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Quasiumbilical anti-invariant submanifolds (**)

1 - Introduction

B. Y. Chen and K. Yano proved that *every totally quasiumbilical submanifold* (1) *of a conformally flat space is conformally flat* [7]. Concerning the converse, it is known for allready a long time that *every conformally flat hypersurface of a conformally flat space is quasiumbilical* [3], [9]. This property was generalized by J. D. Moore and J. M. Morvan as follows: *every conformally flat submanifold* M^n *of a conformally flat space* $\tilde{M}^{n+p}$ *with codimension* $p \leqslant \min \{4, n-3\}$ *is totally quasiumbilical* [8]. For possibly higher codimension B. Y. Chen and one of the authors showed that *every conformally flat submanifold* M^n *of a conformally flat space* $\tilde{M}^{n+p}$ *with* $p \leqslant n-3$ *and flat normal connection is totally quasiumbilical* [6].

Recently one of the authors proved that *every totally quasiumbilical totally real submanifold of a Bochner-Kaehler space is conformally flat* [10]; for totally geodesic submanifolds see [1], and for totally umbilical submanifolds see [11]. In this direction we also mention the following theorem of K. Yano: *every totally real submanifold* M^n *with commutative second fundamental tensors in a Bochner-Kaehler manifold* $\tilde{M}^{2n}$ *is conformally flat* $[11]_2$. Based on a characterization for the conformal flatness of totally real submanifolds of Bochner-Kaehler spaces, this result will be generalized in 3. In 4 we will give some results of this type for anti-invariant submanifolds of Sasakian manifolds with vanishing C-Bochner curvature tensor.

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(1) All manifolds in this paper are assumed to be of dimension $\geqslant 4$.

2 - Preliminaries (see also [4])

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}$. The second fundamental form of M in $\tilde{M}$ will be denoted by h; its components are given by h_{ij}^{α} whereby we agree on the following ranges of indices: $i, j, k, l \in \{1, 2, \dots, n\}$ and $\alpha \in \{1, 2, \dots, p\}$. By $\tilde{R}$ and R we respectively mean the curvature tensors of $\tilde{M}$ and M, and by $R^{\perp}$ the curvature tensor of the normal connection of M in $\tilde{M}$. When the ambient space $\tilde{M}$ is conformally flat, M has flat normal connection ($R^{\perp} \equiv 0$) if and only if all second fundamental tensors A_{ξ} associated with normal sections ξ are simultaneously diagonalizable $[4]_2$. The *equation of Gauss* may be written as

$$\tilde{R}_{ijkl} = R_{ijkl} - D_{ijkl}, \qquad D_{ijkl} = \sum_{\alpha} (h_{il}^{\alpha} h_{jk}^{\alpha} - h_{ik}^{\alpha} h_{jl}^{\alpha}) . \tag{1}$$

Let

$$L_{ij} = -\frac{1}{n-2} S_{ij} + \frac{r}{2(n-1)(n-2)} \delta_{ij} , \tag{2}$$

whereby S and r are respectively the Ricci tensor and the scalar curvature of M. Then with respect to an orthonormal frame the *conformal curvature tensor* C of M is given by

$$C_{ijkl} = R_{ijkl} + \delta_{il} L_{jk} - \delta_{jl} L_{ik} + \delta_{jk} L_{il} - \delta_{ik} L_{jl} , \tag{3}$$

and by a theorem of H. Weyl M is conformally flat if and only if $C \equiv 0$.

A normal section ξ is called *quasiumbilical* if the principal curvatures of M corresponding to ξ, in other words the eigenvalues of A_{ξ}, are given by $\mu_{\xi}, \lambda_{\xi}, \dots, \lambda_{\xi}$ where λ_{ξ} occurs $n-1$ times. In particular, ξ is said to be a *cylindrical, umbilical or geodesic* section when respectively $\lambda_{\xi} = 0$, $\lambda_{\xi} = \mu_{\xi}$ or $\lambda_{\xi} = \mu_{\xi} = 0$. M^n is called a *totally quasiumbilical* submanifold of $\tilde{M}^{n+p}$ if there exist p mutually orthogonal quasiumbilical normal sections on M.

