0\} \\ 1 & k = 0 , \end{cases}$$

where Z *denotes the set of integers.*

(ii) $\displaystyle\int^{\infty} |\Gamma^{-1}(t)\beta^{-1}(t)| \, dt = \infty , \qquad \int_{-\infty} |\Gamma^{-1}(t)\beta^{-1}(t)| \, dt = \infty .$

(iii) *For all* $(t, \psi) \in R \times C[B, R^n]$

$$P_\infty Y^{-1}(t) f(t, \psi) = 0 .$$

(iv) *There exists* $\omega \in C[R \times C[B, R_+], R_+]$, $\omega(t, r)$ *nondecreasing in* r *for fixed* $t \in R$ *and for each* $(t, \psi) \in R \times C[B, R^n]$

$$(4) \qquad |\Gamma^{-1}(t) f(t, \psi)| \leqslant \omega(t, |T(t, \beta) \cdot \psi|_B) ,$$

where $R_+ = \{t \in R \colon t \geqslant 0\}$ *and* $|u|_B = \sup_{t \in B} |u(t)|$.

(v) *There exists a solution* $y = y(t)$ *of* (2) *and two constants* λ, ϱ, *where* $\lambda > \varrho > 0$, *such that* $|\beta(t)y(t)| \leqslant \varrho$ *for all* $t \in R$ *and*

$$\sup_{t\in R} \omega(t, \lambda) \leqslant \frac{\lambda-\varrho}{2Ks(a)}, \tag{5}$$

$$\lim_{|k|\to\infty} \frac{1}{\varphi(k)} \sup_{k\leqslant s\leqslant k+1} \omega(s, \lambda) = 0, \tag{6}$$

where $s(a) = 1 + \sum_{k=1}^{\infty} k^{-a}$ (*convergent series, since* $a > 1$).

Then there exists a solution $x = x(t)$ *of* (1) *such that* $|\beta(t)x(t)| \leqslant \lambda$ *for all* $t \in R$ *and*

$$\lim_{|t|\to\infty} |\beta(t)(x(t) - y(t))| = 0. \tag{7}$$

Remark. If t in (3) is not integer, we include in the three sums of (3) the following three integrals respectively

$$\varphi([t]) \int_{[t]}^{t} |\beta(t)\Phi_{-1}(t; s)\Gamma(s)| \, ds,$$

$$\varphi([t]) \int_{[t]}^{t} |\beta(t)\Phi_0(t; s)\Gamma(s)| \, ds \quad (t \geqslant 0), \qquad \text{or} \qquad \varphi([t]) \int_{t}^{[t+1]} |\beta(t)\Phi_0(t; s)\Gamma(s)| \, ds \quad (t \leqslant 0),$$

and

$$\varphi([t]) \int_{t}^{[t+1]} |\beta(t)\Phi_1(t; s)\Gamma(s)| \, ds,$$

where $[t]$ denotes the greater integer less or equal to t. This remark will hold for the rest of the theorems.

Proof. Let C_β the Banach space of $x \in C[R, R^n]$ such that $\beta(t)x(t)$ is bounded on R. The norm of $x \in C_\beta$ is given by $|x|_\beta = \sup_{t\in R} |\beta(t)x(t)|$. Consider the subset $C_{\beta,\lambda}$ of the Banach space C_β defined as $C_{\beta,\lambda} = \{x \in C_\beta : |x|_\beta \leqslant \lambda\}$. Clearly, $C_{\beta,\lambda}$ is closed and convex. We define an operator F on $C_{\beta,\lambda}$ as follows

$$Fx(t) = y(t) + \int_{-\infty}^{t} \Phi_{-1}(t; s) f(s, T(s, x)) \, ds$$
$$+ \int_{0}^{t} \Phi_0(t; s) f(s, T(s, x)) \, ds - \int_{t}^{\infty} \Phi_1(t; s) f(s, T(s, x)) \, ds.$$

