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Epireflective subcategories and epiclosure (**)

Introduction

Let $\mathscr{C}$ be a category of topological spaces and continuous functions, which is full, closed under homeomorphisms and hereditary. In this paper we introduce a closure operator in $\mathscr{C}$ (called *epiclosure*) and compare it to another operator (called *k-closure*) introduced by S. Salbany in [**8**].

By means of a counterexample we prove that the two operators are different. Besides we show that they are related to the categorical notions of extremal epimorphism and kernel.

Their usefulness comes along mainly in the characterization of epireflective subcategories of a co-well-powered category $\mathscr{C}$. These subcategories are exactly those which are closed under products of spaces and k-closed (or epiclosed) subspaces.

If $\mathscr{C} = \boldsymbol{T}_2$ (Hausdorff spaces), epiclosure and k-closure are both equal to topological closure: this yields the well known theorem characterizing epireflective subcategories of $\boldsymbol{T}_2$.

Comparing k-closure and epiclosure shows clearly that the latter one is the more adeguate notion to deal with problems concerning epireflective subcategories of $\mathscr{C}$.

Using epiclosure, it is possible to express the condition for colocal smallness of $\mathscr{C}$ in a simple way, and to characterize epireflective subcategories of $\mathscr{C}$, and to construct the epireflection in a very natural way.

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1 - K-closure and epiclosure

Let **Top** be the category of topological spaces and continuous functions ([1]).

A *topological category* is a full subcategory of **Top** which is closed under homeomorphisms.

A topological category is said to be *hereditary* if it is closed under subspaces, *productive* if it is closed under products of spaces.

Throughout this paper, $\mathscr{C}$ will be a *hereditary topological category.*

Let A, X be objects in $\mathscr{C}$ such that $A \subset X$. Then we define

(a) the *k-closure* of A in X (which we denote $K_X(A)$) to be the set $\{x \in X \mid f(x) = g(x)$, for each $f, g \colon X \to Y$ in $\mathscr{C}$ such that $f|A = g|A\}$ ([2]);

(b) the *epiclosure* of A in X (which we denote $E_X(A)$) to be the union of all subspaces V of X containing A such that the inclusion of A in V is an epimorphism in $\mathscr{C}$.

The following properties of these two operators can be easily verified:

1.1 $A \subset K_X(A)$, $\qquad A \subset E_X(A)$;

1.2 $A \subset B \Rightarrow K_X(A) \subset K_X(B)$, $\qquad A \subset B \Rightarrow E_X(A) \subset E_X(B)$;

1.3 $K_X(K_X(A)) = K_X(A)$, $\qquad E_X(E_X(A)) = E_X(A)$.

These relations show that both E_X and K_X are « Moore closure operators » (see [**2**], p. 8).

Since the inclusion of A in $E_X(A)$ is an epimorphism, *$E_X(A)$ is the largest subspace V of X such that $A \subset V$ and the inclusion map from A to V is an epimorphism in $\mathscr{C}$.*

From the definition we also have

1.4 $E_X(A) \subset K_X(A)$

and equality holds if and only if the inclusion of A into $K_X(A)$ is an epimorphism. As a special case we have

1.5 $K_X(A) = X \Rightarrow E_X(A) = X$.

([1]) We refer to [**11**] for notations or terms which are not explicitly defined.

([2]) This closure is the same as the one defined by Salbany (see [**8**]).

1.6 - Definition. A subspace A of a space X in $\mathscr{C}$ is *k-closed* in X if $K_X(A) = A$; A is *epiclosed* in X if $E_X(A) = A$.

From 1.1 and 1.4 it follows immediately that each k-closed subspace is also epiclosed.

1.7 - Proposition. *Let X be a space in $\mathscr{C}$, $A \subset B \subset X$, and let B be epiclosed in X; then $E_B(A) = E_X(A)$.*

Proof. Since $A \subset B$ we have $E_X(A) \subset E_X(B) = B$, B being epiclosed in X; hence, by definition of epiclosure, $E_B(A) = E_X(A)$.

A proposition similar to 1.7 does not hold for the k-closure. In fact, as we will show in the last paragraph, the epiclosure and the k-closure do not coincide on every topological category and the following proposition holds.

1.8 - Proposition. *Let X be a space in $\mathscr{C}$. The following conditions are equivalent:*

(a) $E_X = K_X$;

(b) *for each B which is k-closed in X and for each $A \subset B$,* $K_B(A) = K_X(A)$.

