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Integral inequalities of Bihari type for functions of two variables (**)

1 - Introduction

During the last few years the theory of integral inequalities of the type of Gronwall-Bellman and Bihari marks a fast advance. The reason is that these inequalities started to be actively used as a proof apparatus in the qualitative theory of differential equations, in the optimal control theory, for justification of asymptotic methods for various classes of equations and so on.

The present paper is the first to achieve a generalization of the Bihari integral inequality for a scalar function of two arguments in the case when the integration domain is a compact which cannot possibly be represented as a Cartesian product of intervals.

2 - Notations and definitions

Let the point (x_0, y_0) be arbitrary and let in the plane R^2 two continuous rectifiable curves Γ_1 and Γ_2 be given with initial point (x_0, y_0) and parametric equations

$$\Gamma_1\colon\ x = \varphi_1(s)\,,\quad y = \psi_1(s)\,,\qquad \Gamma_2\colon\ x = \varphi_2(s)\,,\quad y = \psi_2(s)\,,$$

where s is the natural parameter.

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It will be assumed that Γ_1 and Γ_2 satisfy the conditions (A) or the conditions (B):

(A)$_1$ For $s_1 \neq s_2$ the relation $(\varphi_i(s_1), \psi_i(s_1)) \neq (\varphi_i(s_2), \psi_i(s_2))$, $i = 1, 2$ is fulfilled and Γ_1 and Γ_2 have no other common point that the point (x_0, y_0).

(A)$_2$ The curves Γ_1 and Γ_2 lie in the semiplane $x > x_0$.

(B)$_1$ For $s_1 \neq s_2$ the relation $(\varphi_i(s_1), \psi_i(s_1)) \neq (\varphi_i(s_2), \psi_i(s_2))$, $i = 1, 2$, holds and Γ_1 and Γ_2 have no other common point that the point (x_0, y_0).

(B)$_2$ The curves Γ_1 and Γ_2 lie in the semiplane $x < x_0$:

Let Γ_1 and Γ_2 satisfy the conditions (A). By $\Gamma_1^+(\Gamma_2^+)$ denote the set of these points in the semiplane $x > x_0$, which remain in the left under the motion along the curve $\Gamma_1(\Gamma_2)$ in the direction of increase of the natural parameter s. The component of the set $\Gamma_1^+ \bigcup \Gamma_1(\Gamma_2^+ \bigcup \Gamma_2)$ to the semiplane $x > x_0$ will be denoted by $\Gamma_1^-(\Gamma_2^-)$.

Analogously, if the curves Γ_1 and Γ_2 satisfy the condition (B), denote by $\Gamma_1^\oplus(\Gamma_2^\oplus)$ the set of these points in the semiplane $x < x_0$ which remain in the left under the motion along the curve $\Gamma_1(\Gamma_2)$ in the direction of increase of the natural parameter s. The component of the set $\Gamma_1^\oplus \bigcup \Gamma_1(\Gamma_2^\oplus \bigcup \Gamma_2)$ to the semiplane $x < x_0$ will be denoted by $\Gamma_1^\ominus(\Gamma_2^\ominus)$.

Let (x_0, y_0) be an arbitrary point which is an initial point for the curves Γ_1 and Γ_2, and let the curves Γ_1 and Γ_2 satisfy the conditions (A).

Def. 1. A right emission zone of a point $(x_0, y_0) \in R^2$ will be called the *set* $S^+(x_0, y_0) = (\Gamma_1^+ \cap \Gamma_2^-) \bigcup (\Gamma_1^- \cap \Gamma_2^+)$. The curves Γ_1 and Γ_2 will be called *boundary lines* of the right emission zone $S^+(x_0, y_0)$.

It is easily seen that if $\Gamma_2 \subset \Gamma_1^-$, then $S^+(x_0, y_0) = \Gamma_1^- \cap \Gamma_2^+$ and if $\Gamma_2 \subset \Gamma_1^+$, then $S^+(x_0, y_0) = \Gamma_1^+ \cap \Gamma_2^-$.