First we shall show that $FC_{\beta,\lambda} \subseteq C_{\beta,\lambda}$. In fact, from (i), (iv), (v) and that $|T(t,\beta)\cdot T(t,x)|_B \leqslant \lambda$ for any $x \in C_{\beta,\lambda}$ we have

$$|\beta(t)Fx(t)| \leqslant |\beta(t)y(t)| + \sum_{k=t}^{-\infty} \int_{k-1}^{k} |\beta(t)\Phi_{-1}(t;s)\Gamma(s)|\omega(s,\lambda)\,ds$$

$$+ \sum_{\substack{k=0 \ \text{if } t\geqslant 0\\ k=-1 \ \text{if } t<0}}^{t} \int_{k}^{k+1} |\beta(t)\Phi_0(t;s)\Gamma(s)|\omega(s,\lambda)\,ds$$

$$+ \sum_{k=t}^{\infty} \int_{k}^{k+1} |\beta(t)\Phi_1(t;s)\Gamma(s)|\omega(s,\lambda)\,ds$$

$$\leqslant |\beta(t)y(t)| + \sum_{k=t}^{-\infty} \varphi^{-1}(k) \sup_{k-1\leqslant s\leqslant k} \omega(s,\lambda) \int_{k-1}^{k} \varphi(k)|\beta(t)\Phi_{-1}(t;s)\Gamma(s)|\,ds$$

$$+ \sum_{\substack{k=0, \ \text{if } t\geqslant 0\\ k=-1, \ \text{if } t<0}}^{t} \varphi^{-1}(k) \sup_{k\leqslant s\leqslant k+1} \omega(s,\lambda) \int_{k}^{k+1} \varphi(k)|\beta(t)\Phi_0(t;s)\Gamma(s)|\,ds$$

$$+ \sum_{k=t}^{\infty} \varphi^{-1}(k) \sup_{k\leqslant s\leqslant k+1} \omega(s,\lambda) \int_{k}^{k+1} \varphi(k)|\beta(t)\Phi_1(t;s)\Gamma(s)|\,ds \leqslant \lambda .$$

We claim that F is continuous on $C_{\beta,\lambda}$. Let $\{x_n\}$, $x \in C_{\beta,\lambda}$, such that $\{x_n\}$ converges to x uniformly on compact intervals of R.

For any $\varepsilon > 0$ and a compact $I = [t_*, t^*] \subset R$, because of (6), we can choose a $k_1 \in Z$ sufficiently large, such that $-k_1 \leqslant t_*$ and $t^* \leqslant k_1$ and

$$\sup_{k\leqslant s\leqslant k+1} \omega(s,\lambda) \leqslant \frac{\varepsilon}{6Ks(a)}\varphi(k) \qquad \text{for all } |k| \geqslant k_1 .$$

Since $\{f(t, T(t,x_n))\}$ converges to $f(t, T(t,x))$ uniformly on $[-k_1, k_1]$, there is a $0 < N \in Z$ such that

$$\sup_{-k_1\leqslant t\leqslant k_1} |\Gamma^{-1}(t)(f(t,T(t,x_n)) - f(t,T(t,x)))| < \frac{\varepsilon}{6Ks(a)} \qquad \text{for all } n \geqslant N .$$

Thus from (i), (iv) and (v), it follows that for all $t \in R$ and $n \geqslant N$

$$|\beta(t)[Fx_n(t) - Fx(t)]| \leqslant 2\sum_{k=k_1}^{-\infty} \sup_{k-1\leqslant s\leqslant k} \omega(s,\lambda) \int_{k-1}^{k} |\beta(t)\Phi_{-1}(t;s)\Gamma(s)|\,ds$$

$$+2\sum_{k=k_1}^{\infty} \sup_{k\leqslant s\leqslant k+1} \omega(s,\lambda) \int_{k}^{k+1} |\beta(t)\Phi_1(t;s)\Gamma(s)|\,ds$$

$$+ \sup_{-k_1\leqslant s\leqslant k_1} |\Gamma^{-1}(s)(f((s,T(s,x_n)) - f(s,T(s,x)))| \cdot \Big\{ \sum_{k=t}^{-\infty} \int_{k-1}^{k} |\beta(t)\Phi_{-1}(t;s)\Gamma(s)|\,ds$$

$$+ \sum_{\substack{k=0, \ \text{if } t\geqslant 0\\ k=-1, \ \text{if } t<0}}^{t} \int_{k}^{k+1} |\beta(t)\Phi_0(t;s)\Gamma(s)|\,ds + \sum_{k=t}^{\infty} \int_{k}^{k+1} |\beta(t)\Phi_1(t;s)\Gamma(s)|\,ds \Big\} \leqslant \varepsilon .$$

