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Positive absolutely continuous solutions of neutral functional-differential equations (**)

The aim of the present paper is to find sufficient conditions for the existence and uniqueness of positive solutions of neutral functional-differential equations. Such a problem occurs when studying some biological, epidemical and physical processes. In [**6**] and [**7**] have been proved local existence theorems for ordinary and delay differential equations by using flow-invariant sets [**2**], [**3**]. Here we shall use a suitable contraction mapping theorem from [**1**] in order to obtain conditions for the existence and uniqueness of the global absolutely continuous solution of the following initial value problem

$$(1)\qquad \begin{aligned} &y'(t) = G\big(t, y(\Delta_1(t)), \dots, y(\Delta_m(t)), y'(\theta_1(t)), \dots, y'(\theta_n(t))\big) \qquad (t > 0)\,, \\ &y(t) = \psi(t)\,, \quad y'(t) = \psi'(t) \quad (t \leqq 0)\,, \end{aligned}$$

where $y(t)$ is the unknown function, while $G(t, u_1, \dots, u_m, v_1, \dots, v_n)$, $\psi(t)$, $\Delta_i(t)$ and $\theta_k(t)$ are the given functions. Further on, we will assume that the initial function $\psi(t)$ is non-negative. Note that depending on the choice of the functions $\Delta_i(t)$ and $\theta_k(t)$ the equation (1) can be of retarded, advanced or mixed type.

Setting $x(t) = y'(t)$ for $t > 0$ and $\varphi(t) = \psi'(t)$ for $t \leqq 0$, we obtain

$$(2)\qquad \begin{aligned} &x(t) = G\Big(t, \psi(0) + \int_0^{\Delta_1(t)} x(s)\,ds, \dots, \psi(0) + \int_0^{\Delta_m(t)} x(s\,ds, x(\theta_1(t)), \dots, x(\theta_n(t))\Big) \quad (t > 0)\,, \\ &x(t) = \varphi(t) \quad (t \leqq 0)\,. \end{aligned}$$

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Introduce the following notations:

$$R^1 = (-\infty, \infty), \quad R^1_+ = [0, \infty), \quad R^1_- = (-\infty, 0],$$

$$R^m = \underbrace{R^1 \times R^1 \times \ldots \times R^1}_{m}, \quad R^m_+ = \underbrace{R^1_+ \times R^1_+ \times \ldots \times R^1_+}_{m}.$$

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, while $H_1 \subset E_1$ and $H_2 \subset E_2$ are convex, closed sets for which $\mu H_k \subseteq H_k$ $(k = 1, 2)$, $0 \leqq \mu \leqq 1$.

Theorem 1 [1]. *Assume that the following conditions hold.*

(1) *On H_k the nonlinear Lipschitz continuous operators $N_k\colon H_k \to H_k$ are defined,* i.e. $\|N_k x - N_k y\|_k \leqq \omega \|x - y\|_k$ *for any $x, y \in H_k$ where the constant $\omega \geqq 1$, $(k = 1, 2)$.*

(2) *On E_1 the linear map $j\colon E_1 \to E_2$ is defined such that if $x \in H_1$, then $jx \in H_2$ and $j(N_1 x) = N_2(jx)$.*

Then for every $x \in H_1$ and $y \in H_2$, for which the relation $jx = ((\omega + 1)I - N_2)y$ is fulfilled, a unique element $z \in H_1$ exists satisfying the equalities $((\omega + 1)I - N_1)z = x$ and $jz = y$.

Theorem 2 [1]. *Let the following conditions hold.*

(1) *The conditions of Theorem 1 are fullfilled.*

(2) *For $x, \tilde{x} \in H_1$ and $y, \tilde{y} \in H_2$ the equalities $jx = ((\omega + 1)I - N_2)y$ and $j\tilde{x} = ((\omega + 1)I - N_2)\tilde{y}$ hold.*

(3) *For $z, \tilde{z} \in H_1$ the following relations hold*

$$((\omega + 1)I - N_1)z = x, \quad ((\omega + 1)I - N_1)\tilde{z} = \tilde{x}.$$

Then the inequality holds, as follows

$$\|z - \tilde{z}\|_1 \leqq \|x - \tilde{x}\|_1.$$

Let us note that the set $D = \{f(t) \in L^1(R^1)\colon f(t)$ is bounded and continuous$\}$ is dense in $L^1(R^1)$ ([4], Lemma 19, p. 298).

Define the sets H_1 and H_2 in the following manner: $H_1 = \{f(t) \in L^1(R^1)\colon \psi(0) + \int_0^t f(s)\,ds \geqq 0$ for every $t \in R^1_+\}$; H_2 consists of all restrictions of the functions $f \in H_1$, on the semiaxis R^1_-.

Lemma 1. *The set H_1 is closed convex and $\mu H_1 \subseteq H_1$ for $0 \leqq \mu \leqq 1$.*

Proof. It is easily verified that the set H_1 is convex and $\mu H_1 \subseteqq H_1$ $(0 \leqq \mu \leqq 1)$. We shall show that H_1 is closed.

