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Stability properties in Banach spaces (**)

1 - Notations, definitions and recalls

Theorems are enumerated by Roman figures and recalled theorems by starred Roman figures. We use $\{n\}$ for the sequence of the natural numbers, R^+ for the positive real semiaxis, $\mathscr{C}$ for the complex field, B for a Banach space and B' for the dual of B.

Let $\{x_n\}$ be a sequence of B, then span $\{x_n\}$ is the linear manifold spanned by $\{x_n\}$, while $[x_n]$ is the closure of span $\{x_n\}$. We say that $\{x_n\}$ is *complete* in B if $[x_n] = B$; moreover we say that $\{y_n\}$ is a *block sequence* of $\{x_n\}$ if, settins $t_0 = 0$, $\exists$ an increasing sequence $\{t_n\}$ of natural numbers so that $y_n \in \text{span}\, \{x_k\}_{k=t_{n-1}+1}^{t_n}$ $\forall n$.

We recall that an $\{x_n\}$ of B is

overfilling if $[x_{n_k}] = [x_n]$ $\forall \{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}$,

non-contractive [7]$_1$ if $\bigcap_{m=1}^{\infty} [x_n]_{n>m} = [x_n]$,

minimal if $x_m \notin [x_n]_{n \neq m}$ $\forall m$.

Let $\{x_n\} \subset B$ and $\{f_n\} \subset B'$, we recall that (x_n, f_n) is a *biorthogonal system* if $f_m(x_n) = \delta_{mn}$ $\forall m$ and n; therefore

$\{x_n\}$ *is minimal* $\Leftrightarrow \exists \{f_n\} \subset B'$ *with* (x_n, f_n) *biorthogonal system.*

Moreover we recall that

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(a) ([6]$_2$, p. 107) a property $\mathscr{P}$ of $\{x_n\}$ is *stable* if $\exists\{\varepsilon_n\}\subset R^+$ so that every $\{u_n\}$ of B, with $\|u_n - x_n\| < \varepsilon_n$ $\forall n$, has property $\mathscr{P}$.

(b) [1] $\{x_n\}$ is *completely stable* if $\exists\{\varepsilon_n\}\subset R^+$ so that, $\forall\{u_n\}$ of B with $\|u_n - x_n\| < \varepsilon_n$ $\forall n$, the operator T induced by $Tx_n = u_n$ $\forall n$ satisfies $\|T - I\| < 1$, where the norm is taken on $[x_n]$.

Finally we recall the following theorems:

I* ([1], see also [5], p. 163) *Completeness is stable.*

II* [1] $\{x_n\}$ *is completely stable* $\Leftrightarrow$ $\{x_n\}$ *is minimal.*

III* ([3], see also [6]$_2$, pp. 84, 87 and 98) *Let* $\{x_n\}$ *be a minimal sequence of* B, *then* $\Rightarrow \exists\{\varepsilon_n\}\subset R^+$ *so that,* $\forall\{u_n\}\subset B$ *with* $\|u_n - x_n\| < \varepsilon_n$ $\forall n$, $\{u_n\}$ *is minimal; moreover* $[u_n] = [x_n]$ *if* $\{u_n\}\subset[x_n]$.

IV* [7]$_4$ *Every overfilling sequence* $\{x_n\}$ *has an infinite subsequence which keeps overfilling for sufficiently near sequence of* $[x_n]$.

V* [7]$_3$ *Let* $\{x_n\}$ *be a sequence of* B *with* $[x_n]$ *of infinite dimension, then:* $\Rightarrow \exists$ *an* $\{y_n\}$, *minimal and complete in* $[x_n]$, *with* $y_n \in \operatorname{span}\{x_k\}_{k\geqslant n}$ $\forall n$.

2 - Introduction

We report in **3** the proofs of the theorems that we state in this paragraph. In what follows $\{x_n\}$ is a general sequence of B.

Firstly we state two definitions.

(D$_1$) $\{x_n\}$ is *uniformly stable as regards the completeness* (more briefly *u-stable*) if $\exists\{\varepsilon_n\}\subset R^+$ so that, $\forall\{x_{n_k}\}\subseteq\{x_n\}$, if $\{u_{n_k}\}\subset[x_{n_k}]$ with $\|u_{n_k} - x_{n_k}\| < \varepsilon_{n_k}$ $\forall k$, then $[u_{n_k}] = [x_{n_k}]$.

(D$_2$) $\{x_n\}$ is *u*-stable* if we have D$_1$, with the condition that all the $\{x_{n_k}\}$ are sequences of infinite elements.

By th. V* we infer the following theorem, that includes th. I* (in (a) $p = 0$ means that $\{x_{n_k}\}_{k=1}^{p}$ does not appear, the same for q of $\{x_{n'_k}\}_{k=1}^{q}$).

I. (a) $\exists\{x_{n_k}\}_{k=1}^{p}\cup\{x_{n'_k}\}_{k=1}^{q}\subseteq\{x_n\}$, *where* $0\leqslant p\leqslant+\infty$, *while* q *is* 0 *or* $+\infty$, *so that* $\{x_{n_k}\}$ *is minimal, moreover* $[x_n] = [\{x_{n_k}\}_{k=1}^{p}\cup\{x_{n'_k}\}_{k=m}^{q}]$ *and* $x_{n'_m}\notin[x_{n_k}]$, *for* $1\leqslant m< q$.

