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$H^{2,p}$ - regularity for the solution of systems of elliptic variational inequalities (**)

1 - Introduction

Let $\Omega \subset R^N$ be a bounded open set with smooth boundary Γ, $A_{ij}^{\alpha\beta} \in H^{1,\infty}(\Omega)$ $\alpha, \beta = 1, \ldots, n$, $i, j = 1, \ldots, N$, be such that

$$\sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} A_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta \geqslant \nu \sum_{\alpha=1}^{n} \sum_{i=1}^{N} |\xi_i^\alpha|^2$$

a.e. in Ω $\forall \xi \in R^{nN}$, $\nu > 0$.

Let ψ: $\Omega \to R^n$ be a mesurable function and $K^\psi = \{v \in (H^1(\Omega))^n,\ v \leqslant \psi$ a.e. in $\Omega\} \neq \emptyset$.

We indicate

$$\langle Au, v\rangle = \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \int_\Omega A_{ij}^{\alpha\beta}(x) \frac{\partial u^\beta}{\partial x_j}(x) \frac{\partial v^\alpha}{\partial x_i}(x) \quad dx$$

$$\forall v \in (H_0^1(\Omega))^n,\ u \in (H^1(\Omega))^n.$$

Suppose now $\psi \in (H^1(\Omega))^n$; we consider the system of variational inequalities

$$(1.1) \qquad \begin{gathered} \langle Au, v-u\rangle \geqslant (f, v-u) \\ \forall v \in K^\psi, \quad v = u \text{ on } \Gamma, \quad u \in K^\psi, \end{gathered}$$

where $f \in (L^2(\Omega))^n$ and (,) indicate the scalar product in $L^2(\Omega)$.

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The $H^2(\Omega)$-regularity for the solution to systems of variational inequalities, also more general than (1.1), has been studied by J. Frehse [3], M. Kucera and J. Necas [8], and J. Necas [9].

J. Frehse gives in [3] also results on $L^{2,\lambda}_{loc}$-regularity $\lambda > 0$ for D^2u (we recall $L^{2,\lambda}(\Omega) = \{v \in L^2(\Omega);\ \forall x_0 \in \Omega \int\limits_{B(R;\,x_0)\cap\Omega} v^2\,dx \leqslant CR^\lambda\}$).

The aim of this paper is to refine the result of J. Frehse by proving the $H^{2,p}$-regularity for solutions of (1.1) for some $p > 2$.

In the first paragraph we deal with (1.1), in the second paragraph we give some results concerning the nonlinear case.

2 - The linear case

We prove in this paragraph the following result.

Th. 1. *There exists* $p_0 > 2$ *such that for* $\psi \in (H^{2,p}(\Omega))^n$, $f \in (L^p(\Omega))^n$, *we have* $u \in (H^{2,p}(\Omega))^n$ *where* u *is a solution to* (1.1) $(2 < p \leqslant p_0)$.

We can suppose in the proof $\psi = 0$.

Suppose, at first, $f \in (L^\infty(\Omega))^n$ and consider the penalized problem

$$Au_\lambda + \frac{1}{\lambda}\,u_\lambda^+ = f\,, \quad u_\lambda|_\Gamma = u|_\Gamma\,. \tag{2.1}$$

Lemma 1. *Let* u_λ *be the solution of* (2.1), *we have* $u_\lambda \in (C^{1,\alpha}(\Omega))^n$, $0 < \alpha < 1$.

Let be $x_0 \in \Omega$, $B(R;x_0) \subset \Omega$ and indicate $A(x) = [A_{ij}^{\alpha\beta}(x)]$, $A^0 = [A_{ij}^{\alpha\beta}(x_0)]$. We consider the problems

$$D(ADu_\lambda) + \frac{1}{\lambda}\,u_\lambda^+ = f \qquad \text{in } \mathscr{D}'(B(R;\,x_0))\,, \tag{2.2}$$

$$D(A_x^0 Du_\lambda^0) = 0 \qquad \text{in } \mathscr{D}'(B(R;\,x_0))\,, \tag{2.3}$$

$$u_\lambda^0 = u_\lambda \quad \text{on} \quad \partial B(R;\,x_0)\,.$$

We set $w = u_\lambda - u_\lambda^0$; we have

$$\int\limits_{B(R;\,x_0)} |Dw|^2\,dx \tag{2.4}$$

$$\leqslant \frac{C_1}{\lambda} \int\limits_{B(R;\,x_0)} u_\lambda^+ |w|\,dx + C_2 \int\limits_{B(R;\,x_0)} fw\,dx + C_3 R \int\limits_{B(R;\,x_0)} |Du_\lambda|\,|Dw|\,dx.$$

We have two possible cases: (1) $-\infty \leqslant \operatorname{Ess\,inf} u_\lambda \leqslant 0$; (2) $\operatorname{Ess\,inf} u_\lambda > 0$.

