Riv. Mat. Univ. Parma (4) 8 (1982), 213-221

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A nonlinear boundary problem on infinite interval with countable set of interface conditions (**)

1 - Introduction

We are interested here in proving the existence of solutions to the boundary value problem with interface conditions

(A) $$\dot{x} - A(t)x = F(t, x) ,$$

(B) $$\mathcal{T}_r(x) = x(t_r^+) - B_r x(t_r^-) = C_r \qquad (r = 1, 2, \ldots) ,$$

(T) $$Tx = Hx ,$$

where A is a continuous $n \times n$ matrix valued function of t on the non compact interval $[a, b[$, $-\infty < a < b \leqslant \infty$, F is a continuous n vector valued function of (t, x) on $[a, b[\times R^n$, B_r are real $n \times n$ non singular matrices, $C_r \in R^n$, for $r = 1, 2, \ldots$, the internal interface points t_r form an infinite point set of first species G, T is a continuous linear operator defined on a subspace of $C[[a, b[, R^n]$, the locally convex space of all continuous R^n-valued functions on $[a, b[$, and H is a continuous operator defined on subspace of $C[[a, b[, R^n]$. Further

$$x(t_r^+) = \lim_{t \to t_r^+} x(t) , \qquad x(t_r^-) = \lim_{t \to t_r^-} x(t) .$$

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(**) Work supported by the Italian C.N.R.. – Ricevuto: 25-IX-1980.

We apply the equivalence theorem for the nonlinear operator equation due to Zezza [**8**] in order to ensure the existence of solutions of the problem (A), (B), (T). Applications of the above theorem to system (A), (T) were given in $[\mathbf{2}]_{1,2}$.

The present results extend to infinite intervals with a countable set of interface points Gonnelli's result [**4**], who considered the same problem on a finite interval with a finite number of interface points. For related discussion on interface problems, the reader is referred to Conti [**3**] and references cited therein.

2 - Preliminaries

Let $G = \{t_r, r = 1, 2, ...\}$ be a countable subset of $[a, b[, -\infty < a < b \leqslant \infty$, such that

$$a < t_1 < t_2 < \ldots < t_r < \ldots < b \,, \tag{2.1}$$

so that $[a, b[\setminus G \equiv \delta$ consist of the countable collection of intervals $[a, t_1[, \ldots$ $\ldots, [t_r, t_{r+1}[, \ldots$. In what follows, $\Delta = [a, b[$. By C_b we denote the Banach space of all bounded $x \in C[\Delta, R^n]$ with the norm

$$\|x\|_{C_b} = \sup_{t \in \Delta} \|x(t)\| \,,$$

where $\|\cdot\|$ denotes the norm in R^n.

We denote by $CG = CG[\delta, R^n]$ the space of all continuous R^n-valued functions for every $t \in \delta$, such that $x(t_r^+)$, $x(t_r^-)$ exist and are finite. By $C_b G$ we denote the space of all bounded $x \in CG$. $C_b G$ is a Banach space under the norm

$$\|x\|_g = \sup_{t \in \delta} \|x(t)\| \,. \tag{2.2}$$

Now let $C_1 G$ (or C_1) denote the space of all functions $x \in C_b G$ (or C_b) such that $\lim_{t \to b} x(t)$ exists and is finite. Then $C_1 G$ (or C_1) is a closed subspace of $C_b G$ (or C_b).

Let $X(t)$ be the fundamental matrix of solutions of the homogeneous system

$$\dot{x} - A(t)x = 0 \,, \tag{A.0}$$

$$x(t_r^+) - B_r x(t_r^-) = 0 \,, \tag{B.0}$$

i.e., let

$$X(t) = U(t)A(t) = U(t)\,U^{-1}(t_i)B_i\,U(t_i)\,\dots\,U^{-1}(t_1)B_1\,U(t_1) \tag{2.3}$$

on the interval $]t_i, t_{i+1}[$, where $U(t)$ is the fundamental solution of (A.0). The solution $Y(t)$ of the nonhomogeneous system (Q), (B),

$$\dot{x} - A(t)x = Q(t) \tag{Q}$$

on $]t_r, t_{r+1}[$ is represented by [7]

$$Y(t) = X(t)X^{-1}(a)\,Y(a) + X(t)\sum_{i=1}^{r} X^{-1}(t_i^+)\,C_i + X(t)\int_a^t X^{-1}(\tau)Q(\tau)\,d\tau\,. \tag{2.4}$$

We assume that the following hypotheses hold:

(i) $A(t)$ is an $n\times n$ real matrix function defined and continuous on Δ and such that $\|X(t)\| \leqslant \varrho$ (a non-negative constant).