Proof. (a) $\Rightarrow$ (b). It follows immediately from 1.7.

(b) $\Rightarrow$ (a). Let A be a subset of X, and $B = K_X(A)$. Since B is k-closed, $K_B(A) = K_X(A) = B$. By 1.5 we have $E_B(A) = B$. By 1.4 B is also epiclosed. So, by 1.7, it follows $E_X(A) = E_B(A) = B = K_X(A)$.

1.9 - Proposition. *If $\mathscr{C} \not\subset \boldsymbol{T}_0$, for any X in $\mathscr{C}$ and for any $A \subset X$, we have $E_X(A) = K_X(A) = A$.*

Proof. Since $\mathscr{C}$ is hereditary, if $\mathscr{C} \not\subset \boldsymbol{T}_0$ there exists a two-point space $Y = \{y_1, y_2\}$ in $\mathscr{C}$, with the indiscrete topology. If X is in $\mathscr{C}$ and $A \subset X$ the maps f_1, f_2 from X into Y defined by $f_1(X) = \{y_1\}$, $f_2(A) = \{y_1\}$, $f_2(X \setminus A) = \{y_2\}$ agree over A; so $K_X(A) = A$ and by 1.4 $E_X(A) = A$.

1.10 - Proposition. *For each X in $\mathscr{C}$ and for each $a \in X$ we have $E_X(\{a\}) = \{a\}$.*

Proof. The identity map of X and the constant map $k\colon X \to X$, such that $k(X) = \{a\}$, agree over $\{a\}$, hence the conclusion.

The closure operators K_X and E_X are not always Kuratowsky operators, as the following example shows.

1.11 - Example. Let X be a rigid space with more than two points (see [3], p. 134), i.e. a space such that the only elements of $C(X, X)$ are the identity map and the constant maps; let $\mathscr{C}$ the « epireflective hull » of X in **Top** (see [4], p. 284). X being rigid, two maps f, g in $C(X, X)$ which coincide on a subset of X with at least two points, are the same map. If Y is in $\mathscr{C}$ there is a set M such that $Y \subset X^M$. Let $f, g \in C(X, Y)$ and suppose they agree on a subset A of X with at least two points. By composing f and g with the natural projections p_j from X^M onto X, we obtain $p_j h = p_j g$ for each $j \in M$, and so $f = g$. Finally, by 1.10,

$$K_X(A) = E_X(A) = \begin{cases} A & \text{if } A \text{ is a singleton} \\ X & \text{if } A \text{ has more than one point.} \end{cases}$$

This proves that neither K_X nor E_X are Kuratowsky closure in this case (3).

Let X, Y spaces in $\mathscr{C}$, $A \subset X$, $f \in C(X, Y)$. The following properties can be easily verified:

1.12 $\quad f(K_X(A)) \subset K_Y(f(A))$;

1.13 $\quad f(E_X(A)) \subset E_Y(f(A))$;

1.14 $\quad f$ is an epimorphism in $\mathscr{C}$ iff $K_X(f(X)) = Y$;

1.15 $\quad f$ is an epimorphism in $\mathscr{C}$ iff $E_Y(f(X)) = Y$.

2 - Epireflective subcategories

Throughout this section $\mathscr{C}$ will be a hereditary topological category.

2.1 - Proposition. *A morphism $f: X \to Y$ in $\mathscr{C}$ is an extremal monomorphism* (see [4], p. 110) *if and only if f is an embedding such that $E_Y(f(X)) = f(X)$.*

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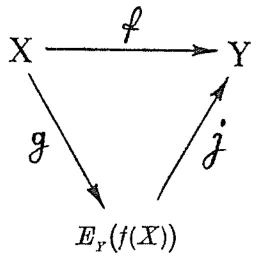
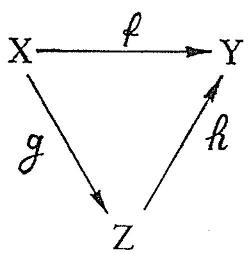
(3) Salbany in [8] suggested to look for such an example.