(b) $\exists\{\varepsilon_n\}\subset R^+$ *so that,* $\forall\{u_n\}\subset[x_n]$ *with* $\|u_n - x_n\| < \varepsilon_n$ $\forall n$, $\{u_{n_k}\}$ *is minimal, moreover* $[x_n] = [\{u_{n_k}\}_{k=1}^{p}\cup\{u_{n'_k}\}_{k=m}^{q}]$ *and* $u_{n'_m}\notin[u_{n_k}]$, *for* $1\leqslant m< q$.

Now we consider th. I* by another point of view, precisely we ask if a sequence is in general u-stable or u^*-stable.

Firstly, by next theorem, we give a negative answer to a problem that we raised in $[7]_1$.

II. *B has always a linearly independent overfilling sequence $\{x_n\}$ such that, if $\{\varepsilon_n\} \subset R^+$, $\exists \{u_n\} \subset [x_n]$ and not overfilling, with $\|u_n - x_n\| < \varepsilon_n$ $\forall n$.*

Now, if $\{x_n\}$ is overfilling, it is obvious that

(1) *$\{x_n\}$ is u^*-stable $\Leftrightarrow \exists \{\varepsilon_n\} \subset R^+$ so that every $\{u_n\}$ of $[x_n]$, with $\|u_n - x_n\| < \varepsilon_n$ $\forall n$, is overfilling.*

Then, by th. II, $\{x_n\}$ is not in general u^*-stable, hence neither u-stable. Next two theorems concern the structure of these sequences.

III. (a) *$\{x_n\}$ is u-stable $\Leftrightarrow$* $\inf \{\text{dist}\,(x_m, [x_{n_k}])$; *$x_m \cup \{x_{n_k}\}$ linearly independent subsequence of $\{x_n\}\}$ is > 0 $\forall m$.*

(b) *Let $\{x_n\}_{n>1}$ be minimal, then $\{x_n\}_{n\geqslant 1}$ is u-stable $\Leftrightarrow \{x_n\}_{n\geqslant 1}$ is minimal, otherwise $x_1 \in$* $\text{span}\,\{x_n\}_{n>1}$.

IV. *Let $\{x_n\}$ be a linearly independent u^*-stable sequence of B, then:*

(a) *$\{x_n\}$ has a complete (in $[x_n]$) minimal subsequence; otherwise $\{x_n\}$ has a complete subsequence which, by removing a finite number of elements at the most, becomes overfilling.*

(b) *$\{x_n\} = \{x_{n_k}\} \cup \{x_{n'_k}\}$ so that, $\forall m$, $x_{n_m} \notin [x_k]_{k \neq n_m}$, moreover $\exists p_m \in \{n\}$ for which $x_{n'_m} \in [x_k]^{p_m}_{k(\neq n'_m)=1} + [x_{n''_k}]$, $\forall \{x_{n''_k}\}^{\infty}_{k=1} \subseteq \{x_n\}_{k>p_m}$.*

By means of th. II we can see that (a) and (b) of th. IV are necessary, but not sufficient, for the u^*-stable sequences.

Moreover, by (a) of th. III, every linearly independent u-stable sequence is minimal; that is, by th. II*,

(2) *minimal = completely stable = u-stable + linearly independent.*

By (2) $\{x_n\}$ has not, in general, an infinite u-stable subsequence; however $\{x_n\}$ has always an u^*-stable subsequence (indeed $\{x_n\}$, if has no infinite minimal subsequence, then has an overfilling subsequence (by th. IV of $[7]_1$) it is now sufficient to use (1) and th. IV*).

Finally, by th. IV, $\{x_n\}$ has not, in general, a complete (in $[x_n]$) u^*-stable subsequence: for example if $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are both overfilling, with $[x_{2n}] \cap [x_{2n-1}] = \{0\}$.

3 - Proofs and remarks

In what follows, if $X \subset B$, then $X^{\perp} = \{f \in B',\ f(x) = 0\ \ \forall x \in X\}$. We recall that

VI* ([**6**]$_2$, p. 99). *In th.* III*, *if* (x_n, f_n) *is a biorthogonal system, we can set* $\varepsilon_n = 1/(2^{n+1}\|f_n\|)$ $\forall n$.

Proof of th. I. (a) We set $x_1 = x_{n_1}$ if $x_1 \notin [x_n]_{n>1}$, otherwise $x_1 = x_{\hat{n}_1}$; so proceeding we find $\{n\} = \{n_k\} \cup \{\hat{n}_k\}$, with $x_{n_k} \notin [x_i]_{i>n_k}$ and $x_{\hat{n}_k} \in [x_i]_{i>\hat{n}_k}$, $\forall k$. Then we set $\{\hat{n}_k\} = \{n'_k\} \cup \{n''_k\}$, so that $x_{n'_m} \notin [x_{n_k}]$, and $x_{n''_m} \in [x_{n_k}]$ $\forall m$.

