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Some remarks on optimal mild solutions of the differential equation $x'=Ax+f$ in Banach spaces (**)

Introduction

We consider in a uniformly convex Banach space X, the non-homogeneous differential equation

$$x'(t) = Ax(t) + f(t)\,, \quad -\infty < t < \infty\,, \tag{1}$$

where the closed linear operator A with domain $D(A)$, dense in X is the infinitesimal generator of a strongly continuous one-parameter operator semigroup T_t, $t \geqslant 0$ (see [2] for definition); $f(t)$: $-\infty < t < \infty \to X$ is a strongly continuous function.

This work is based on recent papers of professor S. Zaidman ($[4]_{1,2,3}$); in Theorem 1 we show the existence and uniqueness of an optimal mild solution of equation (1); and then, assuming $f(t)$ stronlgy almost-periodic, we prove weak almost-periodicity of the optimal mild solution in Theorem 2, generalizing somewhat Theorem 4.2 in $[4]_1$.

Let us recall some useful definitions.

Def. A strongly continuous function $x(t)$: $-\infty < t < \infty \to X$ with integral representation

$$x(t) = T_{t-t_0}x(t_0) + \int_{t_0}^{t} T_{t-\sigma}f(\sigma)\,d\sigma\,,$$

for all $t_0 \in R$ and all $t \geqslant t_0$, is called a *mild solution* of equation (1).

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Now let Ω_f be the set of all mild solutions $x(t)$ of (1) which are bounded over the real line, i.e. $\mu(x) = \sup_{t \in R} \|x(t)\| < \infty$, and assume $\Omega_f \neq \emptyset$.

Def. 2. We call an *optimal mild solution* of (1) every bounded mild solution $x(t)$ such that

$$\mu(\tilde{x}) = \mu^* = \inf_{x \in \Omega_f} \mu(x) .$$

Def. 3. A strongly continuous function $f(t)\colon -\infty < t < \infty \to X$ is called *strongly almost-periodic* if from every real sequence $(s'_n)_1^\infty$ we can extract a subsequence $(s_n)_1^\infty$ such that $\lim_{n \to \infty} f(t + s_n)$ exists in X in the strong sense, uniformly in $-\infty < t < \infty$.

Def. 4. $f(t)$ is *weakly almost-periodic* if from every real sequence $(s'_n)_1^\infty$ we can extract a subsequence $(s_n)_1^\infty$ such that $\lim_{n \to \infty} f(t + s_n)$ exists in X in the weak sense, uniformly in $-\infty < t < \infty$.

1 – Theorem 1. *Let us assume $f(t)$ strongly continuous over the real line and the operator A the infinitesimal generator of a strongly continuous one-parameter operator semi-group T_t such that* $\sup_{t \geqslant 0} \|T_t\| < \infty$.

Suppose also $\Omega_f \neq \emptyset$; then there exists a unique optimal mild solution of equation (1).

Remark. The proof is based on the following elementary fact: in a uniformly convex B-space X, if $K \subset X$ is a non-empty convex and closed subset and $v \notin K$, then there exists one and only one $k_0 \in K$ such that $\|v - k_0\| = \inf_{k \in K} \|v - k\|$ (see [**2**] Corollary 8.2.1).

Proof of Theorem 1. By the above remark, and because the trivial solution $\theta \notin \Omega_f$, it suffices to prove Ω_f is a convex and closed set, then there will exist a unique element $\tilde{x} \in \Omega_f$ such that $\mu(\tilde{x}) = \|\tilde{x}\| \leqslant \|x\| = \mu(x)$ for all $x \in \Omega_f$, i.e. $\mu(\tilde{x}) = \mu^*$.

It is very easy to show convexity of Ω_f. Consider two distinct bounded mild solutions $x_1(t)$ and $x_2(t)$, a number $0 \leqslant \lambda \leqslant 1$ and the continuous function $x(t) = \lambda x_1(t) + (1 - \lambda) x_2(t)$, $t \in R$.

$$x_i(t) = T_{t-t_0} x_i(t_0) + \int_{t_0}^{t} T_{t-\sigma} f(\sigma)\, d\sigma ,$$

for all $t_0 \in R$ and for all $t \geqslant t_0$, $i = 1, 2$. Then

$$x(t) = T_{t-t_0}(\lambda x_1(t_0) + (1-\lambda)x_2(t_0)) + \int_{t_0}^{t} T_{t-\sigma} f(\sigma)\, d\sigma$$

$$= T_{t-t_0} x(t_0) + \int_{t_0}^{t} T_{t-\sigma} f(\sigma)\, d\sigma\,,$$

which shows $x(t)$ is a mild solution. $x(t)$ is bounded over the real line because $\mu(x) = \sup_{t\in R} \|x(t)\| \leqslant \lambda\mu(x_1) + (1-\lambda)\mu(x_2) < \infty$. Therefore $x \in \Omega_f$ and consequently Ω_f is a convex set.

