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M. IRFAN (*)

On limits in a super comma category (**)

1 - Introduzione

C. Pellegrino [6] has proved that if $(F_1 \downarrow F_2)$ is a comma category, with $F_i\colon C_i \to C$, then the canonical functor from $(F_1 \downarrow F_2) \to C_1 \times C_2$ creates those limits (colimits) which the functor $F_2(F_1)$ preserves. Here we study the category $(\text{Cat} \cdot\!\downarrow\!\cdot\, C)$, known as « Super comma category ».

Initially the concept of a super comma category $(\text{Cat} \cdot\!\downarrow\!\cdot\, C)$ was introduced by S. Eilenberg and S. Maclane [2] and was, later on, generalized by D. M. Kan [3]. This category is also known as « Large Diagram Category » in the sense of B. Pareigis [5]. Here Cat denotes the category of small categories and C is an arbitrary category. The objects of this category are pairs (J, F), where J is a small category and $F\colon J \to C$ is a functor, and morphisms $(W, \beta)\colon (J, F) \to (J', F')$ are those pairs consisting of a functor $W\colon J \to J'$ and a natural transformation $\beta\colon F'W \to F$. If morphisms

$$(W, \beta)\colon (J, F) \to (J', F') \quad \text{and} \quad (W', \beta')\colon (J', F') \to (J'', F'')$$

are given, then the composition of these morphisms is given by $(W'W, \beta\beta'W)$.

Define a functor $Q\colon (\text{Cat} \cdot\!\downarrow\!\cdot\, C) \to \text{Cat}$, called *the projection of the comma category*, as: Q sends each object (J, F) of $(\text{Cat} \cdot\!\downarrow\!\cdot\, C)$ to J of cat and each morphism $(W, \beta)\colon (J, F) \to (J', F')$ to $W\colon J \to J'$.

(*) Indirizzo: Dept. of Mathematics, Aligarh Muslim University, Aligarh-202001, India.

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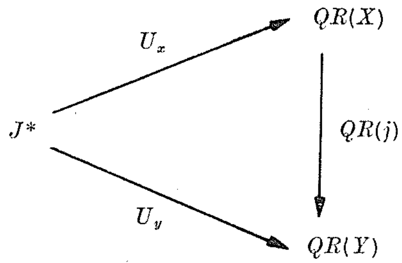
In this note we study the creation, preservation and reflection of limits by the projection Q. However, the corresponding results for colimits do not hold. Prof. G. M. Kelly, University of Sydney, Australia, also agrees with the author's conjecture that such results need not be true.

2 - Limits

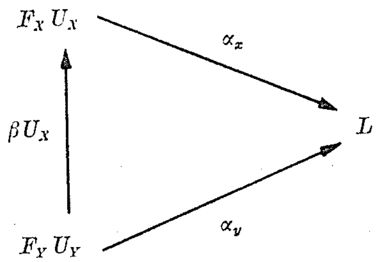
We first prove the following lemma.

Lemma 2.1. *The projection* $Q\colon (\mathrm{Cat}\,\dot{\downarrow}\,C) \to \mathrm{Cat}$ *creates* J*-limits of functors from a small category* J *if* C *is a* J*-cocomplete category* (cfr. [**5**]).

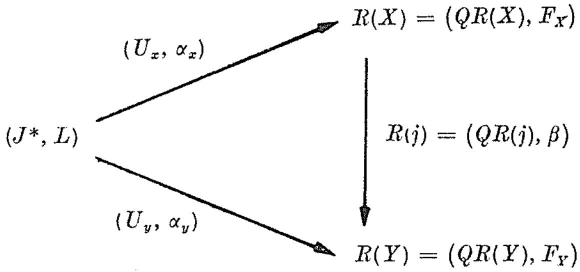
Proof. Let J be a small category and $R\colon J \to (\mathrm{Cat}\,\dot{\downarrow}\,C)$ be a functor. Assume that J^* with $U_X\colon J^* \to QR(X)$, defined for each X in J, is the limit of QR; then for Y in J and $j\colon X \to Y$ the following diagram commutes



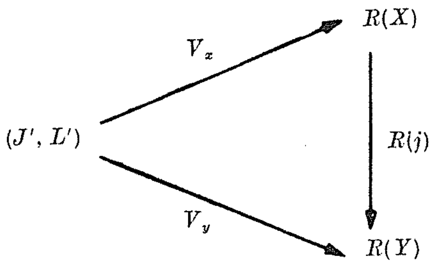
As C is a category with J-colimits, then so is the category C^{J^*}. Let us denote the functors $QR(X) \to C$, $QR(Y) \to C$ by F_X and F_Y respectively and the natural transformation $F_Y\; QR(j) \to F_X$ by β; then we have $\beta U_X\colon F_Y \cdot QR(j)\, U_X \to F_X U_X$. Assume that L with $\alpha_X\colon F_X U_X \to L$, defined for each X in J, is the colimit of $F\_U\_\colon J \to C^{J^*}$ with $X \to F_X U_X$; then if we replace $QR(j)\, U_X$ by U_Y, we obtain the commutative diagram



