Riv. Mat. Univ. Parma (4) 9 (1983), 179-184

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On the existence of $J(4,2)$-structures (**)

1 - Introduction

We study the existence and obstructions to the existence of $J(4,2)$-structures in the sense of Yano, Houh and Chen [3] by using the definitions of Bernard [1], and the Stiefel-Whitney, Chern, Euler and Pontrjagin characteristic classes.

2 - Existence

Let V be a C^∞ differentiable manifold, with $\dim V = n = 2k$, and let J be a C^∞ $(1,1)$ tensor field on V, $J \neq 0$, such that

$$J^4 + J^2 = 0\,, \qquad \operatorname{rank} J = \tfrac{1}{2}(\operatorname{rank} J^2 + n)\,;$$

if J is of constant rank equal to r, J is called a $J(4,2)$-*structure of rank* r and V a $J(4,2)$-*manifold.*

The operators $l = -J^2$, $m = J^2 + 1$ are the projectors of the almost product structure on V given by $H = m - l$. Let L and M be the corresponding complementary distributions. The tangent bundle TV of V admits the decomposition $TV = L \oplus M$, where L and M are vector bundles of respective dimensions $2r - n$ and $2n - 2r$. We have also $\ker J = J(M)$, and $\dim(\ker J)_x = n - r = \frac{1}{2}\dim M_x\,,\ x \in V$.

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(**) Ricevuto: 13-VII-1981.

If rank $J = n$, then $J^2 + 1 = 0$, and thus a $J(4,2)$-structure of maximal rank is an almost complex structure. If rank $J = n/2$, then $J^2 = 0$, and hence a $J(4,2)$-structure of minimal rank is an almost tangent structure.

A $J(4,2)$-structure is nothing but a G-structure on V such that the group G is the group of matrices of the form

$$\begin{pmatrix} A & -B & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & D & C \end{pmatrix}$$

where $A + iB \in Gl(r-k; \mathbf{C})$, $C \in Gl(n-r; \mathbf{R})$, and $D \in M(n-r; \mathbf{R})$, where $M(n-r; \mathbf{R})$ is the set of real $(n-r)\times(n-r)$ matrices. We then have, $N(G)$ being the normaliser of G:

Proposition 2.1. *$N(G) \neq G$, and thus, if there exists such a structure on V, then there exist infinitely many of such structures, all associated, in Bernard's sense* [**1**].

Proof. We consider the matrix

$$h = \begin{pmatrix} I_{r-k} & 0 & 0 & 0 \\ 0 & -I_{r-k} & 0 & 0 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & 0 & Z \end{pmatrix},$$

where I_{r-k} and I_{n-r} denote the identity matrices of degree $r-k$ and $n-r$ respectively, and $Z = pI_{n-r}$, $p \in \mathbf{R} - \{0, 1\}$; obviously, $h \notin G$ and $h \in N(G)$. Indeed, if

$$g = \begin{pmatrix} A & -B & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & D & C \end{pmatrix} \in G, \qquad \text{then} \qquad g' = \begin{pmatrix} A & B & 0 & 0 \\ -B & A & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & Z^{-1}D & C \end{pmatrix}$$

is such that $g' \in G$ and $gh = hg'$. Hence $G \neq N(G)$.

Corollary 2.2. *If there exists a $J(4,2)$-structure on the manifold V, then all the associated G-structures are in bijective correspondence with the set of differentiable sections of the bundle FV/G, where FV denotes the frame bundle of V.*

3 - Obstructions to existence

With the notations of **2**, let K be the subbundle ker J, and S a subbundle of M complementary to K. K and S are isomorphic bundles of dim $n-r$, because J induces a vector bundle isomorphism between them.

The real vector bundle L can be made a complex vector bundle on V of complex dim $= r-k$; it suffices to define

$$(p+iq)X = pX + qJX\,, \qquad p, q \in \mathbf{R},\ X \in L_x,\ x \in V.$$

We shall call L both this bundle and the underlying real vector bundle ($L_{\mathbf{R}}$).