Then we have that $\exists$ a not decreasing sequence $\{s(k)\}$ of natural numbers so that $\{x_n\} = \{x_{n_k}\}_{k=1}^p \cup \{x_{n'_k}\}_{k=1}^q \cup \{x_{n''_k}\}_{k=1}^r$, where $0 \leqslant p$, $r \leqslant +\infty$, q is 0 or $+\infty$, so that:

$$(3)\qquad \begin{aligned} &x_{n_m} \notin [\{x_{n_k}\}_{k(\neq m)=1}^p \cup \{x_{n'_k}\}_{k=s(m)}] && \text{for } 1 \leqslant m \leqslant p\,, \\ &x_{n'_m} \notin [x_{n_k}] \quad \text{and} \quad [x_n] = [\{x_{n_k}\}_{k=1}^p \cup \{x_{n'_k}\}_{k=m}] && \text{for } 1 \leqslant m < q\,. \end{aligned}$$

Suppose in (3) $p < +\infty$, then $[x_{n_k}]_{k=1}^p \cap [x_{n'_k}]_{k>s(p)} = \{0\}$; hence, if we call $\{x_{n'_k}\}_{k=1}^\infty$ again the sequence $\{x_{n'_k}\}_{k>s(p)}$, we can say that

$$(4)\qquad \text{in (3), if } p < +\infty, \quad \{x_{n'_k}\} \text{ is non-contractive.}$$

(b) If in (3) $q = 0$ the thesis follows by th. III*, hence we can suppose that $q = +\infty$ and we have to consider two cases for p.

Let p be finite.

Then by (3) and (4) $\exists \{h_k\}_{k=1}^p \cup \{g_k\} \subset B'$ so that

(5) $(x_{n_k}, h_k)_{k=1}^p$ is biorthogonal system with $\{h_k\}_{k=1}^p \subset [x_{n'_k}]^\perp$, moreover $g_m(x_{n'_m}) = 1$ $\forall m$ and $\{g_k\} \subset [x_{n_k}]^\perp$.

By (4) and by th. VIII of [**7**]$_1$ $\exists \{\varepsilon'_k\} \subset R^+$ so that

$$(6)\qquad \forall \{u_{n_k}\} \subset B \quad \text{with } \|u_{n'_k} - x_{n'_k}\| < \varepsilon'_k \ \forall k, \text{ it follows that } \{x_{n'_k}\} \subset \bigcap_{m=1}^{\infty} [u_{n'_k}]_{k>m}\,.$$

Now, if $\{n_k\}$, $\{n'_k\}$ and $\{n''_k\}$ are the sequences of (3), let us set

$$(7)\qquad \varepsilon_{n_k} = \frac{1}{2^{k+1}\|h_k\|} \quad \text{for } 1 \leqslant k \leqslant p\,, \qquad \varepsilon_{n'_k} = \min\left\{\frac{1}{2^2\|g_k\|}\,, \varepsilon'_k\right\} \ \forall k,$$

$$\varepsilon_{n''_k} = 1 \quad \text{for } 1 \leqslant k \leqslant r\,.$$

By (7) $\{\varepsilon_n\} \subset R^+$; let now $\{u_n\} \subset B$ so that

(8) $$\{u_n\} \subset [x_n] \qquad \text{with } \|u_n - x_n\| < \varepsilon_n \ \forall n .$$

If we consider $\{x_{n'_k} + [x_{n_i}]_{i=1}^p\}$, by (5) $\exists \{\hat{y}_k\} \subset [x_{n'_k}]$ so that $\{x_{n_k}\}_{k=1}^p \cup \{\hat{y}_k\}$ is minimal and complete in $[x_n]$; hence by (4) and by th. III* $\exists \{y_k\} \subset B$ so that

(9) $\{x_{n_k}\}_{k=1}^q \cup \{y_k\}$ is minimal and complete in $[x_n]$, with $\{y_k\}$ block sequence of $\{x_{n'_k}\}$.

By (8), (7), (5) and (9), moreover by th. III*, we have that $\{u_{n_k}\}_{k=1}^p \cup \{y_k\}$ is minimal and complete in $[x_n]$; on the other hand, by (6), (7), (8) and (9), $\{y_k\} \subset [u_{n'_k}]_{k \geqslant m}$ $\forall m$; hence $[\{u_n\}_{k=1}^p \cup \{u_{n'_k}\}_{k\geqslant m}] = [x_n]$ $\forall m$. Finally, by (5), (7) and (8), $u_{n'_m} \notin [u_{n_k}]_{k=1}^p$ $\forall m$.

Suppose now that $p = + \infty$.

By (3) $\exists \{h_k\} \subset B'$ so that

(10) $$h_m(x_{n_m}) = 1 \qquad \text{and} \qquad h_m \in [\{x_{n_k}\}_{k \neq m} \cup \{x_{n'_k}\}_{k=s(m)}^{\infty}]^{\perp} \qquad \forall m .$$

Let us fix $m \geqslant 1$.