(ii) The operator T: $\operatorname{dom} T = C_b G \to R^m$ $(m \leqslant n)$, will be assumed to be continuous and linear such that $T(D) = R^m$, where D is the space of all solutions of (A.0), (B.0).

(iii) $H(u)$ is a continuous (not necessarily linear) operator H: $C_b G \to R^m$ such that

$$\|Hu\| \leqslant h_1\|u\| + h_2\,, \qquad h_1, h_2 \in R\,. \tag{2.5}$$

If we define the linear operator L: $\operatorname{dom} L = C_b \cap C^1[[a, b[, R^n] \cap \operatorname{dom}\mathscr{T}_r \cap \operatorname{dom} T$, L: $\operatorname{dom} L \to C\times R^n \times R^m$, $x \to (\dot{x} - A(t)x, \mathscr{T}_r(x), Tx)$, and the continuous operator N: $\operatorname{dom} N \subset C_b G \to C\times R^n\times R^m$, $x \to (F(t, x(t)), C_r, Hx)$, then the problem (A), (B), (T) may be written as

$$Lx = Nx\,. \tag{2.6}$$

The equation (2.6) is equivalent to [**8**]

$$(2.7)\quad x = Mx = Px + K_P Nx\,,\ x\in \operatorname{dom} M = \{x\in C_b G\colon Nx\in \operatorname{Im} L\} = N^{-1}(\operatorname{Im} L)\,,$$

where P is a projector operator: $C_b G \to \operatorname{Ker} L$, $K_P = [L/\operatorname{dom} L \cap \operatorname{Im}(I-P)]^{-1}$. The following theorem will be crucial.

Theorem A (Zezza [**8**]). *Let Ω be an open region containing $0\in X$, real Banach space. Let $\bar{\Omega}\subset \operatorname{dom} M$ and*

$$Lx \neq \lambda Nx \qquad \lambda\in\,]0,1[\,,\ x\in\partial\Omega\,.$$

Then the operator M has at least one fixed point in $\bar{\Omega}$.

We construct the operators P and K_P in a similar manner as in [**2**]. Let $\varphi_1, \dots, \varphi_k$ be a basis of $\operatorname{Ker} L$ and $\varphi_1, \dots, \varphi_k, \varphi_{k+1}, \dots, \varphi_n$ a basis of D, $\varphi_i\in C_b G$, $i = 1, \dots, n$. The operator $P\colon C_b G \to \operatorname{Ker} L$ may be defined by $P = P_2 \circ P_1$, where

$$P_1\colon\ C_b G \to D\,, \qquad P_1 y = X(t)X^{-1}(a)y(a)\,,$$

$$P_2\colon\ D \to \operatorname{Ker} L\,, \qquad P_2\colon\ y(t) = \sum_{i=1}^{n} \lambda_i\varphi_i \to \sum_{i=1}^{k}\lambda_i\varphi_i \qquad \lambda_i\in R\,.$$

Then from the definition of P since $X(t) = U(t)\Lambda(t)$ we have

Proposition 2.1. The operator P is a topological projector and for fixed $(Q(t), C_r, \chi)\in \operatorname{Im} L$ there exists only one solution $z\in\operatorname{dom} L$ of the system

$$\dot{z} - A(t)z(t) = Q(t)\,, \qquad z(t_r^+) - B_r z(t_r^-) = C_r\,, \qquad Tz = \chi \qquad \chi\in R^m\,,$$

such that $Pz = 0$.

