Riv. Mat. Univ. Parma (4) 9 (1983), 95-103

LUIGI GRASSELLI (*)

Subdivision and Poincaré duality (**)

1 - Introduction

Adapting the subdivision theorem for mock bundles [1] in the case of geometric cocycles [4], we prove the independence of orientations and subdivisions for geometric cohomology groups.

A subdivision theorem for simplicial algebraic cocycles and an alternative description of the subdivision of a geometric cocycle are derived.

A duality map ψ between cohomology and homology of a cycle K [6] is then introduced: by means of an extension to cycles of a Cohen's theorem about cellular dual structures ([2], prop. 5.6), ψ is proved to be, if K is an oriented manifold, the Poincaré duality isomorphism.

2 - Notations and preliminar definitions

See [5] and [4] about ball complexes: $Sd^r(K)$ will be the r-derived subdivision of a ball complex K. An oriented ball complex is a ball complex in which each ball is arbitrarily oriented.

We always consider on $K \times I$ the orientation induced by K.

For definitions and notations about cycles see [4] or [6].

A geometric (g.) q-cocycle ξ over an oriented ball complex K is a pair $\xi/K = (E_\xi, p_\xi)$, where E_ξ is a polyhedron – called total space – and $p_\xi: E_\xi \to |K| - |K^{q-1}|$ is a pl map – called projection –, such that, for each k-ball σ

(*) Indirizzo: Istituto Matematico, Università, Via Campi, 41100 Modena, Italy.
(**) Ricevuto: 12-III-1981.

in K ($k \geqslant q$), $p_\xi^{-1}(\sigma)$ is a ($k-q$) oriented cycle whose boundary is $p_\xi^{-1}(\partial\sigma)$: $p_\xi^{-1}(\sigma)$ is called block over σ and is usually denoted by $\xi(\sigma)$.

Further, we must have

$$\text{(i)} \qquad \tau \in S_{k-1}(\partial\sigma) \Rightarrow \varepsilon(\xi(\tau), \xi(\sigma)) = \varepsilon(\tau, \sigma) ,$$

where ε is the incidence number ([**4**], def. 1).

Two g. q-cocycles ξ_i ($i = 0, 1$) are cohomologous if there exists a g. q-cocycle η over $K \times I$ – called cohomology – such that

$$\eta | K \times \{0\} = \xi_0 , \qquad \eta | K \times \{1\} = -\xi_1 .$$

$Hg^q(K)$ will denote the group of the cohomology classes of g. q-cocycles over K with the disjoint union and ξ will represent both the cocycle ξ and the cohomology class $[\xi]$.

Let σ be a ($q+1$) ball in an oriented ball complex K and let (A, B) be a pair of points with sign ε_A and ε_B; if $p: \{A, B\} \to \partial\sigma$ is a map such that $p(A) \in \mathring{\alpha}_A$, $p(B) \in \mathring{\alpha}_B$, with α_A, α_B q-faces of σ, we say that A and B are connectable in σ if ([**4**], def. 5)

$$\varepsilon(\alpha_A, \sigma) \cdot \varepsilon_A = -\varepsilon(\alpha_B, \sigma) \cdot \varepsilon_B .$$

A singular geometric (sr.g.) q-cycle in a polyhedron X is a pair (P, f), where P is an oriented closed q-cycle and $f: P \to X$ is a pl map.

Two sr.g. q-cycles (P_1, f_1) and (P_2, f_2) in X are homologous if there is an oriented ($q+1$)-cycle Q and a pl map $F: Q \to X$ such that

$$\text{(1)} \qquad \partial Q = P_1 \cup \cdot (-P_2), \qquad\qquad \text{(2)} \qquad F | P_1 = f_1, \; F | P_2 = f_2 .$$

The equivalence classes of sr.g. q-cycles in X given by the homology relation, together with the operation of disjoint union, form a group $H_q(X)$ called the sr.g. homology q-group.

See [**6**] for more information about this geometric description of homology groups.

Every cycle can always be supposed to have a collarable boundary, up to (co)homology.

The simplicial geometric (sl.g.) homology group $H_q^{(s)}(K)$ of a simplicial complex K is defined in the same way starting from sl.g. q-cycles in K, i.e. pairs (L, f), where L is a triangulated q-cycle and $f: L \to K$ is simplicial.