3 - Totally real submanifolds of Bochner-Kaehler spaces

Let $\tilde{M}^{2m}$ be a (real) $2m$-dimensional Kaehler manifold with complex structure J. With respect to an orthonormal frame the *Bochner curvature tensor* $\tilde{B}$ of $\tilde{M}$ is defined by $[11]_2$

$$\begin{aligned} \tilde{B}_{ABCD} = {} & \tilde{R}_{ABCD} + \delta_{AD} N_{BC} - \delta_{BD} N_{AC} + \delta_{BC} N_{AD} - \delta_{AC} N_{BD} + J_{AD} N'_{BC} \\ & - J_{BD} N'_{AC} + J_{BC} N'_{AD} - J_{AC} N'_{BD} - 2(J_{AB} N'_{CD} + J_{CD} N'_{AB}) , \end{aligned} \tag{4}$$

whereby $A, B, C, D \in \{1, 2, \ldots, 2m\}$,

$$N_{AB} = -\frac{1}{2(m+2)}\tilde{S}_{AB} + \frac{\tilde{r}}{8(m+1)(m+2)}\delta_{AB}, \qquad N'_{AB} = \sum_C N_{AC}J_{CB}, \tag{5}$$

and $\tilde{S}$ and $\tilde{r}$ are respectively the Ricci tensor and the scalar curvature of $\tilde{M}$. $\tilde{M}$ is said to be *Bochner flat* or is called a *Bochner-Kaehler space* when $\tilde{B} \equiv 0$.

Let M^n be a *totally real* or *anti-invariant submanifold of* $\tilde{M}^{2m}$, i.e. $\forall x \in M$, $J(T_xM) \subset T_x^\perp M$ [**5**], [**12**] (and therefore essentially $n \leqslant m$). We choose an orthonormal frame $\{E_A\}$ on $\tilde{M}$ such that $\{E_1, \ldots, E_n\}$ is a basis of TM and in this section agree on the following ranges of indices: $i, j, k, l, s, t \in \{1, 2, \ldots, n\}$ and $\alpha, \beta \in \{n+1, n+2, \ldots, 2m\}$. Then, making use of the equation of Gauss, (4) becomes

$$\tilde{B}_{ijkl} = R_{ijkl} - D_{ijkl} + \delta_{il}N_{jk} - \delta_{jl}N_{ik} + \delta_{jk}N_{il} - \delta_{ik}N_{jl}. \tag{6}$$

Contraction of (6) gives

$$\tilde{b}_{jk} = S_{jk} - D_{jk} + (n-2)N_{jk} + N\delta_{jk}, \tag{7}$$

whereby $\tilde{b}_{jk} = \sum_s \tilde{B}_{sjks}$, $D_{jk} = \sum_s D_{sjks}$ and $N = \sum_s N_{ss}$. Contraction of (7) yields

$$\tilde{b} = r - D + 2(n-1)N, \tag{8}$$

whereby $\tilde{b} = \sum_s \tilde{b}_{ss}$ and $D = \sum_s D_{ss}$. From (6), (7) and (8) we find that (see also [**12**])

$$\tilde{B}_{ijkl} = C_{ijkl} - D_{ijkl} - \frac{1}{(n-1)(n-2)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})(D + \tilde{b}) \tag{9}$$

$$+ \frac{1}{n-2}(\delta_{il}\tilde{b}_{jk} - \delta_{jl}\tilde{b}_{ik} + \delta_{jk}\tilde{b}_{il} - \delta_{ik}\tilde{b}_{jl} + \delta_{il}D_{jk} - \delta_{jl}D_{ik} - \delta_{jk}D_{il} - \delta_{ik}D_{jl}).$$

Proposition. *Let M^n, $n \geqslant 4$, be a totally real submanifold of a Bochner-Kaehler manifold $\tilde{M}^{2m}$ with commuting second fundamental tensors. Then M^n is conformally flat if and only if*

$$\sum_\alpha (\varrho_i^\alpha - \varrho_j^\alpha)(\varrho_k^\alpha - \varrho_l^\alpha) = 0, \tag{$*$}$$

for mutually different i, j, k, l where ϱ_i^α are the eigenvalues of $A_\alpha = A_{E_\alpha}$.