Proposition 3.1. *Let V be a $J(4,2)$-manifold, with* $\dim V = n = 2k$. *Then the odd Stiefel-Whitney classes of TV, L and M are all zero.*

Proof. The Whitney product theorem applied to $TV = L \oplus M$, gives

$$w_h(TV) = w_h(L \oplus M) = \sum_{i=0}^{h} w_i(L) \smile w_{h-i}(M)\,. \tag{3.1}$$

Since L can be made a complex vector bundle on V, the mod 2 reduction of the total Chern class $c(L)$ is the Stiefel-Whitney class of the underlying real vector bundle, and thus the odd Stiefel-Whitney classes of L are zero. Hence $w_i(L) = 0$ if i is odd.

On the other hand, $M = K \oplus S$, and hence

$$w_j(M) = w_j(K \oplus S) = \sum_{a=0}^{j} w_a(K) \smile w_{j-a}(S)\,,$$

but, since K and S are isomorphic, we have $w_{j-a}(K) = w_{j-a}(S)$ and

$$w_j(M) = \begin{cases} 2\sum_{a=0}^{[j/2]} w_a(K) \smile w_{j-a}(K) = 0 & \text{if } j \text{ odd} \\ 2\sum_{a=0}^{j/2} w_a(K) \smile w_{j-a}(K) + w_{j/2}(K)^2 = w_{j/2}(K)^2 & \text{if } j \text{ even}\,. \end{cases}$$

Suppose now that h is odd and let i be such that $0 \leqslant i \leqslant h$. Then i is odd or $h-i$ is odd. In the first case, $w_i(L) = 0$; in the second, $w_{h-i}(M) = 0$. Hence, it follows from (3.1) that $w_h(TV) = 0$.

In particular, $w_1(TV) = 0$, and follows the

Corollary 3.2. *A $J(4,2)$-manifold is orientable.*

With regard to the even Stiefel-Whitney classes we have

Proposition 3.3. *Let V be a $J(4,2)$-manifold with* $\dim V = n = 2k$; *then the even Stiefel-Whitney classes have the expression*

$$w_{2h}(TV) = \sum_{i+j=h} (c_i(L))_2 \smile w_j(K)^2,$$

where $h = 1, \ldots, k$, and $(\cdot)_2$ denotes reduction mod 2.

Proof. Since $M = K \oplus S$ and K is isomorphic to S we have

$$w_{2j}(M) = 2\sum_{a=0}^{2j} w_a(K) \smile w_{2j-a}(K) + w_j(K)^2 = w_j(K)^2,$$

from which, since $w_i(L) = w_i(M) = 0$ if i is odd,

$$w_{2h}T(V) = \sum_{2i+2j=2h} w_{2i}(L) \smile w_{2j}(M) = \sum_{i+j=h} (c_i(L))_2 \smile w_j(K)^2.$$

Proposition 3.4. *Let V be an oriented manifold with* $\dim V = n = 2k$, *and suppose that the Euler class $e(TV)$ is non null. If V admits a $J(4,2)$-structure of* rank r, *then $H^{2r-n}(V;\mathbf{Z}) \neq 0$ and $H^{2n-2r}(V;\mathbf{Z}) \neq 0$. Furthermore, if $H^1(V;\mathbf{Z}_2) = 0$, then $H^{n-r}(V;\mathbf{Z}) \neq 0$.*

Proof. If there exists a $J(4,2)$-structure of rank r on V, we have $e(TV) = e(L) \smile e(M)$, where $e(L) \in H^{2r-n}(V;\mathbf{Z})$, and $e(M) \in H^{2n-2r}(V;\mathbf{Z})$. Since $e(TV) \neq 0$, we deduce $e(V) \neq 0$ and $e(M) \neq 0$. If $H^1(V;\mathbf{Z}_2) = 0$, then $w_1(K) = 0$, $w_1(S) = 0$, and thus K and S are orientable. Hence, $e(M) = e(K) \smile e(S)$, where $e(K)$, $e(S) \in H^{n-r}(V;\mathbf{Z})$ and are non null.