By the simplicial approximation theorem it is easy to prove that the map

$\omega_q \colon H_q^{(s)}(K) \to H_q(|K|)$ obtained by considering a sl.g. q-cycle in K as a sr.g. q-cycle in K is a group isomorphism.

A sl.g. q-cycle (L, f) in K such that $\omega_q(L, f)$ is a given sr.g. q-cycle (P, g) is called a simplicial approximation of (P, g).

The relative homology for a pair (X, Y) is obtained by considering (P, f) such that P is boundered and $f(\partial P) \subseteq Y$.

3 - Subdivision theorems

Let K be an oriented ball complex.

Prop. 1. $Hg^q(K)$ is independent of the orientation of K.

Proof. If $\bar{K}$ is a different orientation of K, we denote by $\bar{\sigma}$ the ball $\sigma \in K$ with the orientation of $\bar{K}$.

If ξ/K is a g. q-cocycle over K we build a g. q-cocycle $\bar{\xi}/\bar{K}$ over $\bar{K}$ with the same total space and the same projection by orienting the blocks $\bar{\xi}(\bar{\sigma})$ in the following way. Let ε be 1 or -1 whether or not $\bar{\sigma}$ has the same orientation of σ; then define $\bar{\xi}(\bar{\sigma}) = \varepsilon\xi(\sigma)$.

The map $\xi \to \bar{\xi}$ is a group isomorphism between $Hg^q(K)$ and $Hg^q(\bar{K})$.

By Prop. 1, if $K' \lhd K$ and K is oriented, we may assume, in what follows, K' oriented in such a way that the balls in K' subdividing balls in K of the same dimension have the induced orientation and the other ones are arbitrarily oriented.

Prop. 2. If $K' \lhd K$ and ξ/K' is a g. q-cocycle over K', (E_ξ, p_ξ) is also a g. q-cocycle over K, called *amalgamation* of ξ.

Proof. It is clear that the blocks over the q-balls of K are 0-cycles, since they are formed by isolated points.

The proof is then inductive, considering, as blocks over the generic n-ball σ of K, the blocks over the n-balls σ_i of K' subdividing σ, glued over the boundaries not included in $\partial\sigma$. The glueing is possible, since σ_i are supposed to have the orientations induced by σ, i.e. coherent orientations.

Def. 1. The map $Ag^q \colon Hg^q(K') \to Hg^q(K)$ defined by $Ag^q(\xi/K') = \xi/K$ is called *amalgamation*.

It is easy to see that Ag^q is a well-defined group homomorphism; in order to prove that it is an isomorphism we give the following

Prop. 3. Let ξ_0/K, ξ_1/K be g. q-cocycles having the same total space E_ξ and projections p_{ξ_0}, p_{ξ_1} such that $\xi_0(\sigma) = \xi_1(\sigma)$, for each $\sigma \in K$. Then ξ_0 and ξ_1 are cohomologous.

Proof. We build the cohomology between ξ_0 and ξ_1 by considering the g. q-cocycle ξ over $K \times I$ with $E_\xi \times I$ as total space and the projection p_ξ inductively defined in the following way.

Let σ be a q-ball of $K \times I$; if $\sigma \in K \times \{i\}$ $(i = 0, 1)$, we define $\xi(\sigma) = (-1)^i \cdot (\xi_i(\sigma) \times \{i\})$ and $p_\xi = p_{\xi_i}$, otherwise $\xi(\sigma) = \emptyset$.

Now assume ξ defined over $K \times I^{(s-1)}$ and let σ be an s-ball of $K \times I$; σ may be either a ball of one of the bases of $K \times I$ or the product of an $(s-1)$-ball α of K with I.

In the first case we define p_ξ as the restriction of p_{ξ_0} or p_{ξ_1} respectively; in the second case let $B = \xi_0(\alpha) \times I$ and note that, by induction, we can think p_ξ already defined on ∂B.