Proof. If $\forall\alpha, \beta$: $[A_\alpha, A_\beta] = 0$, then we can choose an orthonormal basis of TM such that $h^\alpha_{ij} = \varrho^\alpha_i \delta_{ij}$. Moreover, by the Bochner flatness of $\tilde{M}$, (9) becomes

$$(10)\qquad C_{ijkl} = D_{ijkl} + \frac{1}{n-2}(\delta_{jl}D_{ik} - \delta_{il}D_{jk} + \delta_{ik}D_{jl} - \delta_{ik}D_{jl}) + \frac{D}{(n-1)(n-2)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) .$$

Putting $a_{ij} = \sum_\alpha \varrho^\alpha_i \varrho^\alpha_j$, we get

$$(11)\qquad D_{ijkl} = a_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) , \qquad D_{jk} = (\sum_{t\neq j} a_{tj})\,\delta_{ik} , \qquad D = \sum_{t\neq s} a_{ts} .$$

Consequently

$$(12)\qquad C_{ijkl} = [(n-1)(n-2)a_{ij} - (n-1)(\sum_{t\neq i} a_{ti} + \sum_{t\neq j} a_{tj}) + 2\sum_{t<s} a_{ts}]\,\frac{\delta_{il}\delta_{jk} - \delta_{ik}\,\delta_{jl}}{(n-1)(n-2)} .$$

Now the proof can be finished in the same way as the proof of Theorem 1 in [**6**].

The proof of the next theorem is based on the following algebraic lemma.

Lemma 2 [**6**]. *Let A_α be $q < n-2$ diagonal matrices of order $n \geqslant 4$ whose eigenvalues $\varrho^\alpha_1, \ldots, \varrho^\alpha_n$ satisfy* (*) *for mutually different i, j, k, l. Then by transformations of the type*

$$\tilde{A}_\alpha = A_\alpha \cos\theta + A_\beta \sin\theta , \qquad \tilde{A}_\beta = -A_\alpha \sin\theta + A_\beta \cos\theta ,$$

matrices $\tilde{A}_\alpha$ can be obtained such that each $\tilde{A}_\alpha$ has an eigenvalue of multiplicity $\geqslant n-1$.

Suppose that M^n is a conformally flat totally real submanifold of a Bochner-Kaehler space $\tilde{M}^{2m}$, $m = n + p$, with $[A_\alpha, A_\beta] = 0$. Choosing an orthonormal frame $\{E_A\}$ on $\tilde{M}$ such that $\{E_i\}$ is a basis of TM which simultaneously diagonalises all A_α and such that $E_{i*} = E_{n+i} = JE_i$, it follows that $E_{1*}, \ldots, E_{n*}$ are cylindrical normal sections (since $h^{k*}_{ij} = h^{i*}_{kj} = h^{j*}_{ik}$ [**5**]). Thus from Proposition 1 and Lemma 2 we have the following

Theorem 3. *Let M^n, $n \geqslant 4$, be a conformally flat totally real submanifold of a Bochner-Kaehler manifold $\tilde{M}^{2(n+p)}$ with commuting second fundamental tensors. Then, if $2p < n-2$, M^n is totally quasiumbilical.*

In case $2p \geqslant n-2$, either a conformally flat totally real submanifold M^n with $[A_\alpha, A_\beta] = 0$ in a Bochner-Kaehler space $\tilde{M}^{2(n+p)}$ is totally quasiumbilical or we can use Proposition 3 of [6] to prove that with respect to a suitable frame the second fundamental tensors take the following form

$$
\begin{aligned}
&A_{n+j} = D(0, \ldots, 0, \varrho^j, 0, \ldots, 0)\,,\\
&A_{2n+u} = D(\varrho^{n+u}, \ldots, \varrho^{n+u}, \varrho^{n+u}_{u+1}, \bar{\varrho}^{n+u}, \ldots, \bar{\varrho}^{n+u}) \qquad (u \in \{1, 2, \ldots, n-2\})\,,\\
(13)\quad &A_{3n-1} = D(\varrho^{2n-1}, \ldots, \varrho^{2n-1}, \varrho^{2n-1}_n)\,,\\
&A_{3n} = D(\varrho^{2n}, \ldots, \varrho^{2n})\,,\\
&A_{3n+v} = 0 \qquad (v \in \{1, 2, \ldots, 2p-n\})\,,
\end{aligned}
$$

whereby ϱ^j and ϱ^{n+u}_{u+1} are the j-th and the $(u+1)$-th element, respectively, and

$$
(14)\qquad D(a_1, \ldots, a_n) = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}.
$$