Let (x_0, y_0) be an arbitrary point which is an initial point for the curves Γ_1 and Γ_2, and let the curves Γ_1 and Γ_2 satisfy the conditions (B).

Def. 2. A left emission zone of the point (x_0, y_0) will be called the *set* $S^-(x_0, y_0) = (\Gamma_1^\oplus \cap \Gamma_2^\ominus) \bigcup (\Gamma_1^\ominus \cap \Gamma_2^\oplus)$. The curves Γ_1 and Γ_2 will be called *boundary lines* of the left emission zone $S^-(x_0, y_0)$.

For any point $(x, y) \in S^+(x_0, y_0)$ we will consider the corresponding left emission zone $S^-(x, y)$ with boundary lines

$$\Gamma_3\colon\ x = \varphi_3(s)\,,\quad y = \psi_3(s)\,, \qquad \Gamma_4\colon\ x = \varphi_4(s)\,,\quad y = \psi_4(s)\,,$$

satisfying the conditions (B).

Without loss of generality we shall consider that

$$S^+(x, y) = \Gamma_1^- \cap \Gamma_2^+ \quad \text{and} \quad S^-(x, y) = \Gamma_3^{\ominus} \cap \Gamma_4^{\oplus} .$$

Let g_1 and g_2 be two curves given by parametric equations by means of their natural parameter.

Def. 3. Minimal common point of the curves g_1 and g_2 will be called that *common point* (if such exists) which is obtained from the equations of the curve g_2 for the smallest value of the parameter.

It will be assumed that for any point $(x, y) \in S^+(x_0, y_0)$ there exists a left emission zone $S^-(x, y)$ with boundary lines Γ_3 and Γ_4 which satisfy the following conditions (C):

(C)$_1$ The curve $\Gamma_3(\Gamma_4)$ has at least one common point with $\Gamma_1(\Gamma_2)$.

(C)$_2$ The inequality $s_1 < s_2$ is fulfilled, where s_1 and s_2 are the values of the parameter s for which in the equations of Γ_3 the minimal common point of Γ_3 with the lines Γ_1 and Γ_2 is obtained. (If Γ_3 and Γ_2 have no common point we set $s_2 = +\infty$).

(C)$_3$ The inequality $s' < s''$ is fulfilled, where s' and s'' are the values of the parameter s for which in the equations of Γ_4 the minimal common point of Γ_4 with the lines Γ_2 and Γ_1 is obtained. (If Γ_4 and Γ_1 have no common point we set $s'' = +\infty$).

Denote the minimal common point of Γ_1 and Γ_3 by (x', y'), while the minimal common point of Γ_2 and Γ_4 will be denoted by (x'', y'').

Consider the closed set $\mathscr{K}xy$ bounded by the arcs from the curves $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4, lying between the points (x_0, y_0), (x', y'), (x, y) and (x'', y'').

Def. 4. The compact set $\mathscr{K}xy$ will be called *zone* of the point $(x, y) \in S^+(x_0, y_0)$.

(C)$_4$ For any point $(\xi, \eta) \in \mathscr{K}xy$ the inclusion $\mathscr{K}\xi\eta \subset \mathscr{K}xy$ is fulfilled, where $\mathscr{K}\xi\eta$ is the zone of the point (ξ, η) and $(x, y) \in S^+(x_0, y_0)$.

For an arbitrary point $(x, y) \in R^2$ denote by Bxy the closed rectangle with vertices (x_0, y_0), (x_0, y), (x, y), (x, y_0).

For any point $(\xi, \eta) \in Bxy$ by $B\xi y$ denote the rectangle with vertices (x_0, y_0), (x_0, y), (ξ, y), (ξ, y_0). Note that in order to define the rectangle $B\xi y$ only the first coordinate ξ is essential since for the points (ξ, η_1) and (ξ, η_2), where $\eta_1 \neq \eta_2$, the respective rectangles $B\xi y$ and $B\xi y$ coincide.

3 - Basic result

Theorem 1. *Let the following conditions hold.*

1 – *The functions* $f(x, y)$ *and* $u(x, y)\colon R^2 \to R^1$ *are non-negative and locally integrable for* $x > x_0$, $y > y_0$.