We deal at first with the case (1). We have

$$\int_{B(R;\,x_0)} |u_\lambda^+|^2\,dx \leq C_4 R^2 \int_{B(R;\,x_0)} |Du_\lambda|^2\,dx\,,$$

$$\int_{B(R;\,x_0)} |w|^2\,dx \leq C_4 R^2 \int_{B(R;\,x_0)} |Dw|^2\,dx\,,$$

then we have easily from (2.4)

$$\begin{aligned}
&\int_{B(R,\,x_0)} |Dw|^2\,dx \\
&\leq \frac{C_5}{\lambda} R^2 \int_{B(R;\,x_0)} |Du_\lambda|^2\,dx + C_6 \int fw\,dx \\
&\leq \frac{C_5}{\lambda} R^2 \int_{B(R;\,x_0)} |Du_\lambda|^2\,dx + C_7 R^{N+2} + \tfrac{1}{2} \int_{B(R;\,x_0)} |Dw|^2\,dx\,,
\end{aligned}$$

then

$$\int_{B(R;\,x_0)} |Dw|^2\,dx \leq \frac{C_8}{\lambda} R^2 \int_{B(R;\,x_0)} |Du_\lambda|^2\,dx + C_9 R^{N+2}\,. \tag{2.5}$$

We have now

$$\begin{aligned}
&\int_{B(\varrho;\,x_0)} |Du_\lambda|^2\,dx \\
&\leq 2 \int_{B(\varrho;\,x_0)} |Du_\lambda^0|^2\,dx + 2 \int_{B(\varrho;\,x_0)} |Dw|^2\,dx \\
&\leq C_{10} \left(\frac{\varrho}{R}\right)^N \int_{B(R;\,x_0)} |Du_\lambda^0|^2\,dx + 2 \int_{B(\varrho;\,x_0)} |Dw|^2\,dx \\
&\leq C_{11} \left(\frac{\varrho}{R}\right)^N \int_{B(R;\,x_0)} |Du_\lambda|^2\,dx + C_{12} \int_{B(R;\,x_0)} |Dw|^2\,dx \\
&\leq \left(\frac{C_{13}}{\lambda} R^2 + C_{11} \left(\frac{\varrho}{R}\right)^N\right) \int_{B(R;\,x_0)} |Du_\lambda|^2\,dx + C_{14} R^{N+2}\,.
\end{aligned}$$

then, [**7**], for $\varrho < R \leq R_0$

$$\int_{B(\varrho;\,x_0)} |Du_\lambda|^2\,dx \leq C_{15} \left(\frac{\varrho}{R}\right)^{N-\eta} \int_{B(R;\,x_0)} |Du_\lambda|^2\,dx \qquad \forall \eta > 0\,. \tag{2.6}$$

(We observe that the constant C_{15} depends on λ, η).

From (2.6) we have easily that for $R < R_0$ we have

$$\int_{B(R;\,x_0)} |Du_\lambda|^2\,dx \leqslant C_{16} R^{N-\eta}, \tag{2.7}$$

then, from (2.5)

$$\int_{B(R;\,x_0)} |Dw|^2\,dx \leqslant C_{17} R^{N+2-\eta}, \tag{2.8}$$

where C_{17} is a constant dependent on λ, η.