In particular, for a real space form $M^n(c)$ of constant sectional curvature c with $[A_\alpha, A_\beta] = 0$ in a complex space form $\tilde{M}^{2(n+p)}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$, for $i \neq j$ we have $a_{ij} = c - \tilde{c}/4$, such that by the same argument as the one used in the proof of Corollary 1 of [6], M is totally quasiumbilical when $2p \leqslant n-2$. Because the normal sections E_{i*} are cylindrical and $2p \leqslant n-2$, a_{ij} being constant for $i \neq j$ implies that the normal sections E_α are cylindrical or umbilical. By transformations as in Lemma 2 we therefore obtain the following (see also [6])

Proposition 4. *Let $M^n(c)$, $n \geqslant 4$, be a real space form immersed in a complex space form $\tilde{M}^{2(n+p)}(c)$ as a totally real submanifold with commuting second fundamental tensors. If $2p \leqslant n-2$, then there exist locally $n+2p$ mutually orthogonal unit normal vector fields $\xi_1, \ldots, \xi_{n+2p-1}, \xi_{n+2p}$ such that $M^n(c)$ is cylindrical with respect to $\xi_1, \ldots, \xi_{n+2p-1}$ and cylindrical or umbilical with respect to ξ_{n+2p}. Thus $c \geqslant \tilde{c}/4$.*

4 - Anti-invariant submanifolds of Sasakian manifolds with vanishing contact Bochner curvature tensor

Let $\tilde{M}^{2m+1}$ be a Sasakian manifold with structure tensors (φ, ξ, η, g) $[1]_1$, $[11]_3$, [12]. Analogeous to the Bochner curvature tensor for a Kaehler manifold, with respect to an orthonormal frame $\{E_A\}$, $(A, B, C, D \in \{1, 2, \ldots, 2m+1\})$, the *contact Bochner curvature tensor* or *C-Bochner curvature tensor* for $\tilde{M}^{2m+1}$ is defined by [**12**]

$$\tilde{B}_{ABCD} \tag{15}$$

$$= \tilde{R}_{ABCD} + (\delta_{AD} - \eta_A\eta_D)P_{BC} - (\delta_{BD} - \eta_B\eta_D)P_{AC} + (\delta_{BC} - \eta_B\eta_C)P_{AD}$$

$$- (\delta_{AC} - \eta_A\eta_C)P_{BD} + \varphi_{AD}P'_{BC} - \varphi_{BD}P'_{AC} + \varphi_{BC}P'_{AD} - \varphi_{AC}P'_{BD}$$

$$- 2(\varphi_{AB}P'_{CD} + \varphi_{CD}P'_{AB}) + \varphi_{AD}\varphi_{BC} - \varphi_{BD}\varphi_{AC} - 2\varphi_{AB}\varphi_{CD}\,,$$

whereby

$$P_{AB} = -\frac{1}{2(m+2)}\,[\tilde{S}_{AB} + (P+3)\,\delta_{AB} - (P-1)\eta_A\eta_B]\,,$$

$$P = \sum_A P_{AA} = -\frac{\tilde{r} + 2(3m+2)}{4(m+1)}\,, \qquad P'_{AB} = \sum_C P_{AC}\varphi_{CB}\,.$$

Let M^n be an *anti-invariant submanifold* of $\tilde{M}^{2m+1}$, i.e. $\forall x \in M$, $\varphi(T_xM) \subset T_x^{\perp}M$, (which since rank $\varphi = 2m$ implies that $n \leqslant m+1$). When M is normal to the structure vector field ξ, then M is automatically anti-invariant and $n \leqslant m$ (in fact, then M is an integral submanifold of the contact distribution defined by $\eta = 0$ [**1**]$_2$; such submanifolds are also called C-totally real submanifolds). When ξ is tangent to M, then M is anti-invariant if and only if ξ is parallel along M [**12**]. We will consider these two cases separately.

(I) *ξ is normal to M.*

Then by computation of the Gauss equation involving the C-Bochner curvature tensor of $\tilde{M}$ and the Weyl conformal curvature tensor of M in an analogeous was as in **3**, the following result can be obtained by a proof similar to the one given in [**10**].

