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On regular rings and Artinian rings (**)

Introduction

This paper is essentially concerned with von Neumann regular rings and Artinian rings. Rings whose right ideals are quasi-injective (called right q-rings) are studied in [5], where they are characterised as right self-injective rings whose essential right ideals are ideals. (As usual, an ideal means a two-sided ideal). Our first theorem contains characteristic properties of regular left q-rings (([5], theorem 2.9) is here improved). Using recent results of Menal [8] on π-regular rings, we consider a class of rings such that for any finitely generated left module M, End (M) is a π-regular ring whose primitive factor rings are Artinian. A few nice characterisations of left continuous regular rings and semi-simple Artinian rings are given. For example, A is semi-simple Artinian iff the sum of any two cyclic projective left A-modules is injective.

Throughout, A represents an associative ring with identity and A-modules are unitary. J, Z will denote respectively the Jacobson radical and the left singular ideal of A. Q will always stand for the maximal left quotient ring of A whenever $Z = 0$ (in that case, Q is a left self-injective regular ring and ${}_AQ$ is the injective hull of ${}_AA$). Following [9], a left A-module M is called semi-simple if the intersection of all the maximal left submodules of M is zero. Thus, A is semisimple iff $J = 0$. Also, a right (left) ideal of A is called reduced if it contains no non-zero nilpotent element.

Recall that: (1) A is unit-regular if, for any $a \in A$, there exists a unit (invertible element) $u \in A$ such that $a = aua$; (2) A is a π-regular ring if,

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for any $a \in A$, there exists a positive integer n such that $a^n \in a^n A a^n$; (3) A is of bounded index if the supremum of the indices of the nilpotent elements of A is finite; (4) A has stable range 1 if whenever $Aa + Ab = A$, $a, b \in A$, there exists $c \in A$ such that $a + cb$ is a unit; (5) A is a left V-ring if every simple left A-module is injective; (6) A left A-module M is p-injective (f-injective) if, for any principal (finitely generated) left ideal I of A and any left A-homomorphism $g: I \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in I$. Following $[\mathbf{12}]_6$, call A a left WP-ring (weak p-injective) if every left ideal not isomorphic to ${}_AA$ is p-injective. A is called a left wq-ring if every left ideal not isomorphic to ${}_AA$ is quasi-injective (wq-rings are studied in $[\mathbf{10}]_{1,2}$). Write A is ELT (MELT) if every essential (maximal essential) left ideal is an ideal of A. ERT (MERT) rings are similarly defined on the right. Since the study of q-rings was initiated in [**5**], ELT rings have been considered by various authors.

It is well-known that there is no inclusion relation between the classes of arbitrary von Neumann regular rings and left V-rings. Since A is regular iff every left A-module is flat iff every left A-module is p-injective while a commutative ring is regular iff it is a V-ring, the notions of V-rings, flatness and p-injectivity are therefore closely connected to von Neumann regularity. Unit-regular rings and regular rings of bounded index have many interesting properties (cfr. [**4**]) and are therefore remarkable generalisations of strongly regular rings.

Let us first consider certain relations between the following generalisations of regular rings: (1) fully left idempotent rings; (2) rings whose simple left modules are flat; (3) left p-V-rings (rings whose simple left modules are p-injective); (4) left p-injective rings.

Lemma 1. *Let M be a maximal right ideal of A which is an ideal. The following are then equivalent.*

(1) *${}_AA/M$ is injective,*
(2) *${}_AA/M$ is p-injective,*
(3) *A/M_A is flat.*

Proof. The equivalence of (1) and (2) may be established by going through the proof of ($[\mathbf{12}]_4$, proposition 2.1(1)).

(2) implies (3). Since A/M is a division ring, then M is a maximal left ideal of A and A/M is a simple p-injective left A-module. For any $0 \neq u \in M$, if $Au = Mu$, then $u \in Mu$. If $Au \neq Mu$, then $Au/Mu \approx A/M$ is a simple left p-injective A-module and the canonical map $Au \to Au/Mu$ yields $u + Mu = ucu + Mu$ for some $c \in A$. Thus $u \in Mu$ again which proves that A/M is right A-flat.