From (2.4) and Proposition 2.1 it is easy to see that

$$z(t) = K_P Nf(t) = X(t)JT_0^{-1}[\chi - TX(\cdot)\sum_{i=1}^{r} X^{-1}(t_i^+)C_i - TR(\cdot, f)] + X(t)\sum_{i=1}^{r} X^{-1}(t_i^+)C_i + R(t, f)\,,$$

where

$$R(t, f) = \int_a^t X(t) X^{-1}(\tau) F(\tau, f(\tau))\, d\tau ,$$

$$J:\ R^m \to R^n , \qquad (\gamma_1, \dots, \gamma_m) \to (0, \dots, 0, \beta_{k+1}, \dots, \beta_n) ,$$

$$\beta_{k+j} = \gamma_j \ (j = 1, 2, \dots, m) \text{ and } T_0 = (T\varphi_{n-m+1}, \dots, T\varphi_n).$$

3 - Main results

In addition to the assumptions (i), (ii), (iii) we also assume that the following ones are satisfied

(iv) $F \in C[\Delta \times R^n, R^n]$ and such that $\|X^{-1}(t) F(t, u)\| \leqslant p(t) \ \|u\| + q(t)$, where $p, q \in C[\Delta, R_+]$ and $W = \int_a^b p(t)\, dt < \infty$, $V = \int_a^b q(t)\, dt < \infty$.

(v) $\|\sum_{i=1}^r X^{-1}(t_i^+)\, C_i\| < K$ (positive constant).

(vi) $\varrho \|J T_0^{-1}\| (h_1 + \varrho \|T\| W) \exp[\varrho W] < 1$.

Theorem 3.1. *Let the assumptions* (i), (ii), (iii), (iv), (v) *and* (vi) *be satisfied. Then there exists at least one solution to the problem* (A), (B), (T) *in* $C_b G$.

Proof. Let $\{d_s\}$ $(s = 1, 2, \dots, a < d_s < d_{s+1})$ be a sequence of points such that $\lim_{s\to\infty} d_s = b = +\infty$, $d_s \neq t_i$ $(s, i = 1, 2, \dots)$. Let $\Delta_s = [a, d_s]$, δ_s be the interval Δ_s without the interface points included within Δ_s, and let the subnorm of $C[\Delta_s, R^n]$, $CG[\delta_s, R^n]$ be $\|\cdot\|_s$ and $\|\cdot\|_{g_s}$ respectively. Assume that $f \in CG[\delta_s, R^n]$ and consider the function

$$\bar f(t) = \begin{cases} f(t) & t \in \Delta_s , \\ f(d_s) & t \in [d_s, b[. \end{cases} \tag{3.1}$$

Then the set of all such functions $\bar f$ is a Banach space $\mathscr{D}_s$ with norm

$$\|\bar f\|_{\mathscr{D}_s} = \|f\|_{g_s} = \sup_{t\in\delta_s} \|f(t)\| . \tag{3.2}$$

Now consider the operator $M_s\colon \mathscr{D}_s \to \mathscr{D}_s$ with $M_s\bar{f} = \bar{x}$, where

$$x(t) = (M\bar{f})(t) = (P\bar{f})(t) + K_P N\bar{f})(t)\,. \tag{3.3}$$

Let

$$z(t) = (K_P N\bar{f})(t) = X(t) J T_0^{-1}[H\bar{f} - TX(\cdot)\sum_{i=1}^{r} X^{-1}(t_i^+)\, C_i - TR(\cdot, \bar{f})] + X(t)\sum_{i=1}^{r} X^{-1}(t_i^+)\, C_i + R(t, \bar{f}) \qquad t \in \delta_s\,. \tag{3.4}$$

Let $\{\bar{f}_n\}$, $\bar{f}$ be in $\mathscr{D}_s$, $\lim_{n\to\infty} \|\bar{f}_n - \bar{f}\|_{\mathscr{D}_s} = \lim_{n\to\infty} \|f_n - f\|_{\sigma_s} = 0$. Then for $\bar{z}_n = K_P N\bar{f}_n$, $\bar{z} = K_P N\bar{f}$ it follows from (3.4), (i), (iii), (iv) and Lebesgue's theorem that

$$\|\bar{z}_n - \bar{z}\|_{\mathscr{D}_s} = \sup_{t\in\delta_s} \|z_n(t) - z(t)\| \to 0\,, \tag{3.5}$$

since P and M_s are continuous.