2 – *The function* $w(t)\colon R^1 \to R^1$ *is non-decreasing, positive and locally integrable for* $t \geqq 0$.

3 – *For any point* $(\xi, \eta) \in Bxy/\{x = x_0\}$ *the inequality*

$$u(\xi, \eta) \leqq c + \int_{B\xi\eta} f(s, \tau)\, w(u(s, \tau))\, ds\, d\tau \tag{1}$$

holds, where $c > 0$ *is a constant and the point* $(x, y) \in R^2$, $x > x_0$, $y > y_0$ *is arbytrary.*

Then for the point $(x, y) \in \Omega$ *the inequality*

$$u(x, y) \leqq G^{-1}\left[G(c) + \int_{Bxy} f(s, \tau)\, ds\, d\tau\right] \tag{2}$$

is fulfilled, where

$$G(t) = \int_{t_0}^{t} \frac{d\varrho}{w(\varrho)}\,, \qquad t \geqq t_0 > 0\,, \tag{3}$$

the function G^{-1} *is reciprocal to* G *and*

$$\Omega = \left\{(x, y)\colon x > x_0,\ y > y_0,\ G(c) + \int_{Bxy} f(s, \tau)\, ds\, d\tau \in \operatorname{Dom} G^{-1}\right\}.$$

Proof. For the proof of Theorem 1 we will apply the method proposed in [**1**].

Let $(x, y) \in \Omega$ be an arbitrary point. Define in the domain $(\xi, \eta) \in Bxy$ the function

$$v(\xi, \eta) = c + \int_{B\xi\eta} f(s, \tau)\, w(u(s, \tau))\, ds\, d\tau\,.$$

The inequality (1) can be written in the form

$$u(\xi, \eta) \leqq v(\xi, \eta)\,. \tag{4}$$

Differentiate the function $v(\xi, \eta)$ with respect to ξ and apply the inequality (4)

$$\frac{\partial v(\xi, y)}{\partial \xi} = \int_{y_0}^{y} f(\xi, \tau) w(u(\xi, \tau)) \, d\tau \tag{5}$$

$$\leqq w(v(\xi, y)) \int_{y_0}^{y} f(\xi, \tau) \, d\tau .$$

The inequality (5) and the definition (3) of the function G imply that

$$\frac{\partial G(v(\xi, y))}{\partial \xi} \leqq \int_{y_0}^{y} f(\xi, \tau) \, d\tau .$$

Integrate the above inequality from x_0 to x and we get

$$G(v(x, y)) - G(v(x_0, y)) \leqq \int_{Bxy} f(s, \tau) \, ds \, d\tau$$

or

$$v(x, y) \leqq G^{-1}\left[G(c) + \int_{Bxy} f(s, \tau) \, ds \, d\tau\right]. \tag{6}$$

The inequalities (4) and (6) imply the validity of the inequality (2) for the point $(x, y) \in \Omega$.

Thus, Theorem 1 is proved.

Theorem 2. *Let the following conditions hold.*

1 - *The functions* $f(x, y)$ *and* $u(x, y): R^2 \to R^1$ *are continuous and non-negative in the domain* $S^+(x_0, y_0)$ *with boundary lines* Γ_1 *and* Γ_2 *satisfying the conditions* (A).

2 - *The function* $w(t): R^1 \to R^1$ *is continuous, positive and non-decreasing for* $t \geqq 0$.

3 - *For any point* $(x, y) \in S^+(x_0, y_0)$ *there exists at least one left emission zone* $S^-(x, y)$ *with boundary lines* Γ_3 *and* Γ_4 *which satisfy the conditions* (B) *and* (C) *and the inequality*

$$u(x, y) \leqq c + \int_{\mathscr{K}xy} f(s, \tau) w(u(s, \tau)) \, ds \, d\tau , \tag{7}$$

where $\mathscr{K}xy$ *is the zone of the point* (x, y).