Since $\alpha \times I$ is contractible to an internal point x, $p_\xi|_{\partial B}$ is homotopic to the constant map x, i.e. there exists a map F from a collar $\partial B \times I$ of ∂B in B to $\alpha \times I$ such that $F|_{\partial B \times \{0\}} = p_\xi|_{\partial B \times \{0\}}$ and $F|_{\partial B \times \{1\}} = x$; defining $F(B - \partial B \times I) = x$, we get an extension of p_ξ on B.

This construction, repeated on each s-ball of $K \times I$, proves this lemma.

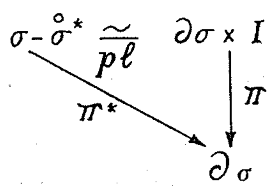
Prop. 4. Let ξ/K be a g. q-cocycle over K; there exists a g. q-cocycle $\tilde{\xi}$ over $Sd^1(K)$, called a *subdivision* of ξ, such that $Ag^q(\tilde{\xi})/K = \xi/K$.

Proof. The proof is by induction on the dimension of K; if $\dim K \leqslant q$ the theorem is trivial.

Assume the statement true for $\dim K \leqslant m$ and let K be an $(m+1)$ dimensional ball complex; we can imagine $\tilde{\xi}/Sd^1(K^m)$ already built with the required properties.

Let σ be a generic $(m+1)$-ball in K and σ^* the isomorphic image of an $(m+1)$-simplex σ_1 of $Sd^1(\sigma)$ such that $\sigma^* \subset \mathring{\sigma}_1$.

Let π^* be the pl map making this diagram commute



There exists a triangulation J of $\sigma - \mathring{\sigma}^*$ such that

(i) $\pi^*: J \to Sd^1(\partial\sigma)$ is simplicial, (ii) $J \cup \sigma^* \lhd Sd^1(\sigma)$,

(iii) $J \cup \sigma^*$ has the orientation induced by $Sd^1(\sigma)$ (and hence by σ).

By induction, $\xi | Sd^1(\partial\sigma)$ has already been built; since π^* is simplicial, $[Hg^q(\pi^*)](\xi)$ is a g. q-cocycle $\#$ over J (see [4], prop. 9); we obviously may assume $E_\# \underset{pl}{\simeq} E_{\tilde{\xi}} \times I$ with $\#(Sd^1(\partial\sigma)) \underset{pl}{\simeq} E_{\tilde{\xi}} \times \{1\}$, $\#(\partial\sigma^*) \underset{pl}{\simeq} E_{\tilde{\xi}} \times \{0\}$ and identify the polyhedra.

We then define $\xi(\sigma) = \xi(\sigma)$ and $p'_\sigma: \xi(\sigma) \to J \cup \sigma^*$ in the following way.

On the collar $E_{\tilde{\xi}} \times I \underset{pl}{\simeq} E_\#$ of $\xi(\sigma)$ let $p'_\sigma = p_\#$; on $\xi(\sigma) - (\overset{\circ}{\widehat{E_{\tilde{\xi}} \times I}})$ let p'_σ be an arbitrary extension of $p_\# |_{E_{\tilde{\xi}} \times \{0\}}$ to σ^* (for example as in Prop. 3).

If we orient the blocks over the balls of $Sd^1(K)$ which subdivide balls of K of the same dimension with the induced orientations, and the other blocks in such a way that the incidence condition works, we obtain a g. q-cocycle over $J \cup \sigma^*$; since $J \cup \sigma^* \lhd Sd^1(\sigma)$, its amalgamation is a g. q-cocycle over $Sd^1(\sigma)$, whose restriction to $Sd^1(\partial\sigma)$ is just the ξ built by induction.

Repeating the construction for each $(m+1)$-ball σ in K, we get a g. q-cocycle ξ over $Sd^1(K)$ having E_ξ as total space and the projection formed by the unions of p'_σ (the construction works because the various ξ_σ coincide on the boundary with the cocycles already built by induction).

Since we have $\xi(\sigma) = (Ag^q(\xi))(\sigma)$ for each $\sigma \in K$, Prop. 3 proves the theorem.

Prop. 5. If $K' \lhd K$, then $Ag^q: Hg^q(K') \to Hg^q(K)$ is an isomorphism.