Now let θ be a bounded set of $\mathscr{D}_s$ with bound μ. The uniform boundedness of the functions $\bar{z} = K_P N\bar{f}$, $\bar{f} \in \theta$ follows from

$$\|\bar{z}\|_{\mathscr{D}_s} = \|z\|_{\sigma_s} \leqslant \varrho\|JT_0^{-1}\|[h_1\mu + h_2 + \|T\|\varrho K + \varrho\|T\|(W\mu + V)] + \varrho K + \varrho(W\mu + V)\,. \tag{3.6}$$

Consequently, since P is a linear operator with a finite dimensional range, $M_s(\theta)$ is uniformly bounded. Moreover the sequence $\{K_P N\bar{f}(t)\}$, which is defined as

$$(K_P N f)(t) = (K_P N f)(t) \qquad t \in]t_i, t_{i+1}[\,,$$

$$(K_P N f)(t_i) = (K_P N f)(t_i^+)\,, \qquad (K_p N f)(t_{i+1}) = (K_p N f)(t_{i+1}^-)\,,$$

is equicontinuous on every $[t_i, t_{i+1}] \subset \Delta_s$. In fact for $t', t'' \in [t_i, t_{i+1}]$ we have

$$\|\bar{z}(t'') - \bar{z}(t')\|_{\mathscr{D}_s} \leqslant \beta\|X(t'') - X(t')\| + K\|X(t'') - X(t')\| + \|X(t'') - X(t')\|(W\mu + V) + \varrho\Big(\mu\int_{t'}^{t''} p(t)\,dt + \int_{t'}^{t''} q(t)\,dt\Big)\,,$$

where $\beta = (h_1\mu + h_2 + \varrho K\|T\| + \varrho\|T\|(W\mu + V))$.

Since P is a linear operator, and applying the criteria for compactness in C_bG, analogous to Ascoli's theorem [7], it follows that $M_s\theta$ is equicontinuous. Consequently M_s is completely continuous in $\mathscr{D}_s$.

In order to apply the fixed point Theorem A, it is enough to prove that $\bar{z}_0 \neq \lambda K_P N\bar{z}_0$, $\bar{z}_0 \in \partial\theta$, $\lambda \in]0, 1[$. If there exists $\bar{z} \in \partial\theta$: $\bar{z} = \lambda K_P N\bar{z}$ then for $\lambda \in]0, 1[$ we have

$$\|z(t)\| \leqslant \varrho\|JT_0^{-1}\|[h_1\|\bar{z}\|_{\mathscr{D}_s} + h_2 + \|T\|\varrho K + \|T\|\varrho(W\|\bar{z}\|_{\mathscr{D}_s} + V)] + \varrho(K+V) + \varrho\int_a^t p(\tau)\|z(\tau)\|\,d\tau \qquad \text{for every } t \in \Delta .$$

Applying Gronwall's inequality we obtain

$$\|z(t)\| \leqslant [\varrho\|JT_0^{-1}\|\{h_1\|\bar{z}\|_{\mathscr{D}_s} + h_2 + \|T\|\varrho K + \|T\|\varrho(W\|\bar{z}\|_{\mathscr{D}_s} + V)\} + \varrho(K+V)] \exp[\varrho W] .$$

Thus

$$\|\bar{z}\|_{\mathscr{D}_s} = \|\lambda K_p N\bar{z}\|_{\mathscr{D}_s} < \|K_p N\bar{z}\|_{\mathscr{D}_s} = \|K_p N z\|_{g_s} = \|z\|_{g_s} \leqslant [1 - \varrho\|JT_0^{-1}\|(h_1 + \varrho\|T\|W)\exp[\varrho W]]^{-1} \cdot [\varrho\|JT_0^{-}\|(h_2 + \|T\|\varrho K + \|T\|\varrho V) + \varrho(K+V)]\exp[\varrho W] .$$

Consequently, according to Theorem A, there exists at least one solution $\bar{x} = M_s\bar{x}$ in $\mathscr{D}_s$.

Then there exists a sequence $\{x_s\}$ $(s = 1, 2, \dots)$ of solutions of (A), (B), (T) such that $\bar{x}_s \in \mathscr{D}_s$. Now it is not difficult to see that ([5], p. 1027), for fixed $c > 0$ and $y(t) = Mx(t)$,

$$\lim_{s\to\infty} \|x_{k_s}(t) - y(t)\| = 0 \qquad t \in [a, c] ,$$

where x_{k_s} is a subsequence of $\{x_s(t)\}$ and converges uniformly to $x(t)$ on every finite interval of Δ. Since c is arbitrary, $y(t) = My(t)$, $t \in \Delta$ which completes the proof of the theorem.

In the result which follows we extend Theorem 3.1 to the case in which T and H are defined on C_1G. Let us now suppose that the following assumptions are satisfied.