We have

$$\begin{aligned}
&\int_{B(\varrho;\,x_0)} |Du_\lambda - (Du_\lambda)_\varrho|^2\,dx \\
&\leqslant \int_{B(\varrho;\,x_0)} |Du_\lambda - (Du^0_\lambda)_\varrho|^2\,dx \\
&\leqslant 2\int_{B(\varrho;\,x_0)} |Du^0_\lambda - (Du^0_\lambda)_\varrho|^2\,dx + 2\int_{B(\varrho;\,x_0)} |Dw|^2\,dx \\
&\leqslant C_{18}\Big(\frac{\varrho}{R}\Big)^{N+2}\int_{B(R;\,x_0)} |Du^0_\lambda - (Du^0_\lambda)_R|^2\,dx + 2\int_{B(\varrho;\,x_0)} |Dw|^2\,dx \\
&\leqslant C_{18}\Big(\frac{\varrho}{R}\Big)^{N+2}\int_{B(R;\,x_0)} |Du^0_\lambda - (Du_\lambda)_R|^2\,dx + C_{20}\int_{B(R;\,x_0)} |Dw|^2\,dx \\
&\leqslant C_{18}\Big(\frac{\varrho}{R}\Big)^{N+2}\int_{B(R;\,x_0)} |Du_\lambda - (Du_\lambda)_R|^2\,dx + C_{21} R^{N+2-\eta},
\end{aligned} \tag{2.9}$$

where $(g)_\varrho$ is the average of g on $B(\varrho;\,x_0)$.

From (2.7), (2.9) we have $u \in \big(C^\alpha(B(R;\,x_0))\big)^n$, $Du \in \big(C^\alpha(B(R;\,x_0))\big)^{nN}$ $\forall\alpha \in (0, 1)$, then $u \in \big(C^{1,\alpha}(B(R;\,x_0))\big)^n$.

Now we consider the case (2); in such a case we can write (2.2) as

$$D(ADu_\lambda) + \frac{1}{\lambda}u_\lambda = f \qquad \text{in} \quad \mathscr{D}'\big(B(R;\,x_0)\big),$$

then we are in the case of a linear system and we can prove by standard methods $u \in \big(C^{1,\alpha}(B(R;\,x_0))\big)$, $\forall\alpha \in (0, 1)$.

From the two cases (1), (2) we have the result.

Consider now again the problem (2.1), we known [**3**], that the solutions of (2.1) are in $H^2_{\text{loc}}(\Omega)$.

Let be $s = \partial u_\lambda/\partial x_k$ we have

$$\sum_{\beta=1}^{n}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(A_{ij}^{\alpha\beta}\frac{\partial s^\beta}{\partial x_j}\right)+\frac{1}{\lambda}\chi_{(u^\beta\geqslant 0)}s^\alpha \tag{2.10}$$

$$=\frac{\partial f}{\partial x_k}+\sum_{\beta=1}^{n}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial A_{ij}^{\alpha\beta}}{\partial x_k}\frac{\partial u^\beta}{\partial x_j}\right)\qquad\text{in}\quad \mathscr{D}'(\Omega)$$

($\chi(u^\beta\geqslant 0)$ is the characteristic function of the set $(u^\beta\geqslant 0)$).

Using the same methods as in [**5**] we prove easily

Lemma 2. *There exists $\bar{p}_0 > 2$, such that*

$$\|u_\lambda\|_{H^{1,p}_{\mathrm{loc}}}\leqslant C\|f\|_{H^{-1,p}}\qquad 2<p\leqslant p_0\,,$$

where C does not depend on λ.

Using the Lemma 2 and (2.10) we have

$$-\sum_{\beta=1}^{n}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(A_{ij}^{\alpha\beta}\frac{\partial s^\beta}{\partial x_j}\right)+\frac{1}{\lambda}\chi_{(u^\beta\geqslant 0)} \tag{2.11}$$

$$=\operatorname{div} g_\lambda^\beta\quad\text{in}\ (\mathscr{D}'(\Omega))^n\,,$$

where g_λ^β is uniformly bounded in $L^p_{\mathrm{loc}}(\Omega)$, $2<p\leqslant p_0$.

Let be now $m^\alpha = s^\alpha(\bar{x})$ where $|s^\alpha(\bar{x})| = \min\limits_{B(2R;\,x_0)}|s^\alpha|$, $B(2R;\,x_0)\subset\Omega$.

We choose $\eta = 0$ such that $\eta = 1$ in $B(R;\,x_0)$, $\eta\geqslant 0$, $\eta = 0$ for $x\notin B(2R;\,x_0)$, $|D\eta|\leqslant CR^{-1}$.