Proof. If ξ/K is a g. q-cocycle over K, let r be the first integer such that $Sd^r(K) \lhd K'$ and let $\xi/Sd^r(K)$ be the g. q-cocycle obtained from ξ/K by means of repeated applications of Prop. 4. Then $\xi'/K' = Ag^q(\xi/Sd^r(K)) \in Hg^q(K')$ is such that $Ag^q(\xi'/K') = \xi/K$ and hence Ag^q is epimorphic.

If $\eta/K \times I$ is a cohomology between $Ag^q(\xi_1)/K$ and $Ag^q(\xi_2)/K$, the same technique works by using $Sd^r(K \times I)$, showing that Ag^q is one-to-one.

Prop. 5 shows the invariance of $Hg^q(K)$ under subdivisions; if K_1 and K_2 are subdivisions of the same polyhedron P, we denote the isomorphism between $Hg^q(K_1)$ and $Hg^q(K_2)$ again with Ag^q.

The subdivision theorem for geometric cocycles, together with the natural equivalence $\varphi_K: Hg^q(K) \to H^q(K)$ defined by (see [4])

$$[\varphi_K(\xi)](\sigma) = \sum_{A \in \xi(\sigma)} \varepsilon_A ,$$

allows to introduce the idea of subdivision for simplicial algebraic (sl.a.) cocycles on simplicial complexes and to deduce an algebraic subdivision theorem.

In fact, defining $A^q\colon H^q(K') \to H^q(K)$ as the isomorphism which makes the following diagram commute

$$\begin{array}{ccc} Hg^q(K') & \xrightarrow{\;Ag^q\;} & Hg^q(K) \\ \varphi_{K'} \big\downarrow & & \big\downarrow \varphi_K \\ H^q(K') & \xrightarrow{\;A^q\;} & H^q(K) \end{array}$$

we can describe amalgamation and subdivision in $H^q(-)$ in a very simple way.

Prop. 6. If $f \in H^q(Sd^1(K))$, then $A^q(f)\colon S_q(K) \to Z$ is defined by $[A^q(f)](\alpha) = \sum\limits_{\alpha^q \subset \alpha} f(\alpha^q)$, for each $\alpha \in S_q(K)$.

Now suppose $f \in H^q(K)$ and choose, for each $\sigma \in S_q(K)$, a q-simplex $\alpha_\sigma \in S_q(Sd^1(\sigma))$; let $\xi(\alpha_\sigma)$ be the union of $|f(\sigma)|$ points all having the sign of $f(\sigma)$ and $p_\xi\colon \xi(\alpha_\sigma) \to \mathring{\alpha}_\sigma$ an arbitrary map. If $\sigma \in S_{q+1}(K)$ and σ_i are its q-faces, let $\alpha_i = \alpha_{\sigma_i}$; since $f(\partial\sigma) = 0$, the points of $\bigcup\limits_{i=0}^{q+1} \xi(\sigma_i) = \bigcup\limits_{i=0}^{q+1} \xi(\alpha_i)$ are pairwise connectable. If (A, B) is a pair of connectable points, connect them by an oriented 1-simplex t and let a_{AB} be the sequence of simplexes of $Sd^1(\sigma)$ $\alpha_A = s_0^q, s_1^{q+1}, \ldots, s_{n-2}^q, s_{n-1}^{q+1}, s_n^q = \alpha_B$, where $s_i^q < s_{i-1}^{q+1}, s_{i+1}^{q+1}$; subdivide t by $(n/2 - 1)$ internal points C_i and define $p_t\colon t \to \sigma$ by setting

$$p_t(C_i) \in \mathring{s}^q_{2i}\,, \qquad p_t(A) = p_\xi(A)\,, \qquad p_t(B) = p_\xi(B)$$

and extending p_t linearly.

Repeating the construction for each pair (A, B) on σ and for each $\sigma \in S_{q+1}(K)$, we get a g. q-cocycle $\tilde{\xi}/Sd^1(K^{q+1})$; set

$$\tilde{f} = \varphi_{Sd^1(K^{q+1})}(\tilde{\xi}) = \varphi_{Sd^1(K)}(\tilde{\xi}) \in H^q(Sd^1(K))\,.$$

Prop. 7. If $f \in H^q(K)$ and $\tilde{f} \in H^q(Sd^1(K))$ is the sl.a. q-cocycle just built, then $A^q(\tilde{f}) = f$, i.e. $\tilde{f}$ is a subdivision of f.