Let $\{f_n(t)\}_{n=1}^{\infty}$ be a sequence of elements from H_1, converging to the function $f(t) \in L^1(R^1)$ and $\|f - f_n\|_{L^1} < \varepsilon$ for $n > n_0$, $\varepsilon > 0$. Then the chain of inequalities

$$\int_0^t f_n(s)\,ds - \int_0^t f(s)\,ds \leqq \int_0^t |f_n(s) - f(s)|\,ds \leqq \int_{-\infty}^{\infty} |f_n(s) - f(s)|\,ds < \varepsilon$$

yields

$$\int_0^t f(s)\,ds \geqq \int_0^t f_n(s)\,ds - \varepsilon \geqq -\psi(0) - \varepsilon\,, \quad \text{i.e.} \quad \psi(0) + \int_0^t f(s)\,ds \geqq 0\,,$$

for every $t \in R_+^1$, which completes the proof of Lemma 1.

The set H_2 also satisfies the conditions of Theorem 1.

If we show that the problem (2) has a unique solution $x(t)$, belonging to the set H_1, then taking into account the setting $y(t) = \psi(0) + \int_0^t x(s)\,ds$, we conclude that the problem (1) has a unique positive absolutely continuous global solution.

Theorem 3. *Let the following conditions hold.*

(1) *The functions* $\Delta_i(t)\colon R_+^1 \to R^1$ $(i = 1, \dots, m)$, $\theta_s(t)\colon R_+^1 \to R^1$ $(s = 1, \dots, n)$ *are measurable, and for any* $f(t) \in D$ $\int_0^{\infty} |f(\theta_s(t))|\,dt \leqq k\int_{-\infty}^{\infty} |f(t)|\,dt$, *where* $k > 0$ *is a constant.*

(2) *The function* $G(t, \dots, v_n)\colon R_+^1 \times R^{m+n} \to R^1$ *satisfies Caratheodory condition (measurable in the first variable and continuous in the other variables) and conditions*

$$|G(t, u_1, \dots, u_m, v_1, \dots, v_n)| \leqq \frac{1}{\omega + 1}\left[\alpha_0(t, |u_1|, \dots, |u_m|) + \sum_{s=1}^{n} \alpha_s(t) v_s\right],$$

$$|G(t, u_1, \dots, u_m, v_1, \dots, v_n) - G(t, \bar{u}_1, \dots, \bar{u}_m, \bar{v}_1, \dots, \bar{v}_n)|$$

$$\leqq \frac{\omega}{\omega + 1}\left[\beta_0(t, |u_1 - \bar{u}_1|, \dots, |u_m - \bar{u}_m|) + \sum_{s=1}^{n} \beta_s(t) |v_s - \bar{v}_s|\right],$$

$$G(t, u_1, \dots, u_m, v_1, \dots, v_n) \geqq \frac{1}{\omega + 1}\,\xi(t)\,,$$

where $\omega \geqq 1$ *is a constant, the functions* $\alpha_0(t, x_1, \dots, x_m)\colon R_+^{m+1} \to R_+^1$, $\beta_0(t, x_1, \dots, x_m)\colon R_+^{m+1} \to R_+^1$ *are non-decreasing in* x_i *and satisfy Caratheodory condi-*

tion, $\alpha_0(\cdot, x_1, \ldots, x_m) \in L^1(R^1_+)$ *for fixed* $(x_1, x_2, \ldots, x_m)$, $\beta_0(t, x, \ldots, x) \leqq \bar{\beta}_0(t)x$, *where* $\bar{\beta}_0(t) \in L^1(R^1_+)$. *The functions* $\alpha_s(t)\colon R^1_+ \to R^1_+$, $\beta_s(t)\colon R^1_+ \to R^1_+$ $(s = 1, \ldots, m)$, $\xi(t)\colon R^1_+ \to R^1$ *are measurable,* $\alpha_s(t)$, $\beta_s(t) \in L^\infty(R^1_+)$, $\psi(0) + \int\limits_0^t \xi(s)\,ds \geqq 0$ *for* $t \in R^1_+$ *and* $\int\limits_0^\infty \bar{\beta}_0(t)\,dt + k\sum\limits_{s=1}^n \bar{\beta}_s \leqq 1$, $\bar{\beta}_s = \operatorname{ess\,sup}\{\beta_s(t)\colon t \in R^1_+\}$ $(s = 1, 2, \ldots, n)$.

(3) *The initial function* $\varphi(t) \in L^1(R^1_-)$.

Then there exists a unique solution $x(t) \in H_1$ *of the problem* (2).

Proof. Let E_1 be the Banach space $L^1(R^1)$, while E_2 is the Banach space $L^1(R^1_-)$ with norms $\|f\|_1 = \int\limits_{-\infty}^{\infty} |f(t)|\,dt$, $\|g\|_2 = \int\limits_{-\infty}^{0} |g(t)|\,dt$ correspondingly.