Then for any point $(x, y) \in D$ *the inequality*

$$u(x, y) \leqq G^{-1}\big[G(c) + \int_{\mathscr{K}xy} f(s, \tau)\, ds\, d\tau\big] \tag{8}$$

is fulfilled, where the function $G(t)$ *is defined by the equality* (3), G^{-1} *is a function reciprocal to* G *and*

$$D = \{(x, y) \in S^{+}(x_0, y_0) \colon G(c) + \int_{\mathscr{K}xy} f(s, \tau)\, ds\, d\tau \in \operatorname{Dom} G^{-1}\} .$$

Proof. Let $(x, y) \in D \subset S^{+}(x_0, y_0)$. Since the zone $\mathscr{K}xy$ is a compact set, then there exists a rectangle $Bx\tau$ with sides parallel to the coordinate axes, for which the inclusion $\mathscr{K}xy \subset Bx\tau$ holds. The conditions (A) and (B) imply that the rectangle with vertices (x_0, τ_0), (x_0, τ), (x, τ), (x, τ_0) (where the points τ_0 and τ are chosen so that the inclusion $\mathscr{K}xy \subset Bx\tau$ should hold) might be chosen so that two of its opposite sides should pass through the points (x_0, y_0) and (x, y), respectively.

Let $(\xi, \eta) \in \mathscr{K}xy$ be an arbitrary point. The conditions (B) and (C) imply that the inclusion $\mathscr{K}\xi\eta \subset B\xi\tau$ is fulfilled, where $\mathscr{K}\xi\eta$ is the zone of the point (ξ, η), while $B\xi\tau$ is a rectangle with vertices (x_0, τ_0), (x_0, τ), (ξ, τ), (ξ, τ_0).

Define the functions

$$\bar{f}(\xi, \eta) = \begin{cases} f(\xi, \eta) & \text{for } (\xi, \eta) \in \mathscr{K}xy \\ 0 & \text{for } (\xi, \eta) \in Bx\tau \setminus \mathscr{K}xy , \end{cases}$$

$$\bar{u}(\xi, \eta) = \begin{cases} u(\xi, \eta) & \text{for } (\xi, \eta) \in \mathscr{K}xy \\ 0 & \text{for } (\xi, \eta) \in Bx\tau \setminus \{\mathscr{K}xy \cup \{x, \tau\} \cup \{x = x_0\}\} \\ u\{x, y\} & \text{for } (\xi, \eta) = (x, \tau) . \end{cases}$$

The inequality (7) and the definitions of the functions $\bar{u}(\xi, \eta)$ and $\bar{f}(\xi, \eta)$ imply that for $(\xi, \eta) \in \mathscr{K}xy \bigcup \{x, \tau\}$ the inequality

$$d(\xi, \eta) \leqq c + \int_{B\xi\tau} \bar{f}(s, \tau)\, w(\bar{u}(s, \tau))\, ds\, d\tau \tag{9}$$

is fulfilled.

Since $\bar{u}(\xi, \eta) \equiv 0$ and $\bar{f}(\xi, \eta) \equiv 0$ for $(\xi, \eta) \in Bx\tau \setminus \{\mathscr{K}xy \bigcup \{(x, \tau)\} \bigcup \{(x = x_0)\}\}$, then the inequality (9) holds for any point $\{(\xi, \eta)\} \in Bx\tau \setminus \{x = y_0\}$. By virtue of Theorem 1, for $(x, \tau) \in \bar{D}$ the following inequality will hold

$$\bar{u}(x, \tau) \leqq G^{-1}\{G(c) + \int_{Bx\tau} \bar{f}(s, \tau)\, ds\, d\tau\} , \tag{10}$$

where

$$\bar{D} = \{(x, \tau): x > x_0,\ \tau > \tau_0,\ G(c) + \int_{Bx\tau} \bar{f}(s, \tau)\, ds\, d\tau \in \operatorname{Dom} G^{-1}\} .$$

But the definition of the function $\bar{f}(\xi, \eta)$ implies that

$$\int_{Bx\tau} \bar{f}(s, \tau)\, ds\, d\tau = \int_{\mathcal{K}xy \cup \{x,\tau\}} f(s, \tau)\, ds\, d\tau = \int_{\mathcal{K}xy} f(s, \tau)\, ds\, d\tau .$$

Therefore, if the point $(x, y) \in D$ then the point $(x, \tau) \in \bar{D}$ and vice versa. In view of the inequality (10) and the definition of the functions $\bar{u}$ and $\bar{f}$ we get that for $(x, y) \in D$ the inequality (8) holds.