The above description of the subdivision of a sl.a. q-cocycle allows an alternative representation of the subdivision of a given g. q-cocycle.

Prop. 8. If ξ/K is a g. q-cocycle over K and $f = \varphi_K(\xi)$, consider $\tilde{\xi}/Sd^1(K^{q+1})$ – as in the above construction – and let $\tilde{\tilde{\xi}}/Sd^1(K)$ be its cone extension ([4], prop. 6); then $\tilde{\tilde{\xi}}$ is a subdivision of ξ.

4 - Poincaré duality

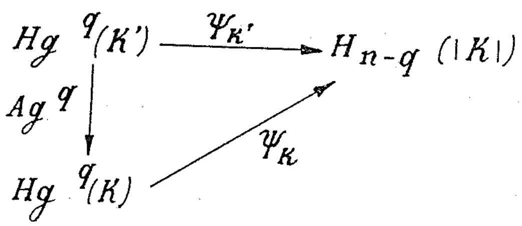
If K is an orientable n-cycle, we may assume a coherent orientation on its n-balls (Prop. 1).

Since a g. q-cocycle $\xi = (E_\xi, p_\xi)$ over an oriented n-cycle K is a sr.g. $(n-q)$-cycle in $|K|$ ([4], prop. 2), we may consider the *duality map* ψ_K: $Hg^q(K) \to H_{n-q}(|K|)$ defined by thinking of $(E_\xi, p_\xi) \in Hg^q(K)$ as an element of $H_{n-q}(|K|)$.

Prop. 9. The duality map ψ_K is a well-defined group homomorphism.

Proof. Let $\eta/K\times I$ be a cohomology between ξ_1/K and ξ_2/K; then E_η is an oriented $(n+1-q)$-cycle in $|K\times I|$ such that $\partial E_\eta = E_{\xi_1} \cup (-E_{\xi_2})$ ([4], prop. 2) and so $(E_\eta, \pi \circ p_\eta)$ – where $\pi: |K\times I| \to |K|$ is the natural projection – is a homology between the sr.g. $(n-q)$-cycles (E_{ξ_1}, p_{ξ_1}) and (E_{ξ_2}, p_{ξ_2}). The proof of $\psi(\xi_1) \cup^\cdot \psi(\xi_2) = \psi(\xi_1 \cup^\cdot \xi_2)$ is evident.

Prop. 10. If K is an oriented n-cycle and $K' \lhd K$, the following diagram is commutative



If M is a combinatorial n-manifold, let M^* denote the dual ball complex and $\alpha^* \in M^*$ the dual of α in M.

Prop. 11. Let (L, f) be a relative sl.g. q-cycle in an oriented n-manifold $(M, \partial M)$ $(q \leqslant n)$, such that $f: Sd^1(L) \to Sd^1(M)$ is also simplicial. Then (L, f) is a g. $(n-q)$-cocycle over M^* such that the $(n-q)$-blocks have the orientation induced by L.

Proof. We have to prove that, if $\dim \alpha = i$, $f^{-1}(\alpha^*)$ is an oriented $(q-i)$-cycle such that $\partial f^{-1}(\alpha^*) = f^{-1}(\partial\alpha^*)$.

A simple extension of ([2], 5.2) proves that $f^{-1}(\alpha^*) = \{b(\sigma_1) \dots b(\sigma_h) \,|\, \alpha \leqslant f(\sigma_1), \sigma_1 < \dots < \sigma_h \in L\}$ and $f^{-1}(\partial\alpha^*) = \{b(\sigma_1) \dots b(\sigma_h) \,|\, \alpha < f(\sigma_1), \sigma_1 < \dots < \sigma_h \in L\} \cup (f^{-1}(\alpha^*) \cap \partial L)$.