Now let us prove Ω_f is a closed set; consider an arbitrary sequence $(x_n(t))_1^\infty$ in Ω_f such that $\lim_{n\to\infty} x_n(t) = x(t) \in X$, $t \in R$; it suffices to show $x \in \Omega_f$. We have

$$x_n(t) = T_{t-t_0} x_n(t_0) + \int_{t_0}^{t} T_{t-\sigma} f(\sigma)\, d\sigma \qquad (n = 1, 2, \ldots).$$

Then
$$x(t) = T_{t-t_0} x(t_0) + \int_{t_0}^{t} T_{t-\sigma} f(\sigma)\, d\sigma$$

because
$$\lim_{n\to\infty} T_{t-t_0} x_n(t_0) = T_{t-t_0} \lim_{n\to\infty} x_n(t_0) = T_{t-t_0} x(t_0)$$

(we use the continuity of T_{t-t_0}). Therefore $x(t)$ is a mild solution. It is also bounded over the real line; in fact there exists a number $M > 0$ such that $\|T_t\| \leqslant M$ for all $t \geqslant 0$. Let us write

$$x(t) = T_{t-t_0} x(t_0) + \int_{t_0}^{t} T_{t-\sigma}\, f(\sigma)\, d\sigma - x_n(t) + x_n(t) = T_{t-t_0}[x(t_0) - x_n(t_0)] + x_n(t).$$

Then we have $\|x(t)\| \leqslant \|T_{t-t_0}[x(t_0) - x_n(t_0)]\| + \|x_n(t)\| \leqslant \|T_{t-t_0}\|\, \|x(t_0) - x_n(t_0)\| + \|x_n(t)\| \leqslant M\|x(t_0) - x_n(t_0)\| + \|x_n(t)\|$ and therefore $\|x(t)\| \leqslant M\|x(t_0) - x_n(t_0)\| + \mu(x_n)$. Choose n large enough such that $\|x(t_0) - x_n(t_0)\| < 1$. Then $\mu(x) \leqslant M + \mu(x_n) < \infty$. The theorem is proved.

2 - Theorem 2. *Let us assume the function $f(t)$ is strongly almost-periodic; the operator A is the infinitesimal generator of a strongly continuous one-parameter operator semi-group T_t such that $\sup_{t\geqslant 0} \|T_t\| < \infty$ and $T_t^* \in L(X^*, X^*)$ for all $t \geqslant 0$, where X^* is the dual space of X and T_t^* the adjoint operator of T_t; then every optimal mild solution of equation* (1) *is weakly almost-periodic.*

We use here a technique similar to the one in [4]$_1$ to prove Theorem 2.

Consider $w(t)$ an optimal mild solution; then $w(t) = T_{t-t_0} w(t_0) + \int_{t_0}^{t} T_{t-\sigma} f(\sigma)\, d\sigma$ for all $t_0 \in R$ and all $t \geqslant t_0$.

Let $(s_n)_1^\infty$ be an arbitrary real sequence; as every uniformly convex B-space is reflexive, using the definition of almost-periodicity of the function $f(t)$ and also properties of a reflexive B-space, we can find a subsequence $(s_{n_p})_1^\infty \subset (s_n)_1^\infty$ such that:

$$\lim_{p\to\infty} f(t+s_{n_p}) = g(t)$$

exists in the strong topology of X, uniformly in $-\infty < t < \infty$;

$$\lim_{p\to\infty} w(t_0+s_{n_p}) = w_0$$

exists in the weak topology of X, t_0 being fixed in R.

Consider the following (strongly) continuous function $\tilde{w}(t) = T_{t-t_0}w_0 + \int_{t_0}^{t} T_{t-\sigma}g(\sigma)\,d\sigma$. Then we have

Lemma 1. *Weak* $\lim_{p\to\infty} w(t+s_{n_p}) = \tilde{w}(t)$, *for every real number* t.