By (3) $[\{x_{n_k}\} \cup \{x_{n'_k}\}_{k\geqslant m}] = [x_n]$; moreover, by lemma 1 of $[7]_3$, $\exists \{\tilde{y}_{mk}\}$ with $\tilde{y}_{mk} \in \operatorname{span}\{x_{n'_i}\}_{i \geqslant m+k}$ $\forall k$ and with $\{x_{n_k}\} \cup \{\tilde{y}_{mk}\}$ complete in $[x_n]$, and $\exists \{h_{mk}\} \subset [\tilde{y}_{mk}]^{\perp}$, with (x_{n_k}, h_{mk}) biorthogonal system; therefore, if we consider the sequence $\{\tilde{y}_{mk} + [x_{n_i}]\}$, by th. V* we have that $\exists \{g_{mk}\}_{k\geqslant m} \subset B'$ and $\{y_{mk}\}_{k \geqslant m}$ so that

(11) $$(x_{n_k}, h_{mk})_{k=1}^{\infty} \cup (y_{mk}, g_{mk})_{k=m}^{\infty} \qquad \text{is biorthogonal system,}$$

moreover $[\{x_{n_k}\}_{k=1}^{\infty} \cup \{y_{mk}\}_{k=m}^{\infty}] = [x_n]$ and $y_{mk} = \sum_{k}^{t_{mk}}{}_i\, \alpha_{mki} x_{n'_i}$ $\forall k \geqslant \mathrm{m}$.

On the other hand by (3) $\exists g_m \cup \{h'_{mk}\}_{k=1}^{\infty} \subset B'$ so that

(12) $$(x_{n'_m}, g_m) \cup (x_{n_k}, h'_{mk})_{k=1}^{\infty} \qquad \text{is biorthogonal system.}$$

Therefore by (10), (11) and (12) we can set

(13) $$h_{mi} = h_i \quad \text{for } 1 \leqslant i \leqslant k \qquad \text{and} \qquad h'_{mk} = h_k \quad \forall k \text{ so that } s(k) \leqslant m .$$

Then, if $\{n_k\}$, $\{n'_k\}$ and $\{n''_k\}$ are the sequences of (3), let us set

$$(14)_1 \qquad \varepsilon_{n_k} = \frac{1}{2^{k+1}(\|h_k\| + \sum_{1}^{s(k)-1}{}_m (\|h_{mk}\| + \|h'_{mk}\|))} \qquad \forall k \quad \text{so that } s(k) > 1$$

$$(14)_2 \qquad \varepsilon_{n_k} = \frac{1}{2^{k+1}\|h_k\|} \qquad \forall k \quad \text{so that } s(k) = 1 \,,$$

$$(14)_3 \qquad \varepsilon_{n'_k} = \frac{1}{2^{k+1}(\|g_k\| + \sum_{1}^{k}{}_m (\sum_{m}^{k}{}_i\, t_{mi} |\alpha_{mik}| \|g_{mi}\|))} \qquad \forall k \,,$$

$$(14)_4 \qquad \varepsilon_{n''_k} = 1 \qquad \forall k \,.$$

By (14) $$\{\varepsilon_n\} = \{\varepsilon_{n_k}\} \cup \{\varepsilon_{n'_k}\} \cup \{\varepsilon_{n''_k}\} \subset R^+.$$

Let now $\{u_n\} \subset B$ so that

$$(15) \qquad \{u_n\} \subset [x_n] \qquad \text{with } \|u_n - x_n\| < \varepsilon_n \ \ \forall n \,.$$

Let us fix again $m \geqslant 1$. If we set, $\forall i \geqslant m$, $v_{mi} = \sum_{i}^{t_{mi}}{}_k\, \alpha_{mik}\, u_{n'_k}$, by (11), (14) and (15) we have that

$$\|v_{mi} - y_{mi}\| \leqslant \sum_{i}^{t_{mi}}{}_k |\alpha_{mik}|\, \|u_{n'_k} - x_{n'_k}\| < \sum_{i}^{t_{mi}}{}_k |\alpha_{mik}|\, \varepsilon_{n'_k}$$
$$< \sum_{i}^{t_{mi}}{}_k |\alpha_{mik}| \frac{1}{2^{k+1}\, t_{mi}\, |\alpha_{mik}| \|g_{mi}\|} \leqslant \frac{1}{2^{i+1}\, \|g_{mi}\|} \,.$$

Therefore, by (11), (14), (15) and by th. III* and VI*, $\{u_{n_k}\} \cup \{v_{mk}\}$ is minimal, and complete in $[x_n]$; hence $[\{u_{n_k}\} \cup \{u_{n'_k}\}_{k \geqslant m}] = [x_n]$. Finally, by (12), (13), (14) and (15), $u_{n'_m} \cup \{u_{n_k}\}_{k=1}^{\infty}$ is minimal $\forall m$, which completes the proof of th. I.

Remarks on th. I. In th. I we considered only near sequences $\{u_n\}$ with the condition that $\{u_n\} \subset [x_n]$; indeed, if $[x_n]$ is an infinite codimensional subspace of B, we recall (th. IV of $[7]_2$) that, $\forall \{\varepsilon_n\} \subset R^+$, $\exists$ a minimal sequence $\{u_n\}$ of B with $\|u_n - x_n\| < \varepsilon_n$ $\forall n$.