The next step is to show that $FC_{\beta,\lambda}$ is bounded and equicontinuous (then Ascoli-Arzela theorem would imply that $FC_{\beta,\lambda}$ is relatively compact). Indeed, since $FC_{\beta,\lambda} \subset C_{\beta,\lambda}$, $FC_{\beta,\lambda}$ is bounded and since $z = Fx$ is a solution of the equation

$$\frac{dz}{dt} = A(t)z + f(t, T(t, x)),$$

we have that $(Fx)'$ is bounded, which implies that Fx is equicontinuous. So using the Schauder-Tychonoff Fixed Point Theorem, we have the existence of a fixed point x of F in $C_{\beta,\lambda}$. Thus

$$x(t) = y(t) + \int_{-\infty}^{t} \Phi_{-1}(t; s) f(s, T(s, x))\, ds$$

$$+ \int_{0}^{t} \Phi_0(t; s) f(s, T(s, x))\, ds - \int_{t}^{\infty} \Phi_1(t; s) f(s, T(s, x))\, ds$$

solves the equation (1).

It remains now to show the asymptotic equivalence between a solution of (2) and the corresponding solution of (1).

First we shall examine the case $t \to +\infty$. Because of (6) we can choose a sufficiently large integer $k_2 > 0$ such that for all $|k| \geqslant k_2$,

$$\sup_{k \leqslant s \leqslant k+1} \omega(s, \lambda) \leqslant \frac{\varepsilon}{4K} \varphi(k).$$

By (3) it is easily seen that the hypotheses of Lemma 1 hold for $q = 1$. Hence we have $\lim_{t\to\infty} |\beta(t) Y(t) P_i| = 0$ $(i = -1, 0)$.

It follows that we can choose an integer $k_3 > k_2$, so that for all $t \geqslant k_3$

$$|\beta(t) Y(t) P_i| \int_{-k_2}^{k_2} |Y^{-1}(s) f(s, T(s, x))|\, ds < \frac{\varepsilon}{4} \qquad i \geqslant k_3.$$

Thus for $t \geqslant k_3$ we obtain from the above relations together with (i), (iv) and (v)

$$|\beta(t)[x(t) - y(t)]| \leqslant \sum_{k=-k_2}^{-\infty} \sup_{k-1 \leqslant s \leqslant k} \omega(s, \lambda) \int_{k-1}^{k} |\beta(t) \Phi_{-1}(t; s) \Gamma(s)|\, ds$$

$$+ |\beta(t) Y(t) P_{-1}| \int_{-k_2}^{k_2} |Y^{-1}(s) f(s, T(s, x))|\, ds + |\beta(t) Y(t) P_0| \int_{0}^{k_2} |Y^{-1}(s) f(s, T(s, x))|\, ds$$

$$+ \sum_{k=t}^{k_2+1} \sup_{k-1 \leqslant s \leqslant k} \omega(s, \lambda) \int_{k-1}^{k} |\beta(t) \Phi_{-1}(t; s) \Gamma(s)|\, ds + \sum_{k=k_2}^{t} \sup_{k \leqslant s \leqslant k+1} \omega(s, \lambda) \int_{k}^{k+1} |\beta(t) \Phi_0(t; s) \Gamma(s)|\, ds$$

$$+ \sum_{k=t}^{\infty} \sup_{k \leqslant s \leqslant k+1} \omega(s, \lambda) \int_{k}^{k+1} |\beta(t) \Phi_1(t; s) \Gamma(s)|\, ds \leqslant \varepsilon.$$

The case $t \to -\infty$ is treated similarly as in Hallam [**3**].

The previous theorem can be generalized through L^q-type conditions by the next two theorems. Their proofs are omitted as they repeat the proof of thm. 1 (see also [**3**]).

Theorem 2. *We suppose that the following conditions hold.*

(i) *There exist supplementary projections* P_i $(i=0, \pm 1, \infty)$ *and constants* K, q *with* $K>0$ *and* $1<q<\infty$, *such that for all* $t \in R$

$$\sum_{k=t}^{-\infty} \Big(\int_{k-1}^{k} |\beta(t)\Phi_{-1}(t;s)\Gamma(s)|^q\,ds\Big)^{1/q} + \sum_{\substack{k=0,\ \text{if } t\geq 0\\ k=-1,\ \text{if } t<0}}^{t} \Big(\int_{k}^{k+1} |\beta(t)\Phi_0(t;s)\Gamma(s)|^q\,ds\Big)^{1/q} \tag{8}$$

$$+ \sum_{k=t}^{\infty} \Big(\int_{k}^{k+1} |\beta(t)\Phi_1(t;s)\Gamma(s)|^q\,ds\Big)^{1/q} \leq K .$$