Proof. Consider the following representation (see [4]$_2$ Lemma 1)

$$w(t+s_{n_p}) = T_{t-t_0}w(t_0+s_{n_p}) + \int_{t_0}^{t} T_{t-\sigma}f(\sigma+s_{n_p})\,d\sigma \quad (p=1,2,\ldots).$$

Let x^* be arbitrary in X^*; then we get the equality

$$\langle x^*, T_{t-t_0}w(t_0+s_{n_p})\rangle - \langle x^*, T_{t-t_0}w_0\rangle = \langle T^*_{t-t_0}x^*, w(t_0+s_{n_p}) - w_0\rangle,$$

which shows the sequence $(T_{t-t_0}w(t_0+s_{n_p}))_1^\infty$ converges to $T_{t-t_0}w_0$ in the weak topology of X. We have also

$$\Big\|\int_{t_0}^{t} T_{t-\sigma}f(\sigma+s_{n_p})\,d\sigma - \int_{t_0}^{t} T_{t-\sigma}g(\sigma)\,d\sigma\Big\| = \Big\|\int_{t_0}^{t} T_{t-\sigma}[f(\sigma+s_{n_p}) - g(\sigma)]\,d\sigma\Big\|$$

$$\leqslant \int_{t_0}^{t} \|T_{t-\sigma}[f(\sigma+s_{n_p}) - g(\sigma)]\|\,d\sigma \leqslant \int_{t_0}^{t} \|T_{t-\sigma}\|\,\|f(\sigma+s_{n_p}) - g(\sigma)\|\,d\sigma$$

$$\leqslant M_{t,t_0}\cdot \int_{t_0}^{t} \|f(\sigma+s_{n_p}) - g(\sigma)\|\,d\sigma,$$

where $\|T_{t-\sigma}\| \leqslant M_{t,t_0}$ a constant which may depend on t and t_0, two fixed real numbers. Therefore

$$\lim_{p\to\infty} \int_{t_0}^{t} T_{t-\sigma} f(\sigma + s_{n_p})\, d\sigma = \int_{t_0}^{t} T_{t-\sigma} g(\sigma)\, d\sigma \text{ in the strong topology of } X.$$

The lemma is proved.

Lemma 2. $\mu(\tilde{w}) = \mu^*$.

Proof. $w(t)$ is an optimal mild solution, consequently we have $\mu^* = \mu(w) = \sup_{t\in R} \|w(t)\|$. By Lemma 1, we have for arbitrary $x^* \in X^*$ $\lim_{p\to\infty} \langle x^*, w(t+s_{n_p})\rangle = \langle x^*, \tilde{w}(t)\rangle$ for every $t \in R$. But for every $p = 1, 2, 3, \ldots$

$$|\langle x^*, w(t + s_{n_p})\rangle| \leqslant \|x^*\| \|w(t + s_{n_p})\|$$

$$\leqslant \|x^*\| \cdot \sup_{t\in R} \|w(t + s_{n_p})\| = \|x^*\| \cdot \sup_{t\in R} \|w(t)\| = \|x^*\| \cdot \mu^*.$$

Therefore $|\langle x^*, \tilde{w}(t)\rangle| \leqslant \|x^*\| \mu^*$, for every $t \in R$ and consequently $\|\tilde{w}(t)\| \leqslant \mu^*$, for every $t \in R$; finally we have $\mu(\tilde{w}) \leqslant \mu^*$.

Let us suppose $\mu(\tilde{w}) < \mu^*$.

Remark $\lim_{p\to\infty} g(t - s_{n_p}) = f(t)$ uniformly in $t \in R$. By the properties of a reflexive B-space we can extract a subsequence of $(s_{n_p})_1^\infty$ (we write it the same way) such that the sequence $(\tilde{w}(s_{n_p}))_1^\infty$ converges weakly to $z \in X$; then we have

$$\lim_{p\to\infty} \tilde{w}(t - s_{n_p}) = T_{t-t_0} z + \int_{t_0}^{t} T_{t-\sigma} f(\sigma)\, d\sigma = z(t),$$

in the weak topology of X, for every real t. The function $z(t)$ is a mild solution and, for the same reasons as above we have $\mu(z) \leqslant \mu(\tilde{w})$ therefore $\mu(z) < \mu^*$ which is absurd by definition of μ^*.

Lemma 3. *$\tilde{w}(t)$ is an optimal solution*, i.e. $\mu(\tilde{w}) = \inf_{v\in\Omega_g} \mu(v)$.