Proof. Let $f: X \to Y$ be an extremal monomorphism in $\mathscr{C}$. f has the following factorization



where g satisfies $g(x) = f(x)$ for every $x \in X$, and j is the inclusion of $E_Y(f(X))$ into Y. By 1.7 and 1.15, g is an epimorphism, hence it is an isomorphism, since f is an extremal epimorphism. Thus f is an embedding as a composition of two embeddings, moreover $E_Y(f(X)) = f(X)$.

Conversely, let f be an embedding such that $E_Y(f(X)) = f(X)$. Let $f = hg$ in



where Z is in $\mathscr{C}$ and g is an epimorphism in $\mathscr{C}$. Therefore $h(Z) = h(E_Z(g(X)))$ $\subset E_Y(hg(X)) = E_Y(f(X)) = f(X)$. As f is an embedding, we can define the map $k: Z \to X$ in the following way: $k(z) = f^{-1}(h(z))$, for every $z \in Z$. Clearly $kg = \mathrm{id}_X$, and since g is an epimorphism, g must be an isomorphism.

2.2 - Proposition. *If $\mathscr{C}$ is productive, a morphism $f: X \to Y$ in $\mathscr{C}$ is a kernel if and only if f is an embedding such that $K_Y(f(X)) = f(X)$.*

Proof. Let f be the kernel in $\mathscr{C}$ of the morphisms $g, h: Y \to Z$. Since $\mathscr{C}$ is hereditary, f is an embedding. So, let j be the embedding of $K_Y(f(X))$ into Y. Being $gj = hj$, there exists a map $k: K_Y(f(X)) \to X$ such that $fk = j$. Then $K_Y(f(X)) = f(X)$.

Conversely, let f be an embedding such that $K_Y(f(X)) = f(X)$. If $f(X) = Y$, f is obviously a kernel. Otherwise, if $f(X) \neq Y$, for each $y \in Y \setminus f(X)$ there exist a space Z_y in $\mathscr{C}$ and two maps $g_y, h_y\colon Y \to Z_y$ such that $g_y|f(X) = h_y|g(X)$ and $g_y(y) \neq h_y(y)$. Let $Z = \prod_y Z_y$ be the product of the spaces Z_y and $p_y\colon Z \to Z_y$ the projection of Z onto Z_y. It can be easily verified that f is the kernel of the maps $g, h\colon Y \to Z$ defined by the relations $p_y g = g_y$, $p_y h = h_y$.

2.3 - Definition. We say that the category has a *controlled epiclosure* if, for every X, Y in $\mathscr{C}$ and for every embedding $f\colon X \to Y$, we have

2.4 - $|E_Y(f(X))| \leqslant \eta(X)$,

where $\eta(X)$ is a cardinal number which depends only on X.

2.5 - Proposition. *$\mathscr{C}$ is co-well-powered* (see [**4**], p. 44) *if and only if it has a controlled epiclosure.*

Proof. If $\mathscr{C}$ is co-well-powered and $\{e_i\colon X \to A_i\}$ is a set of representative for the epimorphisms with domain X in $\mathscr{C}$ (see [**4**], p. 43), 2.4 is verified by setting $\eta(X) = |\bigcup_i A_i|$.

Conversely, if 2.4 is verified and $e\colon X \to Y$ is an epimorphism in $\mathscr{C}$, we must have $|Y| \leqslant \eta(e(X))$. Since $|e(X)| \leqslant |X|$, it follows that $\eta(e(X))$, and so also $|Y|$, cannot exceed a given cardinality, only depending on X. So $\mathscr{C}$ must be co-well-powered.

The relations among the notions of k-closure, epiclosure and the corrisponding categorical concepts allow us to prove, more satisfactory, some general theorems which characterize epireflective subcategories of given topological categories.

2.6 - Proposition. *Let $\mathscr{D}$ be a topological epireflective subcategory of a hereditary and productive topological category $\mathscr{C}$; $\mathscr{D}$ is closed under products and epiclosed subspaces.*

Proof. It follows from [**4**] (p. 282) and from 2.1.

2.7 - Proposition. *If $\mathscr{C}$ is a productive and hereditary topological category, with controlled epiclosure, and $\mathscr{D}$ is a topological subcategory, closed under products and epiclosed subspaces, than $\mathscr{D}$ is epireflective in $\mathscr{C}$.*

Proof. It follows from [**4**] (p. 282), from 2.1 and from 2.5. Using the epiclosure operator it is possible to build the epireflection in a very simple way. Let X be a space in $\mathscr{C}$ and $\{e_i: X \to D_i\}$, with D_i in $\mathscr{D}$, a set of representative epimorphisms of X (in $\mathscr{D}$). Let $r: X \to \prod_i D_i$ be the product of the morphisms e_i and R_X the epiclosure of $r(X)$ in $\prod_i D_i$; the map $r': X \to R_X$ induced by r, is the epireflection of X.

