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GIUSTINA PICA (*)

Generalized chromatic numbers of some graphs (**)

1 - Introduction

Generalizations of the chromatic number of a graph have been given by G. Chartrand, D. P. Geller and S. Hedetniemi [2], F. Speranza [5], S. Antonucci [1] and M. Gionfriddo [3].

We will adopt the definition in [1]$_1$ which includes almost all the definitions mentioned above. Let A be a set of natural numbers. The A-chromatic number of G, $\gamma_A(G)$, is the smallest number of colours needed to colour the vertices of the graph G so that the distance between two vertices with the same colour is not in A.

If d denotes the diameter of G and $D = \{1, 2, ..., d\}$, then of course, $\gamma_A(G)$ depends just on $A \cap D$. It is trivially verified that, for every graph G, $\gamma_A(G)=1$ iff $A \cap D = \emptyset$ and that $\gamma_A(G) = |V(G)|$ iff $A \cap D = D$ and G is connected. In particular, if G is the clique K_n, then $d = 1$ and $\gamma_A(G)=n$ or 1 (depending on whether or not $1 \in A$).

We observe that if G is either the clique with a hamiltonian cycle removed $K_n(-1)$, or G is a cycle C_n, it is easy to find D. If G is one of these two graphs and $A \cap D = \{1, 2, ..., s\}$, $\gamma_A(G)$ is known [1], [5].

We now will determine the A-chromatic numbers of $K_n(-1)$ if $n > 4$ and

$$A \cap D = \{2, 4, ...\}, \qquad A \cap D = \{1, 3, ...\}, \qquad A \cap D = \{c\}$$

of C_n if $A \cap D = \{c\}$ and $n > 3$. Let $\gamma'(G)$, $\gamma''(G)$ and $\gamma_c(G)$ denote these numbers, respectively.

(*) Indirizzo: Istituto di Matematica, Facoltà di Ingegneria, Via Claudio 21, 80125 Napoli, Italy.

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We call the corrispondent colourings even-colourings, odd-colourings and c-colourings, respectively.

Furthermore, if G is a tree, $\gamma'(G)$ and $\gamma''(G)$ are known [**1**]. We will determine $\gamma_c(G)$ for trees.

2 - The chromatic numbers of $K_n(-1)$ and C_n

Since $D(K_n(-1)) = \{1, 2\}$, then $\gamma_2 = \gamma'$ and $\gamma_1 = \gamma''$. Let $\lceil k \rceil$ and $\lfloor k \rfloor$ denote the least integer greater than or equal to k, and the greatest integer not greater than k, respectively. Then we have the following theorem.

Theorem 1.
$$\gamma'(K_n(-1)) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd}, \end{cases}$$

$$\gamma''(K_n(-1)) = \lceil n/2 \rceil .$$

Proof. Let the vertices of $K_n(-1)$ be labeled $1, 2, \dots, n$ according to their order in the removed hamiltonian cycle. Then for each i, $d(v_i, v_{i+1}) = 2$, while $d(v_i, v_j) = 1$ for all j such that $j \neq i-1, i, i+1 \pmod n$. Thus in a odd-colouring at most two vertices (whose two indices differ by 1 modulo n) may have the same colour, so that $\gamma''(K_n(-1))$ is not less than $\lceil n/2 \rceil$. On the other hand, if we assign the colour $\lfloor (i-1)/2 \rfloor$ to the vertices v_i a $\lceil n/2 \rceil$ colouring results.

Similarly, for $\gamma'(K_n(-1))$, at most $\lfloor n/2 \rfloor$ vertices may have the same colour (two vertices in the same colour class must have non-consecutive indices) so that $\gamma'(K_n(-1)) \geqslant 2$ or 3 according to n is even or odd. To achieve such a colouring, assign to vertex v_{2t+1} the colour 0, to vertex v_{2t+2} the colour 1, for $0 \leqslant t \leqslant \lfloor n/2 \rfloor - 1$, and to vertex v_n the colour 2 when n is odd.

Consider now the n-cycle C_n. If v_i is a vertex of C_n, then $d(v_i, v_{i+1}) = 1$, $d(v_i, v_{i+2}) = 2, \dots, d(v_i, v_{i+h}) = h$, where the addition is mod n and $h = \lfloor n/2 \rfloor$. So $D(C_n) = \{1, 2, \dots, \lfloor n/2 \rfloor\}$.

Theorem 2. *Let c be an integer non greater than h and let $m = n/(n, c)$* (1).

(1) (n, c) is the greatest common divisor of n and c.

Then

$$\gamma_c(C_n) = \begin{cases} 2 & \textit{if } m \textit{ is even} \\ 3 & \textit{if } m \textit{ is odd}. \end{cases}$$

Proof. Label the vertices $v_1, v_2, \ldots, v_n$ according to their order in the circuit and for any i consider the cycle $C(i)$ of distinct vertices: $C(i) = (v_i, v_{i+c}, v_{i+2c}, \ldots, v_{i+mc} = v_i)$. $C(i)$ has length $m = n/(n, c)$ and any pair of consecutive vertices in the cycle must have different colours. Thus if m is odd [respectively even], three [resp. two] colours are necessary for an usual colouring of $C(i)$.