We multiple (2.11) by $\eta^2(s^\alpha - m^\alpha)$ (it is possible being $s\in(H^1(\Omega))^n$). We have

$$\nu\int_\Omega \eta^2|Ds|^2\,\mathrm{d}x \tag{2.12}$$

$$\leqslant C_1\int_\Omega \eta|D\eta|\,|s-m|\,|Ds|\,\mathrm{d}x + C_2\int_\Omega g\eta^2|Ds|^2\,\mathrm{d}x + C_3\int_\Omega g\eta|D\eta|\,|s-m|\,\mathrm{d}x\,,$$

then

$$\int_\Omega \eta^2|Ds|^2\,\mathrm{d}x \tag{2.13}$$

$$\leqslant C_4R^{-2}\int_{B(2R;\,x_0)}|s-m|^2\,\mathrm{d}x + C_5\int_{B(2R;\,x_0)}|g|^2\,\mathrm{d}x\,.$$

Repeating the proof of Poincare's inequality we have

$$R^{-(N+2)} \int_{B(2R;\,x_0)} |s-m|^2\, dx \leqslant C_6 \left(\fint |Ds|^p \right)^{p/2}, \tag{2.14}$$

$p = 2N/(N+2)$ $\fint_{B(R,\,x_0)} q\, dx$ indicates the average of q on $B(R; x_0)$).

From (2.13), (2.14) we have

$$\fint_{B(R,\,x_0)} |Ds|^2\, dx \leqslant C_7 \left(\fint_{B(2R;\,x_0)} |Ds|^p \right)^{2/p} + C_8 \fint_{B(2R;\,x_0)} |g_\lambda|^2\, dx. \tag{2.15}$$

From (2.15) and [4] we have

$$\|u_\lambda\|_{H^{2,p}_{\text{loc}}} \leqslant C_9 \|g_\lambda\|_{L^p_{\text{loc}}} \tag{2.16}$$

($2 < p \leqslant p_0 \leqslant \bar{p}_0$, $\bar{p}_0$ suitable).

Using the Lemma 2, (2.10), (2.11), (2.16) we have

$$\|u_\lambda\|_{H^{2,p}_{\text{loc}}} \leqslant C_{10} \|f\|_{L^p_{\text{loc}}},$$

where C_6 does not depend on λ; then passing to the limit for $\lambda \to 0$ we have the result in the case $f \in (L^\infty(\Omega))^n$.

A regularisation on f give the result in the general case.

3 - The nonlinear case

(a) *The sublinear case.*

Let $H(x, u, p)$: $\Omega \times R \times R^N \to R^n$ be a function measurable for $x \in \Omega$ and continuous in (u, p) such that $|H(x, u, p)| \leqslant K_1|p| + K_2$ and consider the non-linear system of variational inequalities

$$\begin{gathered} \langle Au, v-u\rangle + \int_\Omega H(x, u, Du)(v-u)\, dx \geqslant 0 \\ \forall v \in K^\psi, \quad v = u \text{ on } \Gamma, \quad u \in K^\psi. \end{gathered} \tag{3.1}$$

Theorem 2. *Let u be a solution of* (3.1); *then there exists $p_0 > 2$ such that $u \in (H^{2,p}_{\text{loc}}(\Omega))^n$, $2 < p \leqslant p_0$.*

We observe that, using the methods of [5], we can prove the following

Meyers estimate

$$\|u\|_{H^{1,p}_{\mathrm{loc}}} \leqslant C \qquad 2 < p \leqslant p_0 ,$$

$p_0 > 2$ suitable, then $H(x, u, Du) \in \left(L^p_{\mathrm{loc}}(\Omega)\right)^n$.

The Th. 1 gives now the result.

(b) *The case of quadratic growth.*

Let $H(x, u, p)\colon \Omega \times R \times R^n \to R^n$ a function measurable in x continuous in (u, p).

We consider in the following two types of hypothesis on $H(x, u, p)$

$$(3.2) \qquad |H(x, u, p)| \leqslant K_1 |p|^2 + K_2 ,$$

$$(3.3) \qquad H(x, u, p)u \geqslant -K_3 |p^2| + K_4(1 + |u|) , \qquad K_3 < \nu .$$

We recall also the following result [2],

Lemma 1. *Let $v \in H^2_{\mathrm{loc}}(\Omega) \cap C^\beta(\Omega)$, $\beta \in (0, 1)^n$ then for every $B(R; x_0) \subset\subset \Omega$ we have*

$$(3.4) \qquad \begin{gathered} Dv \in L^s_{\mathrm{loc}}(\Omega) , \\ \int_{B(R;\, x_0)} |Dv - (Dv)_R|^s \, dx \leqslant C R^{N(1-s/q)} \|v\|_{H^2(B(R;\, x_0))} , \end{gathered}$$

where $q = 4N/(N - 2\beta)$, $s \in (1, q)$ and C depends on $\|v\|_{(\beta(K)}$ $(B(R; x_0) \subset K$, K compact$)$.