Thus, Theorem 2 is proved.

Theorem 3. *Let the following conditions hold.*

1 – *The functions $u(x, y)$, $f(x, y): R^2 \to R^1$ are continuous and non-negative in the domain $S^+(x_0, y_0)$ with boundary lines Γ_1 and Γ_2 which satisfy the conditions* (A).

2 – *The function $g(x, y): R^2 \to R^1$ is continuous and $g(x, y) \geqq 1$ for $(x, y) \in S^+(x_0, y_0)$.*

3 – *The function $\varphi(t): R^1 \to R^1$ is continuous, positive and non-decreasing for $t \geqq 0$, while for $t \geqq 0$ and $r > 0$ the inequality $(1/r)\, \varphi(t) \leqq w(t/r)$ is fulfilled, where the function $w(t): R^1 \to R^1$ is continuous, positive and non-decreasing for $t \geqq 0$.*

4 – *The function $q(x, y): R^2 \to R^1$ is continuous, positive and non-decreasing for $(x, y) \in S^+(x_0, y_0)$.*

5 – *For any point $(x, y) \in S^+(x_0, y_0)$ there exists at least one left emission zone $S^-(x_0, y_0)$ with boundary lines Γ_3 and Γ_4 which satisfy the conditions* (B) *and* (C) *and such that*

$$u(x, y) \leqq q(x, y) + g(x, y) \int_{\mathcal{K}xy} f(s, \tau)\, \Phi(u(s, \tau))\, ds\, d\tau , \tag{11}$$

where $\mathcal{K}xy$ is the zone of the point (x, y).

Then for $(x, y) \in D_1$ the inequality

$$u(x, y) \leqq q(x, y)\, g(x, y)\, G^{-1}\big[G(1) + \int_{\mathcal{K}xy} f(s, \tau)\, g(s, \tau)\, ds\, d\tau\big] , \tag{12}$$

holds, where the function G is defined by the equality (3), *G^{-1} is the reciprocal* *function of G and*

$$D_1 = \{(x, y) \in S^+(x_0, y_0) : G(1) + \int_{\mathcal{K}_{xy}} f(s, \tau) g(s, \tau) \, ds \, d\tau \in \operatorname{Dom} G^{-1}\} .$$

Proof. From the conditions of Theorem 3 it follows that

$$\frac{u(x, y)}{q(x, y)} \leqq 1 + g(x, y) \int_{\mathcal{K}_{xy}} \frac{f(s,\tau) \circ / \circ (u(s,\tau))}{q(x, y)} \, ds \, d\tau .$$

Therefore,

$$\frac{u(x, y)}{q(x, y) g(x, y)} \leqq \frac{1}{g(x, y)} + \int_{\mathcal{K}_{xy}} \frac{f(s,\tau) \circ / \circ (u(s,\tau))}{q(x, y)} \, ds \, d\tau$$

$$\leqq 1 + \int_{\mathcal{K}_{xy}} f(s, \tau) g(s, \tau) w\Big(\frac{u(s, \tau)}{q(s, \tau) g(s, \tau)} \Big) \, ds \, d\tau . \tag{13}$$

We apply Theorem 2 for the function $u(x, y)/q(x, y)g(x, y)$ and we obtain the inequality (12).

Thus, Theorem 3 is proved.

References

[1] S. G. HRISTOVA and D. D. BAINOV, *On a generalization of Bihari inequalities*, Izv. Acad. Sci. Kazah. SSR **5** (1978), 88-89 (in Russian).

Summary

The paper considers two nonlinear generalizations of integral inequalities of Gronwall-Bellman type for scalar functions of two arguments.

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