Proof. Let us suppose this is false; remark $\Omega_g \neq \emptyset$ for $w \in \Omega_g$, and there is uniqueness of the optimal solution by Theorem 1. Let $w_0(t)$ be this unique optimal mild solution, then $\mu(w_0) < \mu(\tilde{w})$, with

$$w_0(t) = T_{t-t_0} w_0(t_0) + \int_{t_0}^{t} T_{t-\sigma} g(\sigma)\, d\sigma.$$

Exactly as in Lemma 2, we can find a subsequence $(s_{n_p})_1^\infty$ and a function $V(t)$ such that

$$\lim_{p\to\infty} w_0(t-s_{n_p}) = T_{t-t_0}z + \int_{t_0}^{t} T_{t-\sigma}f(\sigma)\,d\sigma = V(t)\,,$$

in the weak topology of X.

Moreover we have $\mu(V)\leqslant\mu(w_0)<\mu(\tilde{w})$ with $V\in\Omega_f$, which is absurd.

Proof of Theorem 2. It suffices to prove

$$\lim_{p\to\infty} w(t+s_{n_p}) = \tilde{w}(t) \quad \text{in the weak topology of } X, \text{ uniformly in } t\in R\,.$$

In fact if this would not be true, there will exist $x^*\in X^*$ such that the limit $\lim\limits_{p\to\infty}\langle x^*, w(t+s_{n_p})\rangle = \langle x^*, \tilde{w}(t)\rangle$ is not uniform in t. And consequently we can find a number $\alpha>0$, a real sequence $(t_p)_1^\infty$ and two subsequences $(s'_{n_p})_1^\infty$, $(s''_{n_p})_1^\infty$ of $(s_{n_p})_1^\infty$ such that

$$(*) \qquad |\langle x^*, w(t_p+s'_{n_p}) - w(t_p+s''_{n_p})\rangle| > \alpha \qquad (p=1,2,\dots)\,.$$

Again extract two subsequences without changing the notations; using the almost-periodicity of $f(t)$, we get

$$\lim_{p\to\infty} f(t+t_p+s'_{n_p}) = g_1(t)\,, \qquad \lim_{p\to\infty} f(t+t_p+s''_{n_p}) = g_2(t)$$

uniformly in $t\in R$. As in the beginning of the proof we extract two subsequences and get the sequences $(w(t+t_p+s'_{n_p}))_1^\infty$ and $(w(t+t_p+s''_{n_p}))_1^\infty$ which converge respectively in the weak topology of X to the optimal mild solutions in Ω_{g_1} and Ω_{g_2}

$$\tilde{w}_1(t) = T_{t-t_0}\tilde{w}_1 + \int_{t_0}^{t} T_{t-\sigma}g_1(\sigma)\,d\sigma\,, \qquad \tilde{w}_2(t) = T_{t-t_0}\tilde{w}_2 + \int_{t_0}^{t} T_{t-\sigma}g_2(\sigma)\,d\sigma\,.$$

Now we have $g_1(\sigma)=g_2(\sigma)$, $\sigma\in R$; in fact $\lim\limits_{p\to\infty} f(t+s_{n_p})$ exists uniformly in $t\in R$ and $(s'_{n_p})_1^\infty\subset(s_{n_p})_1^\infty$, $(s''_{n_p})_1^\infty\subset(s_{n_p})_1^\infty$, therefore $\sup\limits_{\tau\in R}\|f(\tau+s'_{n_p})-f(\tau+s''_{n_p})\|<\varepsilon$ if $p\geqslant p_0(\varepsilon)$, and consequently $\sup\limits_{t\in R}\|f(t+t_p+s'_{n_p})-f(t+t_p+s''_{n_p})\|<\varepsilon$, $p\geqslant p_0(\varepsilon)$ which shows the equality $g_1(\sigma)=g_2(\sigma)$, $\sigma\in R$.

By the uniqueness of optimal mild solution we have $\tilde{w}_1(t) = \tilde{w}_2(t)$, $t \in R$. But $\tilde{w}_1(0) = \text{weak}\lim_{p\to\infty} w(t_p + s'_{n_p})$ and $\tilde{w}_2(0) = \text{weak}\lim_{p\to\infty} w(t_p + s''_{n_p})$.

The equality $\tilde{w}_1(0) = \tilde{w}_2(0)$ contradicts then inequality (*). Theorem is proved.

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References

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