Theorem 5. *Let M^n, $n \geqslant 4$, be a totally quasiumbilical submanifold of a Sasakian space $\tilde{M}^{2m+1}$ with vanishing C-Bochner curvature tensor such that M is normal to the structure vector field of $\tilde{M}$. Then M is conformally flat.*

Next we assume that M has commuting second fundamental tensors. If we choose the orthonormal frame $\{E_A\}$ such that $E_1, \ldots, E_n$ are principal directions on M such that $E_{n+i} = \varphi E_i$ and $E_{2m+1} = \xi$ then each E_{n+i} is a cylindrical normal section and E_{2m+1} is a geodesic section [**12**]. Therefore the following results can be obtained in the same way as those in 3, (now $\alpha \in \{n+1, n+2, \ldots, 2m+1\}$).

Proposition 6. *Let M^n, $n \geqslant 4$, be a submanifold of a Sasakian space $\tilde{M}^{2m+1}$ normal to the structure vector field ξ. If $\tilde{M}$ has vanishing C-Bochner curvature tensor and M has commuting second fundamental tensors, then M is conformally flat if and only if* (*) *holds for mutually different i, j, k, l.*

Theorem 7. *Let M^n, $n \geqslant 4$, be a conformally flat submanifold of a Sasakian manifold $\tilde{M}^{2m+1}$ normal to the structure vector field ξ, $m = n + p$. If $\tilde{M}$ has vanishing C-Bochner curvature tensor, M has commuting second fundamental tensors and $2p < n - 2$, then M is totally quasiumbilical.*

When $2p \geqslant n - 2$ in Theorem 7 M is totally quasiumbilical or the second fundamental tensors take particular forms as in 3. Moreover if M is a real space form of constant sectional curvature c immersed in a Sasakian space form of constant φ-sectional curvature $\tilde{c}$ normal to ξ and $\forall \alpha, \beta$: $[A_\alpha, A_\beta] = 0$, then $a_{ij} = c - \tilde{c}/4$ for $i \neq j$ such that $M^n(c)$ is totally quasiumbilical in $\tilde{M}^{2(n+p)+1}(\tilde{c})$ if $2p \leqslant n - 2$. In this case $\{E_\alpha\}$ can be chosen such that all E_α are cylindrical except possibly the last one which may be umbilical.

(II) *ξ is tangent to M.*

Let M^{n+1} be an anti-invariant submanifold which is tangent to the structure vector field ξ of a Sasakian manifold $\tilde{M}^{2m+1}$, $m = n + p$. By the parallellism of ξ along M, M locally is a Riemannian direct product $M'^n \times \mathscr{C}$ where M' is a totally geodesic hypersurface of M and $\mathscr{C}$ is a curve generated by ξ. By choosing an orthonormal frame $\{E_A\}$ on $\tilde{M}$, $(A \in \{0, 1, \ldots, 2m\})$, such that $E_0 = \xi$ and $\{E_w\}$ is a basis of TM', $(w, y, z \in \{1, 2, \ldots, n\})$, we can prove in the same way as above that when $\tilde{M}^{2m+1}$ has vanishing C-Bochner curvature tensor and M'^n, $n \geqslant 4$, is totally quasiumbilical, then M' is conformally flat. If we take $\{E_A\}$ such that $E_{x^*} = E_{n+x} = \varphi E_x$, then we have $(\lambda \in \{2n+1, 2n+2, \ldots, 2m\})$

$$A_{x^*} = \begin{bmatrix} 0 & 0 & \ldots & 1 & \ldots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 1 & & & H_{x^*} & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}, \qquad A_\lambda = \begin{bmatrix} 0 & 0 \\ 0 & H_\lambda \end{bmatrix}, \tag{17}$$

where $H_{x^*} = (h^{x^*}_{yz})$, $H_\lambda = (h^\lambda_{yz})$ and 1 is the $(x+1)$-th element of the first row and column of A_{x^*}. In particular, if $\forall \alpha, \beta \in \{n+1, n+2, \ldots, 2m\}$,

$[H_\alpha, H_\beta] = 0$, we can choose an orthonormal frame such that

$$H_{x^*} = D(0, \ldots, 0, \varrho_x, 0, \ldots, 0) . \tag{18}$$

Therefore, when in this case $\tilde{M}$ has vanishing C-Bochner curvature tensor, $2p < n - 2$ and M' is conformally flat, then M' is totally quasiumbicical.

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