(ii) $$\int^{\infty} |\Gamma^{-1}(t)\beta^{-1}(t)|^{-q}\,dt = \infty , \qquad \int_{-\infty} |\Gamma^{-1}(t)\beta^{-1}(t)|^{-q}\,dt = \infty .$$

(iii) *For all* $(t, \psi) \in R \times C[B, R^n]$

$$P_\infty Y^{-1}(t) f(t, \psi) = 0 .$$

(iv) *There exists* $\omega \in C[R \times C[B, R_+], R_+]$, $\omega(t, r)$ *nondecreasing in* r *for fixed* $t \in R$ *and such that for all* $(t, \psi) \in R \times C[B, R^n]$

$$|\Gamma^{-1}(t) f(t, \psi)| \leq \omega(t, |T(t, \beta)\cdot \psi|_B) .$$

(v) *There exists a solution* $y = y(t)$ *of* (2) *and two constants* λ, ϱ, *where* $\lambda > \varrho > 0$, *such that for all* p *given from* $p^{-1} + q^{-1} = 1$ *and for all* $t \in R$, $|\beta(t) y(t)| \leq$ *and*

$$\Big(\int_{t}^{t+1} \omega^p(s, \lambda)\,ds\Big)^{1/p} \leq \frac{\lambda - \varrho}{K} . \tag{9}$$

Then there exists a solution $x = x(t)$ *of* (1) *such that* $|\beta(t) x(t)| \leq \lambda$ *for all* $t \in R$ *and* (7) *holds.*

Theorem 3. *Suppose that the following conditions hold.*

(i) *There exist supplementary projections* $P_i (i = 0, \pm 1, \infty)$ *and positive constants* $K_j, \alpha_j (j = 1, 2, 3)$ *and* q, $1 < q < \infty$, *such that for* $t, s \in R$

$$(10)_1 \qquad |\beta(t)\Phi_{-1}(t; s)\Gamma(s)| \leqslant K_1 \exp[-\alpha_1(t-s)],$$

for all $s \in [k-1, k]$ and for all $k \in Z$, $k \leqslant t$;

$$(10)_2 \qquad |\beta(t)\Phi_0(t; s)\Gamma(s)| \leqslant K_2 \exp[-\alpha_2|t-s|], \qquad \text{for all } s \in [k, k+1]$$

and for all $k \in Z$, $0 \leqslant k \leqslant t-1$, when $t \geqslant 0$, $k \in Z$, $0 \geqslant k \geqslant t+1$, when $t < 0$;

$$(10)_3 \qquad |\beta(t)\Phi_1(t; s)\Gamma(s)| \leqslant K_3 \exp[-\alpha_3(s-t)]$$

for all $s \in [k, k+1]$ and for all $k \in Z$, $k \geqslant t$.

(ii) *Let conditions* (ii), (iii) *and* (iv) *of Theorem* 2 *hold.*

(iii) *There exists a solution* $y = y(t)$ *of* (2) *and constants* λ, ϱ *with* $\lambda > \varrho > 0$ *such that* $|\beta(t)y(t)| \leqslant \varrho$ *for* $t \in R$ *and condition* (9) *is satisfied with*

$$K = \sum_{j=1}^{3} \frac{K_j(1 - \exp[-q\alpha_j])^{1/q}}{(q\alpha_j)^{1/q}(1 - \exp[-\alpha_j])}.$$

Then there exists a solution $x = x(t)$ *of* (1) *such that* $|\beta(t)x(t)| \leqslant \lambda$ *for all* $t \in R$ *and* (7) *holds.*

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Riassunto

In questa Nota abbiamo studiato il comportamento asintotico della soluzione di una equazione differenziale nonlineare con perturbazione funzionale di una equazione differenziale lineare. A questo scopo usiamo condizioni del tipo di Stepanoff per teoremi L^q *con* $1 \leq q < \infty$. *Questo studio viene applicato su tutto l'asse reale con l'introduzione di quattro proiezioni e di un ammissibile grado di nonlinearità della perturbazione funzionale.*

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