It is clear that each simplex of $f^{-1}(\alpha^*)$ is face of a $(q-i)$-simplex of $f^{-1}(\alpha^*)$. If $\beta = b(\sigma_1) \dots b(\sigma_{q-i-1})$ is a $(q-i-1)$-simplex of $f^{-1}(\alpha^*)$, β is face of exactly one $(q-i)$-simplex of $f^{-1}(\alpha^*)$ if $\beta \in Sd^1(\partial L)$ or $\alpha \neq f(\sigma_1)$, i.e. if $\beta \in f^{-1}(\partial\alpha^*)$; in any other case β is face of exactly two $(q-i)$-simplexes of $f^{-1}(\alpha^*)$. This is proved by the fact that there are exactly two h-faces in a $(h+1)$-simplex containing a given $(h-1)$-face.

Since L is oriented, all the blocks over M^* are orientable and so we may orient the balls of M^* and the blocks over them in such a way that the incidence condition holds.

Note that, since the n-balls of M^* are coherently oriented, the blocks over them have the orientation induced by L.

Prop. 11 is an extension of prop. 5.6 in [**2**] to cycles.

Prop. 12 (*Poincaré duality theorem*). If M is an oriented n-manifold, the duality map $\psi_M: Hg^q(M) \to H_{n-q}(|M|)$ is a group isomorphism.

Proof. (i) ψ_M is an epimorphism.

Let (L, f) be a simplicial approximation of a given $(P, g) \in H_{n-q}(|M|)$ in M such that $f: Sd^1(L) \to Sd^1(M)$ is also simplicial. Then, by Prop. 11, (L, f) is a g. q-cocycle over M^* and so, by Prop. 10, $Ag^q(L, f)$ is a g. q-cocycle ξ over M such that $\psi_M(\xi) = (P, g)$.

(ii) ψ_M is a monomorphism. Let (Q, F) be a homology between $\psi_M(\xi_1) = (L_1, f_1)$ and $\psi_M(\xi_2) = (L_2, f_2)$ such that $L_1 \cup^{\cdot} (-L_2) = \partial Q$ is collared in Q.
If $\bar{L}_1 = L_1 \times [0, \frac{1}{2}[$ and $\bar{L}_2 = L_2 \times]\frac{1}{2}, 1]$ are open collars of L_1 and L_2, consider $\bar{F}: Q \to |M \times I|$ defined by

$$\bar{F} = f_1 \times id: \bar{L}_1 \to M \times [0, \tfrac{1}{2}[\,,$$

$$\bar{F} = f_2 \times id: \bar{L}_2 \to M \times]\tfrac{1}{2}, 1]\,,$$

$$\bar{F} = F: Q - (\bar{L}_1 \cup \bar{L}_2) \to M \times \{\tfrac{1}{2}\}\,.$$

Let R, S be such that $|R| = Q$, $S \lhd M \times I$ and $\bar{F}: R \to S$ is simplicial; then, by Prop. 11, $(R, \bar{F})$ is a g. q-cocycle over S^* and, since $(R, \bar{F})\,|\,M \times \{0\} = \xi_1$ and $(R, \bar{F})\,|\,M \times \{1\} = -\xi_2$, the amalgamation of $(R, \bar{F})$ over $M \times I$ is a cohomology between ξ_1 and ξ_2.

References

[1] S. Buoncristiano, C. P. Rourke and B. J. Sanderson, *A geometric approach to homology theory*, London Math. Society, Lecture Note Series **18** (1975).

[2] M. M. Cohen, *Simplicial structures and transverse cellularity*, Ann. of Math. **85** (1967), 218-245.

[3] L. C. Glaser, *Geometrical combinatorial topology*, Van Nostrand Reinhold Math Studies, New York 1970.

[4] L. Grasselli, *Cocicli simpliciali geometrici*, Atti Sem. Mat. Fis. Univ. Modena **27** (1978), 67-77.

[5] J. F. P. Hudson, *Piecewise linear topology*, Benjamin, Inc., New York, Amsterdam 1969.

[6] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer-Verlag 1972.

[7] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, Leipzig 1934.

[8] E. C. Zeeman, *Seminar on combinatorial topology*, Inst. Hautes Etudes Sci. Publ. Math. 1963.

Riassunto

Si prova, in modo costruttivo, l'invarianza per suddivisioni dei gruppi di coomologia geometrica ed algebrica e se ne deriva una dimostrazione geometrica del teorema di dualità di Poincaré.

* * *