Moreover the property $x_{n_m} \notin [\{x_{n_k}\}_{k(\neq m)=1}^{p} \cup \{x_{n'_k}\}_{k=s(m)}^{q}]$ $\forall m$ of (3) is not considered in th. I, because it does not keep for sufficiently near sequences, precisely:

(c) *Suppose in* (3) $s(k)=1$ $\forall k$ (it is possible), *then*: $\Rightarrow \forall\{\varepsilon_n\}\subset R^+$ $\exists$ *a non-contractive sequence* $\{u_n\}$ *complete in* $[x_n]$, *with* $\|u_n - x_n\| < \varepsilon_n$ $\forall n$.

Indeed by hypothesis and by lemma 3 of $[7]_3$ we have that $\{x_{n'_k}\}\subset\bigcap_{m=1}^{\infty}[x_n]_{n>m}$, hence by th. VIII of $[7]_1$ $\exists\{\eta_n\}\subset R^+$ so that

$$\text{(16)}\qquad \forall\{u_n\}\subset B \quad \text{with } \|u_n - x_n\| < \eta_n \ \forall n,\ \{x_{n'_k}\}\subset\bigcap_{m=1}^{\infty}[u_n]_{n>m}\,.$$

Then, if $\{v_k\}$ is a non-contractive sequence complete in $[x_n]$, let us set

$$\text{(17)}\qquad u_{n_k}=x_{n_k}\,,\qquad u_{n''_k}=x_{n''_k}\,,\qquad u_{n'_k}=x_{n'_k}+\min\{\varepsilon_{n'_k},\eta_{n'_k}\}\frac{v_k}{2\|v_k\|}\quad \forall k\,.$$

By (17) $\|u_n - x_n\| < \varepsilon_n$ $\forall n$; on the other hand, $\forall m\geqslant 1$, by (16) $\{x_{n'_k}\}\subset[u_n]_{n>m}$, hence by (17) $\exists p_m\in\{n\}$ with $\{v_k\}_{k>p_m}\subset[u_n]_{n>m}$, that is $[x_n]=[u_n]_{n>m}$.

(d) *Suppose in* (3) $\{x_{n'_k}\}$ *non-contractive, then*: $\Rightarrow \forall\{\varepsilon_k\}\subset R^+$ $\exists$ *a non-contractive sequence* $\{u_k\}$ *complete in* $[x_n]$, *with* $\|u_k - x_{n'_k}\| < \varepsilon_k$ $\forall k$.

Indeed, always by th. VIII of $[7]_1$, $\exists\{\varepsilon'_k\}\subset R^+$ so that

$$\forall\{u_k\}\subset B\,,\quad \text{with } \|u_k - x_{n'_k}\|<\varepsilon'_k \ \forall k,\quad \{x_{n'_k}\}\subset\bigcap_{m=1}^{\infty}[u_k]_{k>m}\,.$$

Then, if $\{v_k\}$ is the sequence of (c), it is sufficient to set

$$u_k = x_{n'_k}+\min\{\varepsilon'_k,\varepsilon_k\}\frac{v_k}{2\|v_k\|}\quad \forall k\,.$$

Finally we mention particular cases of the sequence $\{\varepsilon_n\}$: for example, if B is separable and if $\{x_n\}$ is dense in B, we have that every $\{u_n\}$ of B, with $\lim_{n\to\infty}\|u_n - x_n\| = 0$, is complete in B.

Let us now recall a result of Klee.

VII* ([2], pp. 193-194) *Let* $\{x_n\}$ *be a minimal sequence complete in* B, *with* $\|x_n\|=1$ $\forall n$, *then*: $\Rightarrow$ *every infinite subset of* $\left\{\sum_1^{\infty}{}_k\, t^k x_k;\ 1/6\leqslant t\leqslant 1/3\right\}$ *is complete in* B.

Proof of th. II. Let $\{x_n\}$ be a minimal sequence of B, with $\|x_n\|=1$ $\forall n$, moreover let us set $x_n=\varphi(t_n)$ $\forall n$, where $\varphi(t)=\sum_1^{\infty}{}_k\, t^k x_k$, while $\{t_n\}$ is the sequence of the rational numbers of $J = 1/6 \mapsto 1/3$. Then, by th. VII*, we have that $\{x_n\}$ is overfilling.

Let now $\{\varepsilon_n\}$ be a fixed sequence of R^+; we can choose a sequence $\{J_k\}$ of subintervals of J and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ so that

(18) $$t_{n_k} \in J_k \text{ and } \|\varphi(t) - x_{n_k}\| < \varepsilon_{n_k} \quad \forall t \in J_k, \quad \text{moreover}$$
$$J_{k+1} \subset J_k \text{ with } l_{k+1} \ (= \text{length of } J_{k+1}) < l_k/2 \quad \forall k.$$

Indeed we can start with an arbitrary natural number n_1; then suppose to have found $\{J_i\}_{i=1}^k$, we can choose $t_{n_{k+1}} \in J_k$, now J_{k+1} follows by continuity of $\varphi(t)$.