Since, in these conditions, the bundles K and S are orientable, we have $e(M) = e(K) \smile e(K) = e(K)^2$; on the other hand, the top Chern class of a complex vector bundle coincides with the Euler class of the underlying real vector bundle, and hence $e(L) = c_{r-k}(L)$. Thus we obtain

Proposition 3.5. *If V is a $J(4,2)$-manifold with* $\dim V = n = 2k$, *the Euler class $e(TV)$ has the expression $e(TV) = c_{r-k}(L) \smile e(M)$.*

Furthermore, if $H^1(V; \mathbf{Z}_2) = 0$, *then*

$$e(TV) = c_{r-k}(L) \smile e(K)^2 .$$

Proposition 3.6. *Let* V *be a* $J(4, 2)$*-manifold with* $\dim V = n = 2k$. *Then the Pontrjagin classes have the expression*

$$2p_h T(V) = \sum_{i+j=h} \{c_i(L)^2 + 2 \sum_{a=1}^{i} (-1)^a c_{i-a}(L) \smile c_{i+a}(L)\} \smile 2 \sum_{b=0}^{j} p_b(K) \smile p_{j-b}(K) .$$

Proof. Let $\bar{L}$ be the conjugate bundle of L. Then $L \otimes_{\mathbf{R}} \mathbf{C}$ is isomorphic to $L \oplus \bar{L}$; hence, since $c_j(\bar{L}) = (-1)^j c_j(L)$, we have

$$p_i(L) = (-1)^i c_{2i}(L \otimes_{\mathbf{R}} \mathbf{C}) = (-1)^i c_{2i}(L \oplus \bar{L}) = (-1)^i \sum_{a+b=2i} c_a(L) \smile c_b(\bar{L})$$

$$= (-1)^i \{2c_0(L) \smile c_{2i}(L) - 2c_1(L) \smile c_{2i-1}(L)$$

$$+ 2c_2(L) \smile c_{2i-2}(L) - \ldots + \ldots + (-1)^{i+1} 2c_{i-1}(L) \smile c_{i+1}(L) + (-1)^i c_i(L)^2\}$$

$$= c_i(L)^2 + 2 \sum_{a=1}^{i} (-1)^a c_{i-a}(L) \smile c_{i+a}(L) .$$

On the other hand, since $M = K \oplus S$,

$$2p_j(M) = 2p_j(K \oplus S) = 2 \sum_{b+c=j} p_b(K) \smile p_c(S)$$

$$= 2 \sum_{b+c=j} p_b(K) \smile p_c(K) = 2 \sum_{b=0}^{j} p_b(K) \smile p_{j-b}(K) .$$

The result follows from the relation

$$2p_h(TV) = 2p_h(L \oplus M) = \sum_{i+j=h} p_i(L) \smile 2p_j(M) .$$

Proposition 3.7. *Let* V be a $J(4, 2)$*-manifold with* $\dim V = n = 2k$.

Then

$$\big(p_h(TV)\big)_2 = \sum_{i+j=h} w_{2i}(L)^2 \smile w_j(K)^4 .$$

Proof. Since reduction mod 2 of the Pontrjagin class $p_h(TV)$ is equal to the square of the Stiefel-Whitney class $w_{2h}(TV)$, we have

$$(p_h(TV))_2 = w_{2h}(T(V))^2 = (w_{2h}(L \oplus M))^2 = \Big(\sum_{i+j=2h} w_i(L) \smile w_j(M)\Big) \smile \Big(\sum_{a+b=2h} w_a(L) \smile w_b(M)\Big)$$

$$= \sum_{i+j=2h} \; \sum_{a+b=2h} \big(\{w_i(L) \smile w_a(L)\} \smile \{w_j(M) \smile w_b(M)\}\big) .$$

But, since the odd Stiefel-Whitney classes of L are zero by Prop. 3.1, we have

$$\sum_{i=0}^{2h} \; \sum_{a=0}^{2h} w_i(L) \smile w_a(L) = 1 + w_1(L) \smile w_1(L) + \ldots + w_{2h}(L) \smile w_{2h}(L)$$

$$= \sum_{i=a=0}^{h} w_{2i}(L) \smile w_{2a}(L) ,$$

and hence

$$(p_h(TV))_2 = \sum_{j=h-i} \; \sum_{b=h-a} \Big(\Big\{ \sum_{i=a=0}^{h} w_{2i}(L) \smile w_{2a}(L)\Big\} \smile \{w_{2j}(M) \smile w_{2b}(M)\}\Big)$$

$$= \sum_{i+j=h} w_{2i}(L)^2 \smile w_{2j}(M)^2 = \sum_{i+j=h} w_{2i}(L)^2 \smile w_j(K)^4 .$$

References

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