(3) implies (2). For any $0 \neq b \in A$, let $f\colon Ab \to A/M$ be a non-zero left A-homomorphism. Then $Ab \not\subseteq M$ (otherwise, since A/M_A is flat and M is an ideal of A, $b = vb$ for some $v \in M$ and for any $a \in A$, $f(ab) = f(avb) = avf(b) = 0$ contradicting f non-zero) which implies $A = M + Ab$. If $1 = u + db$, $u \in M$, $d \in A$, then $b = bu + bdb$ so that $f(b) = f(bu) + bf(db)$. But $bu = cbu$ for some $c \in M$ which implies $f(bu) = cf(bu) = 0$. Thus $f(ab) = abf(db)$ for all $a \in A$ which proves that ${}_AA/M$ is p-injective.

Prop. 2. (1) *A* MELT *fully left idempotent ring is an* ELT *ring whose simple left modules are either injective or projective.*

(2) *A* MELT *fully left and right idempotent ring is von Neumann regular.*

Proof. (1) Let M be a maximal left ideal which is essential in ${}_AA$. Since A is fully left idempotent, then A/M is right A-flat which implies ${}_AA/M$ injective by Lemma 1. This proves that every simple left A-module is either projective or injective whence any proper essential left ideal E of A is an intersection of maximal essential left ideals of A. Therefore E is an ideal of A (since A is MELT) which proves that A is ELT.

(2) follows from ([**12**]$_1$, proposition 9).

([**12**]$_2$, lemma 2.1) and Proposition 2 yield the following generalization of ([**5**], theorem 2.19).

Corollary 2.1. *If A is a* MELT *left continuous ring whose simple right modules are flat, then $A = B \oplus C$, where B is a (left and right) continuous strongly regular ring and C is a semi-simple Artinian ring.*

Corollary 2.2. *A is* ELT *regular iff A is a* MELT *ring whose simple right modules are p-injective and flat.*

Corollary 2.3. *A is an* ELT *and* ERT *regular ring iff A is a semi-prime* MELT *and* MERT *ring whose simple left and right modules are flat.*

The next result contains an improvement of ([**5**], theorem 2.9).

Theorem 3. *The following conditions are equivalent.*

(1) *A is an* ELT *left and right self-injective regular left and right V-ring of bounded index;*

(2) *A is a regular ring whose left ideals are quasi-injective;*

(3) *A is a semi-prime π-regular left wq-ring such that the maximal left ideals of Q are quasi-injective;*

(4) *any left ideal I of A satisfies one of the following conditions:* (a) *I is p-injective quasi-injective;* (b) *I is a proper p-injective left annihilator;* (c) *$r(I)$ is a non-zero direct summand of A_A;*

(5) *A is a semi-prime ring whose faithfull left ideals are p-injective quasi-injective;*

(6) *A is a* MELT *left wq left and right V-ring;*

(7) *A is a semi-prime ring whose maximal left ideals and left annihilators are quasi-injective.*

Proof. (1) implies (2) by ([**5**], theorem 2.3).

(2) implies (3) obviously.

Assume (3). Then A is left non-singular semi-simple $[\mathbf{10}]_2$. If M is any maximal ideal of Q, then Q/M is a MELT simple ring which implies Q/M Artinian and proves that Q is of bounded index. A is therefore a semi-simple π-regular ring of bounded index and by ([**8**], theorem C), any primitive factor ring of A is Artinian. If A is not simple, then A contains a non-trivial central idempotent ([**8**], lemma 1) and therefore A is a left q-ring ($[\mathbf{10}]_1$, lemma 1.5). In any case, A is a regular left q-ring and (3) implies (4).

Assume (4). Then A is left self-injective. Let K be a complement left ideal such that $E = Z \oplus K$ is an essential left ideal. If E is p-injective, then Z is p-injective and for any $z \in Z$, if $i\colon Az \to Z$ is the canonical injection, then $z = i(z) = zw$ for some $w \in Z$ which implies $Az \cap l(w) = 0$, whence $z = 0$ and therefore $Z = 0$. If $r(E) = eA$, $0 \neq e = e^2 \in A$, then $0 \neq e \in Z$, which yields a contradiction. Thus $Z = 0$ in any case and A is therefore regular. Then any proper essential left ideal of A must be quasi-injective which implies that A is a left q-ring and proves that (4) implies (5).