By (18) $\lim_{k\to\infty} t_{n_k} = \bar{t}$, hence $\bar{t} \in J_k \ \forall k$; therefore, if $\{n'_k\}$ is the subsequence of $\{n\}$ complementary to $\{n_k\}$, let us set $u_{n'_k} = x_{n'_k}$ and $u_{n_k} = \varphi(\bar{t}) \ \ \forall k$; by (18) $\|u_n - x_n\| < \varepsilon_n \ \forall n$, on the other hand $[u_{n_k}]_{k=1}^\infty = \text{span}\,\{\varphi(\bar{t})\}$, that is $\{u_n\}$ is not overfilling, which completes the proof of th. II.

Remarks on th. II. Let us consider the regularity properties of a sequence $\{x_n\}$.

If $\{x_n\}$ is minimal, or more than minimal (for example uniformly minimal, M-basic, basic with brackets, basic), it is known that $\{x_n\}$ keeps its property for sufficiently near sequences ([**3**] and [**4**], see also [**6**]$_2$ p. 98, and [**6**]$_1$ p. 171, see moreover [**7**]$_2$, corollary II). While the properties less than minimal, for example the ω-linear independence, do not keep for sufficiently near sequences: indeed, if $\{x_n\}$ is not minimal, it is immediate to see that, $\forall \{\varepsilon_n\} \subset R^+$, $\exists$ a not linearly independent sequence $\{u_n\}$ of $[x_n]$ with $\|u_n - x_n\| < \varepsilon_n \ \forall n$.

Let us now consider another category of properties, in the opposite direction: the non-contractive and the overfilling sequences.

Both the non-contractive and the minimal $\overline{Y}$-overfilling sequences $\{x_n\}$ (that is with $\bigcap_{m=1}^{\infty} [x_{n_k}]_{k>m} = \overline{Y}$, $\forall \{x_{n_k}\}_{k=1}^\infty \subseteq \{x_n\}$) keep their properties for sufficiently near sequences (th. VIII of [**7**]$_1$ and th. III of [**7**]$_2$); while, by th. II, this is not true for the general overfilling sequence, but it is true for particular subsequences (see th. IV*).

Proof of th. III. (a) Let us prove $\Rightarrow$.

Suppose that the thesis is not true, then $\exists \bar{n} \in \{n\}$ so that

(19) $\forall \varepsilon \in R^+$ $\exists$ a linearly independent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that $0 < \text{dist}\,(x_{\bar{n}}, [x_{n_k}]) < \varepsilon$.

Let us fix $\{\varepsilon_n\} \subset R^+$, by (19) $\exists \{x_{n_k}\}_{k=1}^p \subset \{x_n\}$ and $\{\alpha_k\}_{k=1}^p \subset \mathscr{C}$, with $p < +\infty$, so that

$$x_{\bar{n}} \cup \{x_{n_k}\}_{k=1}^p \text{ is linearly independent and } \Big\|x_{\bar{n}} + \sum_1^p{}_k \alpha_k x_{n_k}\Big\| < \varepsilon_{\bar{n}}\,.$$

Then, setting $u_{\bar{n}} = \sum_1^p {}_k \alpha_k x_{n_k}$ and $x_n = x_n$ for $n \neq \bar{n}$, we have $\|u_n - x_n\| < \varepsilon_n$ $\forall n$, but $[u_{\bar{n}} \cup \{u_{n_k}\}_{k=1}^p] = [x_{n_k}]_{k=1}^p \neq [x_{\bar{n}} \cup \{x_{n_k}\}_{k=1}^p]$, hence $\{x_n\}$ would not be u-stable.

Let us prove $\Leftarrow$.

By hypothesis every linearly independent subsequence of $\{x_n\}$ is minimal, then let us set

$$\varepsilon_m = \frac{1}{2^{m+1}} \inf \{\operatorname{dist}(x_m, [x_{n_k}]);\ x_m \cup \{x_{n_k}\} \tag{20}$$

minimal subsequence of $\{x_n\}\}$ $\quad \forall m$.

By hypothesis $\{\varepsilon_n\} \subset R^+$; moreover, if $\{x_{n_k}\}$ is a minimal subsequence of $\{x_n\}$, by (20) $\exists \{f_k\} \subset B'$ (see [**6**]$_2$ cor. 2.1, p. 255) so that

$$(x_{n_k}, f_k) \text{ is a biorthogonal system, with } \|f_k\| \leqslant \frac{1}{\varepsilon_{n_k} 2^{n_k+1}} \quad \forall k . \tag{21}$$

Let now $\{x_{n_k'}\} \subseteq \{x_n\}$ and $\{u_{n_k'}\} \subset [x_{n_k'}]$ with $\|u_{n_k'} - x_{n_k'}\| < \varepsilon_{n_k'}$ $\forall k$; then if $\{x_{n_k}\}$ is a linearly independent subsequence of $\{x_{n_k'}\}$ with $\operatorname{span}\{x_{n_k}\} = \operatorname{span}\{x_{n_k'}\}$ by hypothesis it follows that $\{x_{n_k}\}$ is minimal; therefore, by (21),

$$\|u_{n_k} - x_{n_k}\| < \varepsilon_{n_k} \leqslant 1/(2^{n_k+1}\|f_k\|) \leqslant 1/(2^{k+1}\|f_k\|) \qquad \forall k ;$$

that is, by th. III* and VI*, $[u_{n_k'}] \subseteq [x_{n_k'}] = [x_{n_k}] = [u_{n_k}] \subseteq [u_{n_k'}]$.