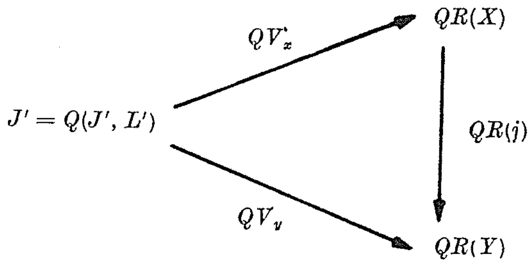
It, therefore, follows that (J^*, L) is an object in $(\text{Cat} \cdot\!\downarrow\!\cdot\, C)$ with (U_X, α_X), (U_Y, α_Y) as morphisms such that the following diagram is commutative



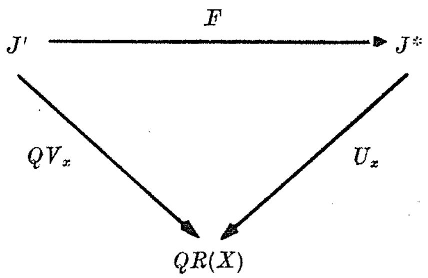
It is to be proved that (J^*, L) with $(U_X, \alpha_X): (J^*, L) \to R(X)$ is the limit of R. In view of this consider the commutative diagram



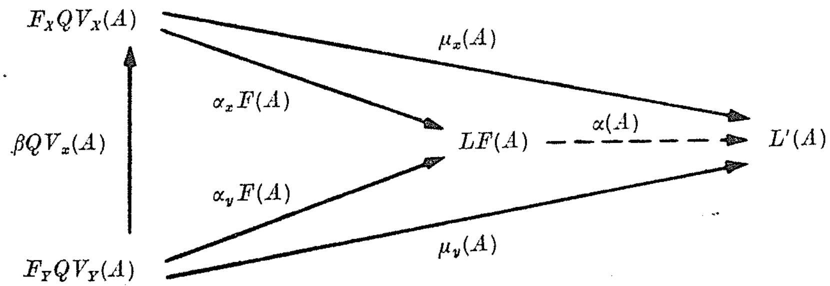
This yields the commutative diagram



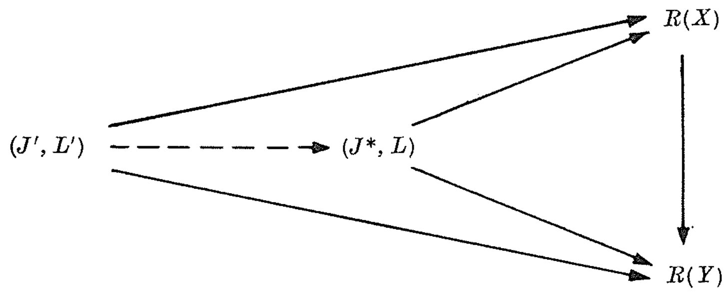
So that there is a unique morphism, say, $F: J' \to J^*$ making the following diagram commutative



Set $V_X = (QV_X, \mu_X)$, $V_Y = (QV_Y, \mu_Y)$ where $\mu_X: F_XQV_X \to L'$ and $\mu_Y: F_YQV_Y \to L'$. Also, we have $\alpha_X F: F_XU_XF \to LF$, $\alpha_Y F: F_YU_YF \to LF$. Replace U_XF by QV_X, U_YF by QV_Y, so that we obtain $\alpha_X F: F_XQV_X \to LF$, $\alpha_Y F: F_YQV_Y \to LF$. For some $A \in J'$. If we consider the diagram



then there exists a unique morphism, say, $\alpha(A): LF(A) \to L'(A)$ such that $\alpha(A)\alpha_X F(A) = \mu_X(A)$ and $\alpha(A)\alpha_Y F(A) = \mu_Y(A)$. This all means that we have thus obtained a unique morphism $(F, \alpha): (J', L') \to (J^*, L)$ making the following diagram commutative.



This completes the proof of the lemma.

The following corollary is an immediate consequence of the above result.

Corollary 2.2. *If C is a J-cocomplete category then the functor Q: (Cat $\cdot\!\downarrow\!\cdot$ C) $\to$ Cat reflects J-limits.*

Corollary 2.3. *If C is a J-cocomplete category then the functor Q: (Cat $\cdot\!\downarrow\!\cdot$ C) $\to$ Cat preserves J-limits.*

Proof. It is trivial (cfr. [4], th. 5.4.2).

Theorem 2.4. *If C is a cocomplete category then the functor Q*: (Cat $\dot{\downarrow}$ C) $\to$ Cat *creates limits.*

Proof. It is obvious in view of the above lemma.

References

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[**3**] D. M. Kan, *Adjoint functors,* Trans. Amer. Math. Soc. **87** (1958), 294.

[**4**] S. MacLane, *Categories for the working mathematician,* Springer-Verlag, New York 1971.

[**5**] B. Pareigis, *Categories and functors,* Academic Press, New York and London 1970.

[**6**] C. Pellegrino, *Creazione di limiti nelle categorie-comma,* Riv. Mat. Univ. Parma (3) **3** (1974), 201-204.

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