Consider now the following nonlinear system of variational inequalities

$$(3.1)' \qquad \begin{gathered} \langle Au, v - u\rangle + \int_\Omega H(x, u, Du)(v - u)\, dx \geqslant 0 \\ \forall v \in K^\psi, \quad v - u \in L^\infty(\Omega) , \quad u \in K^\psi, \quad v = u \text{ on } \Gamma, \end{gathered}$$

where $\psi \in H^{2,p}_{\mathrm{loc}}(\Omega)$.

From Th. 1 and Lemma 1 above, we have

Theorem 3. *Let $u \in \left(H^2_{\mathrm{loc}}(\Omega)\right)^n \cap \left(C^\beta(\Omega)\right)^n$, $\beta \in (0, 1)$, be a solution of $(3.1)'$ and suppose (3.2) holds; there exists $p_0 > 2$ such that $u \in \left(H^{2,p}_{\mathrm{loc}}(\Omega)\right)$, $2 < p \leqslant p_0$.*

If $N = 2, 3$ it is enough to suppose $u \in H^2_{\mathrm{loc}}(\Omega)$, being $C^{\beta}(\Omega) \subset H^2_{\mathrm{loc}}(\Omega)$, $\beta \in (0, 1, 2)$.

In the case $N = 2$ the H^2_{loc}-regularity for a weak solution of (3.1′) with the condition (3.2) has been proved by S. Hildebrandt [**6**], then

Corollary 1. *Let $N = 2$ and u be a weak solution, then there exists $p_0 > 2$ such that $u \in (H^{2,p}_{\mathrm{loc}}(\Omega))^n$, $2 < p \leqslant p_0$.*

We consider now the case $N = 3$, $u|_{\Gamma} = 0$, $\psi|_{\Gamma} > 0$, $\psi \in H^2(\Omega)$ let be

$$\tau_M(t) = \begin{cases} t & |t| \leqslant M \\ M & t > M \\ -M & t < -M \end{cases} \qquad \omega_M(p) = \{\tau_M(p_i)\} \qquad p \in R^N$$

and
$$H_M(x, u, p) = \tau_M(H(x, u, \omega_M(p))).$$

We suppose (3.2), (3.3) hold for $H(x, u, p)$; then $H_M(x, u, p)$ is sublinear and (3.2), (3.3) hold also for $H(x, u, p)$.

Consider the systems of variational inequalities

$$(3.1)' \qquad \begin{gathered} \langle A u_M, v - u_M \rangle + \int_{\Omega} H_M(x, u_M, Du_M)(v - u_M)\, dx \geqslant 0 \\ \forall v \in K^{\psi}, \qquad v = u = 0 \text{ on } \Gamma, \qquad u \in K^{\psi}. \end{gathered}$$

We have easily the existence of a solution u_n to (3.1)′ and, if $\nu > \nu_0(K_4)$ we have also $\|u_M\|_{H^1} \leqslant C$.

We have also

$$\|u_M\|_{H^2_{\mathrm{loc}}} \leqslant \|H(x, u_M, Du_M)\|_{L^2} \leqslant K_1' \|Du_M\|^2_{L^4} + K_2'$$
$$\leqslant K_3 \|u_M\|_{H^2} \|u_M\|_{H^1} + K_2' \leqslant \eta \|u_M\|^2_{H^2} + K_\eta \qquad (\eta < 1)\,,$$

then $\|u_M\|_{H^2} \leqslant C'$, where C, C' does not depend on M.

Passing to the limit we have the existence of a solution of (3.1) in $H^2(\Omega) \cap H^1_0(\Omega)$, then

Theorem 3. *Let be $\psi \in H^{2,p}_{\mathrm{loc}}(\Omega) \cap H^2(\Omega)$, $\psi|_{\Gamma} > 0$ we suppose* (3.2), (3.3) *hold and $\nu > \nu_0(K_4)$ ($\nu_0 > 0$ suitable).*

There exists a solution $u \in H^1_0(\Omega) \cap H^2(\Omega) \cap H^{2,p}_{\mathrm{loc}}(\Omega)$ of (3.1), $2 < p \leqslant p_0$.

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