(b) $\Rightarrow$ follows by (a), hence let us prove $\Leftarrow$.

If $\{x_n\}_{n\geqslant 1}$ is minimal the thesis follows by th. III*, hence suppose $x_1 \in \operatorname{span}\{x_n\}_{n>1}$.

Therefore, by hypothesis, $\exists \{f_n\}_{n>1} \subset B'$ and $\{\alpha_n\}_{n=2}^p \subset \mathscr{C}$ so that

$$(x_n, f_n)_{n>} \text{ is biorthogonal system, } x_1 = \sum_2^p {}_n \alpha_n x_n,\ p < +\infty . \tag{22}$$

By (22), $\forall m \in \{n\}_{=2}$ with $\alpha_m \neq 0$, we have that

$$\left(x_1, \frac{f_m}{\alpha_m}\right) \cup \left(x_n,\ f_n - \frac{\alpha_n}{\alpha_m} f_m\right)_{n(\neq m)=2}^p \cup (x_n, f_n)_{n>p} \text{ is biorthogonal system.} \tag{23}$$

It is now sufficient to set

$$\varepsilon_1 = \frac{1}{2^2(\sum_2^p {}_m (\|f_m\|/|\alpha_m| \text{ so that } \alpha_m \neq 0))}, \qquad \varepsilon_n = \frac{1}{2^{n+1}\|f_n\|} \qquad \text{for } n > p,$$

(24)

$$\varepsilon_n = \frac{1}{2^{n+1}(\|f_n\| + \sum_2^p {}_m(\|f_n - (\alpha_n/\alpha_m) f_m\|, m \neq n \text{ and such that } \alpha_m \neq 0))} \quad \text{for } 2 \leqslant n \leqslant p.$$

Now, if $\{x_{n_k}\} \subseteq \{x_n\}$ and if $\{u_{n_k}\} \subset [x_{n_k}]$ with $\|u_{n_k} - x_{n_k}\| < \varepsilon_{n_k}$ $\forall k$, by th. III* and VI* and by (22), (23) and (24) it is easy to check that $[u_{n_k}] = [x_{n_k}]$. This completes the proof of th. III.

Proof of th. IV. (b) Let us set

(25) $\{x_n\} = \{x_{n_k}\} \cup \{x_{n'_k}\}$, with $x_{n_m} \notin [x_k]_{k \neq n_m}$ and $x_{n'_k} \in [x_k]_{k \neq n'_m}$ $\forall m$.

Let us fix m.

Suppose that the thesis is not true, hence

(26) $\forall i \in \{n\}$ $\exists \{x_{n(i)_k}\}_{k=1}^\infty \subset \{x_k\}_{k>i}$ so that $x_{n'_m} \notin [x_k]_{k(\neq n'_m)=1}^i + [x_{n(i)_k}]_{k=1}^\infty$.

By hypothesis $\{x_n\}$ is linearly independent, hence by (25) we have that

(27) $\forall p \in \{n\}$ $\exists r(p) \in \{n\}$ so that $0 < \operatorname{dist}(x_{n'_m}, [x_k]_{k(\neq n'_m)=1}^{r(p)}) < 1/p$.

Let us fix $\{\varepsilon_n\} \subset R^+$, by (27) $\exists p' \in \{n\}$ so that

(28) $\exists u' \in [x_k]_{k(\neq n'_m)=1}^{r(p')}$ with $\|u' - x_{n'_m}\| < \varepsilon_{n'_m}$.

Now let us set

(29)
$$r' = r(p') \text{ and } n''_k = n(r')_k \ \forall k;$$
$$u_k = x_k \text{ for } 1 \leqslant k(\neq n'_m) \leqslant r',\ u_{n'_m} = u',\ u_{n''_k} = x_{n''_k}\ \forall k.$$

By (26), (28) and (29) it is easy to check that

$$[u_{n'_m} \cup \{u_k\}_{k(\neq n'_m)=1}^{r'} \cup \{u_{n''_k}\}_{k=1}^\infty] = [x_k]_{k(\neq n'_m)=1}^{r'} + [x_{n''_k}]_{k=1}^\infty$$
$$\neq [x_{n'_m} \cup \{x_k\}_{k(\neq n'_m)=1}^{r'}] + [x_{n''_k}]_{k=1}^\infty,$$

that is $\{x_n\}$ would not be u^*-stable.