2.8 - Corollary. *Let $\mathscr{C}$ be a productive and hereditary topological category with controlled epiclosure. A topological subcategory of $\mathscr{C}$ is epireflective in $\mathscr{C}$ if and only if it is closed under products and epiclosed subspaces.*

3 - Examples

Let $\mathscr{C}$ be a hereditary topological category, X be a space of $\mathscr{C}$ and $A \subset X$.

If $\mathscr{C} \not\subset \boldsymbol{T}_0$ or $\mathscr{C} = \boldsymbol{T}_1$ we have $E_X(A) = A$. So $\mathscr{C}$ has a controlled epiclosure and the epiclosed subspaces are the subspaces.

If $\mathscr{C}$ is either $\boldsymbol{T}_2$ or $\boldsymbol{T}_3$ or the category of Tychonoff spaces or the category of 0-dimensional T_1 spaces or the category of totally disconnected T_2 spaces, it is $E_X(A) = \overline{A}$. So $\mathscr{C}$ has a controlled epiclosure and the epiclosed subspaces are exactly the closed subsets.

If $\mathscr{C} = \boldsymbol{T}_0$, the epiclosure is the b-closure (see [**6**]). Furthermore in this case the epiclosure is controlled (see [**5**]).

The following example shows that K_X and E_X are not always the same.

Let $\boldsymbol{T}_{2a}$ the category of Urysohn spaces. The interval $[0, 1]$ with the natural topology τ is in $\boldsymbol{T}_{2a}$. In $[0, 1]$ we consider the topology τ' generated by the open sets of τ and by the sets

$$U_k = \{0\} \cup \Big(\bigcup_{n=k}^{\infty} \Big]\frac{1}{n+1}, \frac{1}{n}\Big[\Big) \qquad (k = 1, 2, 3, \ldots),$$

and denote this topological space with $X = ([0, 1], \tau')$. Being τ' a finer topology than τ, it is clear that X is a space of $\boldsymbol{T}_{2a}$.

Let $A = \{1/n \mid n = 1, 2, 3, \ldots\}$ and suppose $f, g: X \to Y$ are two morphisms of $\boldsymbol{T}_{2a}$ such that $f|A = g|A$ and $f(0) \neq g(0)$. So in Y there exist two closed disjoint neighbourhoods V_1 and V_2 of $f(0)$ and $g(0)$ respectively. So the intersection $f^{-1}(V_1) \cap g^{-1}(V_2)$ is a closed neighbourhood of 0. The family $\{U_k \mid k = 1, 2, 3, \ldots\}$ is a fundamental system of neighbourhoods of 0. So for some k, $U_k \subset f^{-1}(V_1) \cap g^{-1}(V_2)$ and hence

$$\overline{U}_k = \Big[0, \frac{1}{k}\Big] \subset \overline{f^{-1}(V_1) \cap g^{-1}(V_2)} = f^{-1}(V_1) \cap g^{-1}(V_2).$$

Therefore, for any $n > k$, $f(1/n) \in V_1 \cap V_2$ and a contradiction follows. Then $0 \in K_X(A)$. More precisely: $K_X(A) = \{0\} \cup A$.

Since A is closed in X the topology induced on the set $\{0\} \cup A$ is discrete. It follows that $E_X(A) = A$ (4).

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Sommario

Tramite l'introduzione di un particolare operatore di chiusura, detto «epichiusura», vengono studiate e, in certe condizioni, caratterizzate le sottocategorie epiriflessive di una qualsiasi categoria di spazi topologici e funzioni continue che sia piena, chiusa per omeomorfismi e ereditaria.

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(4) In [9] the author studies the epimorphisms of the category $\boldsymbol{T}_{2a}$ introducing a closure operator $\hat{A}$ of a subset A of a space X. It can be easily proved that such a closure is the k-closure. Mistaking the k-closure for the epiclosure, the author classifies the kernels of $\boldsymbol{T}_{2a}$ instead of classifying the extremal epimorphisms.