(ii)′ $T: C_1G \to R^m$ is a continuous and linear operator such that $T(D) = R^m$.

(iii)′ $H: C_1G \to R^m$: $\|Hu\|_g \leqslant h_1\|u\|_g + h_2$.

(vii) $\lim\limits_{t\to b} X(t) = \psi_1$ exists and is finite.

(viii) $\sum\limits_{i=1}^{\infty} X^{-1}(t_i^+)\,C_i$ converges and $\|\sum\limits_{i=1}^{\infty} X^{-1}(t_i^+)\,C_i\| < K_1$ (positive constant).

Theorem 3.2. *Assume that the hypotheses* (i), (ii)′, (iii)′, (iv), (v), (vi), (vii) *and* (viii) *hold. Then the problem* (A), (B), (T) *admits at least one solution in* C_1G.

Proof. Consider the operator $M: C_1G \to C_1G$ with $(Mf)(t) = (Pf)(t) + (K_PNf)(t)$.

It is known that a set $\Gamma \subset C_1$ is relatively compact if and only if it is uniformly bounded, equicontinuous, and uniformly convergent in the following sense: for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\|\lim_{t\to\infty} f(t) - f(t)\| < \varepsilon \qquad \text{for every} \quad t > \delta(\varepsilon)\,,\ f \in \Gamma$$

(cfr. Avramescu [**1**]).

Let Φ be a bounded set of C_1G with bound μ. $M\Phi$ is uniformly bounded and equicontinuous. We have

$$\|\lim_{t\to b}(K_PNf)(t) - (K_PNf)(t)\|$$

$$\leqslant \|\psi_1 - X(t)\|\,\|JT_0^{-1}[Hf - TX(\cdot)\sum_{i=1}^{r} X^{-1}(t_i^+)\,C_i - TR(\cdot, f)]\|$$

$$+ \|\lim_{t\to b} X(t)\sum_{i=1}^{r} X^{-1}(t_i^+)\,C_i - X(t)\sum_{i=1}^{r} X^{-1}(t_i^+)\,C_i\| + \|\lim_{t\to b} R(t, f) - R(t, f)\|$$

$$\leqslant \|\psi_1 - X(t)\|[\|JT_0^{-1}\|(\|Hf\| + \|T\|\varrho K + \|T\|\varrho(W\mu + V))]$$

$$+ \|\psi_1 - X(t)\|\,\|\sum_{i=1}^{\infty} X^{-1}(t_i^+)\,C_i\| + \varrho\|\sum_{r}^{\infty} X^{-1}(t_i^+)\,C_i\| + \|\psi_1 - X(t)\|(W\mu + V)$$

$$+ \varrho(\mu\int_t^b p(\tau)\,d\tau + \int_t^b q(\tau)\,d\tau)$$

$$< \|\psi_1 - X(t)\|\{\|JT_0^{-1}\|(\|Hf\| + \|T\|\varrho K + \|T\|\varrho(W\mu + V)) + K_1 + (W\mu + V)\}$$

$$+ \varrho\|\sum_{r}^{\infty} X^{-1}(t_i^+)\,C_i\| + \varrho(\mu\int_t^b p(\tau)\,d\tau + \int_t^b q(\tau)\,d\tau)\,.$$

It follows that given $\varepsilon > 0$ there exists $t_0(\varepsilon) > 0$ such that $\| \lim_{t \to b} (K_P Nf)(t) - (K_P Nf)(t) \| < \varepsilon$ for every $t > t_0(\varepsilon)$ and every $f \in \Phi$. Consequently $\{K_P Nf\}$ is a uniformly convergent family. Then $\{Mf\}$ is relatively compact in $C_1 G$ [1]. The rest of the proof follows as in Theorem 3.1.

Remark. We remark that the extension to an infinite interval with countable set of interface points, of the interface problems considered in [6] and [7] are special cases of Theorem 3.1. In this case $m = n$, $P = 0$ and equation (2.7) yields

$$x = K_0 Nx = X(t)\big(TX(t)\big)^{-1}\Big[Hx - TX(\cdot)\sum_{i=1}^{r} X^{-1}(t_i^+)\, C_i - TR(\cdot, x)\Big] + X(t)\sum_{i=1}^{r} X^{-1}(t_i^+)\, C_i + R(t, x)\,.$$

Acknowledgement. I would like to thank Prof. R. Conti for reading the manuscirpt and for his kind help.

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