(a) Suppose that $\{x_n\}$ has no complete minimal subsequence, then we shall prove that

(30) $\{x_n\}$ has not two infinite subsequences $\{x_{n_k}\}$ and $\{x_{n'_k}\}$ with $\{x_{n_k} + [x_{n'_i}]_{i=1}^{\infty}\}_{k=1}^{\infty}$ linearly independent.

In fact suppose that (30) is not true, hence

(31) $\exists \{x_{n_k}\} \cup \{x_{n'_k}\} \subset \{x_n\}$ with $\{x_{n'_k} + [x_{n_i}]_{i=1}^{\infty}\}_{k=1}^{\infty}$ linearly independent.

Then $\{x_{n'_k} + [x_{n_i}]_{i=1}^{\infty}\}_{k=1}^{\infty}$ is minimal, because, if $\exists m \in \{n\}$ with $x_{n'_m} \in [\{x_{n'_k}\}_{k(\neq m)=1}^{\infty} \cup \{x_{n_k}\}_{k=1}^{\infty}]$, by (b) $\exists s(m) \in \{n\}$ so that $x_{n'_m} \in [\{x_{n'_k}\}_{k(\neq m)=1}^{s(m)} \cup \{x_{n_k}\}_{k=1}^{\infty}]$, impossible by (31). On the other hand, if we consider $\{x_{n_k} + [x_{n'_i}]_{i=1}^{\infty}\}_{k=1}^{\infty}$, we have that $[x_{n_k} + [x_{n'_i}]_{i=1}^{\infty}]_{k=1}^{\infty}$ has infinite dimension, because $\{x_n\}$ has no complete minimal subsequence; hence $\exists$ an infinite subsequence $\{x_{n''_k}\}$ of $\{x_{n_k}\}$, so that $\{x_{n''_k} + [x_{n'_i}]_{i=1}^{\infty}\}_{k=1}^{\infty}$ is linearly independent and complete in $[x_{n_k} + [x_{n'_i}]_{i=1}^{\infty}]_{k=1}^{\infty}$; moreover, by preceding arguments, $\{x_{n''_k} + [x_{n'_i}]_{i=1}^{\infty}\}_{k=1}^{\infty}$ is minimal; consequently $\{x_{n'_k}\} \cup \{x_{n''_k}\}$ would be minimal and complete in $[x_n]$, which is not possible; that is (31) is not possible and (30) is proved.

Now we affirm that

(32) $\{x_n\}$ has no infinite minimal subsequence.

Indeed, if $\exists$ a minimal subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ and if $\{n'_k\}$ is the subsequence of $\{n\}$ complementary to $\{n_k\}$, by (30) $\{x_{n'_k} + [x_{n_i}]_{i=1}^{\infty}\}$ cannot have an infinite linearly independent subsequence, hence it would follow that $\{x_n\}$ has a complete minimal subsequence, contrary to hypothesis; therefore (32) is proved.

Then, by (32) and by th. IX of $[7]_1$, $\exists$ an overfilling subsequence $\{x_{n_k}\}$ of $\{x_n\}$. On the other hand, by (30), if $\{n'_k\}$ is the subsequence of $\{n\}$ complementary to $\{n_k\}$, $\exists \{x_{n''_k}\}_{k=1}^{p} \subset \{x_{n'_k}\}$, with $p < +\infty$, so that $\{x_{n''_k} + [x_{n_i}]_{i=1}^{\infty}\}_{k=1}^{p}$ is complete in $[x_{n'_k} + [x_{n_i}]_{i=1}^{\infty}]$; that is $\{x_{n''_k}\}_{k=1}^{p} \cup \{x_{n_k}\}$ is complete in $[x_n]$, which completes the proof of th. IV.

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Riassunto

Un teorema di Gurarii afferma che, se $\{x_n\}$ è una successione di uno spazio di Banach B, esiste una $\{\varepsilon_n\}$ di numeri positivi tale che ogni $\{u_n\} \subset [x_n]$, con $\|u_n - x_n\| < \varepsilon_n \ \forall n$, sia completa in $[x_n]$.

Mediante una tecnica diversa ripresentiamo questo teorema, in una forma in cui viene considerata anche la struttura della $\{x_n\}$.

Definiamo poi le $\{x_n\}$ « u-stabili » e « u-stabili »: per le prime esiste una $\{\varepsilon_n\}$ tale che, $\forall \{x_{n_k}\} \subseteq \{x_n\}$ e $\forall \{u_{n_k}\} \subset [x_{n_k}]$ con $\|u_{n_k} - x_{n_k}\| < \varepsilon_n \ \forall k$, sia $[u_{n_k}] = [x_{n_k}]$; le seconde hanno la stessa proprietà, con la condizione che ogni $\{x_{n_k}\}$ sia infinita.*

Esaminando la struttura di tali successioni facciamo vedere che le u-stabili sono un'estensione delle successioni minimali; mentre le u-stabili (ma non u-stabili) sono collegate alle successioni « overfilling ». Dimostriamo inoltre che la proprietà di essere overfilling non si mantiene per successioni abbastanza vicine; ne segue che una successione non è in generale u*-stabile, però ha sempre una sottosuccessione infinita u*-stabile.*

* * *