Assume (5). If K is a left ideal such that $E = Z \oplus K$ is essential, then ${}_AE$ is faithful (since A is semi-prime) which implies E (and hence Z) a p-injective left ideal. Then $Z = 0$ as before and since A is left self-injective, A is regular. Since ${}_AL$ is faithful for any essential left ideal L, then ${}_AL$ is quasi-injective whence A is a left q-ring and (5) implies (6).

(6) implies (7) by ([**4**], lemma 6.20) and ($[\mathbf{10}]_1$, lemma 1.5).

Since a quasi-injective maximal essential left ideal in a left self-injective ring is an ideal, then (7) implies (1) by ([**4**], corollary 6.22) and Proposition 2.

We now look more closely at rings whose simple right modules are flat.

Lemma 4. *If every simple right A-module is flat, then*

(1) *any reduced right ideal of A is a strongly regular ring,*

(2) *any reduced finitely generated right ideal is a direct summand of A_A.*

Proof. (1) Let R be a reduced right ideal of A, $a \in R$. Since aA is reduced, then $l(a) \subseteq r(a)$. If $aA + r(a) \neq A$, let M be a maximal right ideal containing $aA + r(a)$. Then A/M is right A-flat which implies $a = ba$ for some $b \in M$. Therefore $1 - b \in l(a) \subseteq r(a) \subseteq M$ which yields $1 \in M$, a contradiction. Thus $aA + r(a) = A$ and $a = a^2c$ for some $c \in A$. Now $a = a^2d$, where $d = ac^2 \in R$ and since $a - ada \in R$, then $(a - ada)^2 = 0$ implies $a = ada$, whence R is a strongly regular ring.

(2) Let $F = aA + bA$ be a reduced right ideal generated by two elements a, b of A. From (1), $a = ada$ for some $d \in F$ and therefore $aA = eA$, where $e = ad$ is idempotent. Now $aA + bA = eA + (1 - e)bA = eA + uA$, where $(1 - e)bA = uA$, $u = u^2 \in F$ since $(1 - e)b \in F$. If we set $v = u(1 - e)$, then $v \in F$, $vu = u$, $v^2 = v$ and $uA = vA$. Therefore $F = eA + vA = (e + v)A$ is a direct summand of A_A as before. Then (2) follows by induction on the number of generators.

Call A *densely nil* if every non-zero right ideal contains a non-zero nilpotent element [**2**].

Corollary 4.1. *If A is a prime ring whose simple right modules are flat, then A is either a division ring or densely nil.*

Apply (([**12**]$_5$, proposition 6)).

Since (1) A fully left idempotent ring whose maximal left ideals are ideals is strongly regular; (2) A MELT left p-injective ring whose simple right modules are flat is an ELT left p-V-ring, then using the proof of ([**12**]$_5$, proposition 8(2)), we get the following decomposition result.

Theorem 5. *The following conditions are equivalent.*

(1) *A is a direct sum of a semi-simple Artinian ring and a strongly regular ring with zero socle;*

(2) *A is a* MELT *ring with finitely generated left socle such that every cyclic semi-simple right A-module is flat;*

(3) *A is a* MELT *left p-injective ring with finitely generated left socle such that every simple right A-module is flat.*

If A is a right continuous regular ring (in the sense of Utumi [**11**]) and $A = B \oplus C$, where B, C are ideals of A, then both B and C are right continuous regular rings with identity. A densely nil right continuous regular ring is right self-injective ([**11**], theorem 3). Following [**2**], write MDSN for the minimal direct summand of A_A containing the nilpotent elements of A. The next decomposition theorem extends that of right continuous regular rings given by Utumi ([**11**], p. 604).