As Lemma 1 shows, the set H_k $(k = 1, 2)$ are closed, convex and $\mu H_k \subseteqq H_k$. The set $H_1^D = H_1 \cap D$ is dense in H_1.

Let the operators $N_1\colon H_1^D \to H_1$, $N_2\colon H_2 \to H_2$ be defined by the formulas

$$(N_1 f)(t) = \begin{cases} (\omega + 1)G\Big(t, \psi(0) + \int\limits_0^{\Delta_1(t)} f(s)\,ds, \ldots, \psi(0) + \int\limits_0^{\Delta_m(t)} f(s)\,ds, f(\theta_1(t)), \ldots & \\ f(\theta_n(t))\Big) & t > 0 \\ 0 & t \leqq 0 \end{cases}$$

for $f \in H_1^D$; $(N_2 g)(t) = 0$, $t \leqq 0$, $g \in H_2$.

The map j is defined as a restriction of the function $f(t) \in H_1$ on the semi-t axis R^1_-.

We shall show that if the function $f(t) \in H_1^D$, then the function $(N_1 f)(t) \in H_1$.

The function $(N_1 f)(t)$ is measurable since $f(t) \in D$, i.e. $f(t)$ is continuous and $f(\theta_s(t))$ is measurable. Besides

$$|(N_1 f)(t)|$$
$$\leqq \alpha_0\Big(t, \Big|\psi(0) + \int\limits_0^{\Delta_1(t)} f(s)\,ds\Big|, \ldots, \Big|\psi(0) + \int\limits_0^{\Delta_m(t)} f(s)\,ds\Big|\Big) + \sum_{s=1}^n \alpha_s(t)|f(\theta_s(t))|$$
$$\leqq \alpha_0(t, \psi(0) + \|f\|_1, \ldots, \psi(0) + \|f\|_1) + \sum_{s=1}^n \bar{\alpha}_s |f(\theta_s(t))|\,,$$

where $\bar{\alpha}_s = \operatorname{ess\,sup}\{\alpha_s(t)\colon t \in R^1_+\}$ $(s = 1, 2, \ldots, n)$.

It is easy to verify that $(N_1 f)(t) \geqq \xi(t)$ and therefore

$$\int\limits_0^t (N_1 f)(t)\,dt \geqq \int\limits_0^t \xi(t)\,dt \geqq -\psi(0)\,, \quad \text{i.e. } \psi(0) + \int\limits_0^t (N_1 f)(s)\,ds \geqq 0\,.$$

The Lipschitz continuity of the operator N_1 (for N_2 this is quite obvious) follows from the inequalities

$$\int_0^\infty |(N_1 f)(t) - (N_1 g)(t)|\,dt$$

$$\leqq \omega\Big[\int_0^\infty \beta_0\Big(t, \Big|\int_0^{\Delta_1(t)} |f(s) - g(s)|\,ds\Big|, \dots, \Big|\int_0^{\Delta_m(t)} |f(s) - g(s)|\,ds\Big|\Big)\,dt$$

$$+\int_0^\infty \sum_{s=1}^{n} \beta_s(t)\,|f(\theta_s(t)) - g(\theta_s(t))|\,dt\Big]$$

$$\leqq \omega\Big[\int_0^\infty \beta_0(t, \|f-g\|_1, \dots, \|f-g\|_1)\,dt + k\sum_{s=1}^{n} \bar{\beta}_s \|f-g\|_1\Big]$$

$$\leqq \Big[\int_0^\infty \bar{\beta}_0(t)\,dt + k\sum_{s=1}^{n} \bar{\beta}_s\Big]\|f-g\|_1 \leqq \omega\|f-g\|_1 .$$

Appling Theorem 17, p. 23 [4] we extend the operator N_1 over H_1, i.e. $N_1\colon H_1 \to H_1$.

So, in this way we have established that all the conditions of Theorem 1 are fulfilled. If we define the function $h(t)$ by the formula

$$h(t) = \begin{cases} 0 & t > 0 \\ (\omega + 1)\varphi(t) & t \leqq 0 \end{cases}$$

then there exists a unique function $x(t) \in H_1$ for which $\big((\omega+1)I - N_1\big)x(t) = h(t)$ and $(jx)(t) = \varphi(t)$, i.e. $x(t)$ is a solution of the problem (2).

The Theorem is thus proved.

As a consequence of Theorem 2 we obtain the following stability result.

Theorem 4. *If the conditions of Theorem* 1 *are fulfilled and* $x(\varphi_1)(t)$ *and* $x(\varphi_2)(t)$ *are solution of a problem* (2) *with initial functions* $\varphi_1(t)$ *and* $\varphi_2(t)$, *then*

$$\int_{-\infty}^{\infty} |x(\varphi_1)(t) - x(\varphi_2)(t)|\,dt \leqq (\omega + 1)\int_{-\infty}^{0} |\varphi_1(t) - \varphi_2(t)|\,dt .$$

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Summary

In the present paper the authors, by the help of a contraction mapping theorem, have proved a theorem for the existence and uniqueness of the global positive absolutely continuous solution of neutral type functional-differential equations.

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