Theorem 6. *Let A be a ring whose simple right modules are flat such that any reduced right ideal is essential in a finitely generated right ideal. Then $A = B \oplus C$, where B is the densely nil* MDSN *and C is a (left and right) continuous strongly regular ring.*

Proof. If I is a reduced right ideal then any essential extension of I_A in A_A is also reduced. By Lemma 4, any reduced right ideal is therefore essential in a direct summand of A_A and by ([**2**], theorem 12), $A = B \oplus C$, where B is the densely nil MDSN and C is a reduced abelian Baer ring with identity. Now C is a reduced right ideal of A and by Lemma 4, C is strongly regular. Since C is a Baer ring, then C is left and right continuous.

It is well-known that if $E = \mathrm{End}\,_A(M)$, where $_AM$ is injective, $J(E)$ the Jacobson radical of E, then $E/J(E)$ is a left self-injective regular ring and $J(E) = \{f \in E/\mathrm{ker} f$ is essential in $_AM\}$. With suitable modifications, the proof of that result yields the next lemma. Set $B = \mathrm{End}\,(_AA)$ and let $J(B)$ denote the Jacobson radical of B.

Lemma 7. *If A is a left p-injective ring whose complement left ideals are principal, then $B/J(B)$ is von Neumann regular and $J(B) = \{f \in B/\mathrm{ker} f$ is essential in $_AA\}$.*

We are now in a position to give some new characteristic properties of left continuous regular rings.

Theorem 8. *The following conditions are equivalent.*

(1) *A is left continuous regular;*

(2) *A is a left non-singular left p-injective ring whose complement left ideals are principal;*

(3) *A is a left non-singular left f-injective ring whose complement left ideals are finitely generated;*

(4) *A is a right f-injective ring whose complement left ideals coincide with left annihilators of elements of A.*

Proof. It is easy to see that (1) implies (2), (3) and (4).

Assume (2). Since $Z = 0$, then $J(B) = 0$ in Lemma 7 and $A \approx B$ is therefore regular. Then any complement left ideal is a direct summand of ${}_AA$ which implies A left continuous and therefore (2) implies (1).

Similarly, (3) implies (1).

Assume (4). Then $Z = 0$. For any $a \in A$, Aa is a left annihilator by Ikeda-Nakayama's theorem which then implies that $Aa = l(u)$, $u \in A$. If K is a complement left ideal such that $E = Aa \oplus K$ is an essential left ideal, since $K = l(v)$ for some $v \in A$, then $E = l(uA) + l(vA) = l(uA \cap vA)$ which yields $E = A$ (since $Z = 0$) and proves A regular. Since any complement left ideal is the left annihilator of an element (and therefore a direct summand of ${}_AA$), then (4) implies (1).

We now know from [**3**] that prime left non-singular left p-injective rings need not be primitive. However, the next remark holds (([**12**]$_2$, proposition 1.6) is thereby improved).

Remark. A prime left p-injective ring is primitive with non-zero socle iff it has a maximal left annihilator.

The next proposition is motivated by [**8**].

Prop. 9. *Let A be a semi-simple left WP, π-regular ring such that every maximal left ideal of Q is quasi-injective. Then A is a regular left and right V-ring of bounded index such that for any finitely generated left A-module M.* $\mathrm{End}_A(M)$ *is a π-regular ring of bounded index. Consequently,* $\mathrm{End}_A(M)$ *has stable range* 1 *and M cancels from direct sums of left A-modules.*

Proof. Since $Z = 0$, Q is left self-injective regular of bounded index by Theorem 3, which implies that A is of bounded index. If A is simple, then A is Artinian by ([**8**], theorem C). If not, then A contains a non-trivial central idempotent by ([**8**], lemma 1) which implies that A is von Neumann regular ([**12**]$_6$, Lemma 1.3). Thus A is a left and right V-ring such that for any finitely generated left A-module M, $\mathrm{End}_A(M)$ is a π-regular ring whose primitive factor rings are Artinian and the last part of the proposition then follows ([**8**], theorem D).

Similarly, ([**8**], theorems C and D) yield the next result which is related to a question of J. W. Fisher: Are π-regular V-rings regular?

Prop. 10. *Let A be a* MELT *π-regular fully right idempotent ring such that every maximal left ideal of Q is quasi-injective. Then A is an* ELT *unit-regular left and right V-ring whose primitive factor rings are Artinian.*

Finally, we turn to characteristic properties of semi-simple Artinian rings.

Theorem 11. *The following conditions are equivalent.*

(1) *A is semi-simple Artinian;*

(2) *A is a semi-prime ring whose faithful left modules are injective;*

(3) *A is a semi-prime ring whose faithful proper left ideals have non-zero right annihilators;*

(4) *A is a regular ring whose faithful non-singular left modules are projective;*

(5) *A is a left self-injective ring such that the sum of any two cyclic injective left A-modules is injective;*

(6) *the sum of any two cyclic projective submodules of every left A-module is injective;*

(7) *every semi-simple left A-module is p-injective and quasi-injective;*

(8) *A is a* MELT *ring whose projective and quasi-injective left modules coincide;*

(9) *A is a left p.p. ring whose essential right ideals are right annihilators.*

Proof. Obviously, (1) implies (2) while (2) implies (3).

The proof of Theorem 3 shows that (3) implies (4).

Assume (4). Since A is regular, then a theorem of I. Kaplansky asserts that any finitely generated submodule of a projective left A-module is a direct summand. Therefore, ${}_AQ$ projective implies that ${}_AA$ is a direct summand of ${}_AQ$ whence $A = Q$ is left self-injective regular. Then A is left hereditary (cfr. the proof of Theorem 3 (4)) and we may conclude that (4) implies (5).

It is easy to see that (5) implies (6).

Assume (6). Then A is left self-injective. Let $B = A_1 \oplus A_2$ be a direct sum of two copies of A, $B_1 = \{(-a, 0) \in B / a \in A_1\}$, $B_2 = \{(0, a) \in B / a \in A_2\}$. For any left ideal I of A, write $K = \{(a, a) \in B / a \in I\}$. Then $B_1 \approx A_1$, $B_2 \approx A_2$ and the canonical homomorphism $B \to \bar{B} = B/K$ yields $B_1 \approx \bar{B}_1$ and $B_2 \approx \bar{B}_2$. Since $\bar{B} = \bar{B}_1 + \bar{B}_2$, then $\bar{B}$ is injective by hypothesis and $\bar{B} = \bar{B}_1 \oplus T$ implies $\bar{B}/\bar{B}_1$ left A-injective. If $\overline{(b_1, b_2)} + \bar{B}_1 \in \bar{B}/\bar{B}_1$, then $\overline{(b_1, b_2)} + \bar{B}_1 = \overline{(b_2, b_2)} + \bar{B}_1$ which yields $A/I \approx \bar{B}/\bar{B}_1$, whence A/I is a cyclic injective left A-module. Therefore (6) implies (7) by ([**9**], theorem 3.2).

Assume (7). The fact that every semi-simple left A-module is p-injective implies A von Neumann regular whence every left ideal is semi-simple. This implies that A is a left q-ring and by Theorem 3, A is a left V-ring. But then every left A-module is semi-simple by ([**9**], theorem 2.1) which proves that every left A-module is quasi-injective and therefore every left A-module is injective and projective. Thus (7) implies (8).

Assume (8). Then A is left self-injective and any maximal essential left ideal (being an ideal of A) is quasi-injective and therefore projective. Thus every maximal left ideal of A is projective and (8) implies (9) by ([**12**]$_3$, remark 6).

Finally, assume (9). Since a maximal right ideal is either a direct summand of A_A or an essential right ideal, then every maximal right ideal of A is a right annihilator. If I is a finitely generated proper right ideal, M a maximal right ideal containing I, then $l(M) \neq 0$ implies $l(I) \neq 0$. Since A is a left p.p. ring, by ([**1**], theorem 5.4(, every principal left ideal is a direct summand of ${}_AA$ which proves A regular. Then every maximal right ideal, being the right annihilator of an element, is a direct summand of A_A which proves that (9) implies (1).

Semi-group analogues of certain results on injectivity and p-injectivity have been considered in [**6**] and [**7**].

Question. Are there semi-group analogues of Theorems 3 and 11?

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Abstract

Some characteristic properties of the following classes of rings are given: (1) *Regular rings whose left ideals are quasi-injective*. (2) *Left continuous regular rings*. (3) *Semi-simple Artinian rings. Connections between several generalisations of von Neumann regular rings are also considered. Known results are improved.*

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