On the other hand, we note that if $C(i) \neq C(j)$ no vertex in $C(i)$ has distance c from any vertex in $C(j)$. So that after colouring $C(1)$ we may use the same two (or three) colours to colour $C(2)$ in the same way and so on, achieving the required colouring.

3 - The c-chromatic number $\gamma_c(T_n)$ of a tree T_n

Let $P(v, w)$ be the path of T_n between the vertices v and w, $J_h(v) = \{v_h \in V(T_n)/d(v, v_h) = h\}$, and $I_h(v)$ a subset of $J_h(v)$ such that $P(v, v_h) \cap P(v, v_h') = \{v\}$ for every two vertices, v_h and v_h'. Furthermore, let $\bar{v}$ be one of the two central vertices of T_n and $r(T_n)$ be its radius [4]. For every two paths, P and Q, $P \cap Q$ denotes the vertex set intersection.

Lemma. If $\bar{k}' \leqslant \bar{k} \leqslant k$, *then the distance between two vertices* $v_{\bar{k}} \in J_{\bar{k}}(v)$ *and* $v_{\bar{k}'} \in J_{\bar{k}'}(v)$, *having distance* c *from a given vertex* $v_k \in J_k(v)$, *is less than* c.

Proof. Set $P(v, v_k) \cap P(v, v_{\bar{k}}) = \{v, v_1, \ldots, v_h\}$, $P(v, v_k) \cap P(v, v_{\bar{k}'}) = \{v, v_1, \ldots, v_{h'}\}$ and $P(v, v_{\bar{k}}) \cap P(v, v_{\bar{k}'}) = \{v, v_1, \ldots, v_{h''}\}$. Then $h'' = \min\{h, h'\}$. If $h'' = h$, then $d(v_{\bar{k}}, v_{\bar{k}'}) = \bar{k}' + \bar{k} - 2h$. Analogously, if $h'' = h'$, then $d(v_{\bar{k}}, v_{\bar{k}'}) = \bar{k}' + \bar{k} - 2h'$. On the other hand, being $\bar{k}' < k$ [resp. $\bar{k} < k$] and $d(v_k, v_{\bar{k}}) = k + \bar{k} - 2h = c$ [resp. $d(v_k, v_{\bar{k}'}) = k + \bar{k}' - 2h' = c$], then $d(v_{\bar{k}'}, v_{\bar{k}}) < c$.

Remark. If c is even, the lemma implies that $|P(\bar{w}, v_{\bar{k}}) \cap P(\bar{w}, v_{\bar{k}'})| \geqslant 2$, where $\bar{w}$ is the vertex at distance $c/2$ from v_k and $v_{\bar{k}}$.

We set $r'(T_n) = r(T_n)$ if $r(T_n)$ and $c/2$ are both even or odd, $r'(T_n) = r(T_n) - 1$, otherwise, and $c' = c$ or $c - 1$ depending on whether or not $c/2$ is even.

Theorem 3. *If c is even, then $\gamma_c(T_n) \geqslant \max |I_{c/2}(v)|$ and if $r'(T_n) \leqslant c'$, then $\gamma_c(T_n) \leqslant \max |I_{c/2}(v)| + [\min \{r'(T_n), c'\} - c/2]/2$, if $r(T_n) > c$, then $\gamma_c(T_n) \leqslant \max |I_{c/2}(v)| + (c-2)/2$.*

Proof. We start by proving the first inequality. Let v_t and $v_{t'}$, be two vertices such that $P(v, v_t) \cap P(v, v_{t'}) = \{v\}$. Evidently v_t and $v_{t'}$ are at distance $t + t'$. This distance is c iff $t' = c - t$. If there exists another vertex, $v_{t''}$, such that $P(v, v_{t''})$ intersects $P(v, v_t)$ and $P(v, v_{t'})$ just in the vertex v, any two of v_t, $v_{t'}$ and $v_{t''}$ have distance c iff $t = t' = t'' = c/2$. Thus the vertices of $I_{c/2}(v)$ have different colours in a c-colouring of $V(T_n)$.

In order to obtain the other inequalities, we assume, without loss of generality, that $|I_{c/2}(\bar{v})|$ is maximum.

If we assign the same colour to the vertices of $J_h(\bar{v})$, $0 \leqslant h < c/2$, we obtain a c-colouring of these vertices, because $h \leqslant h' < c/2$ implies $d(v_h, v_{h'}) < c$.

For every $k \geqslant c/2$, any two vertices, $v_k \in J_k(\bar{v})$, are at distance c iff they belong to a same set $I_{c/2}(\bar{w})$. Let us assign different colour to vertices v_k, $k = c/2$, $c/2 + 2, \ldots$, belonging to the set $I_{c/2}(\bar{w})$. Furthermore we assign to the other vertices of $J_k(\bar{v}) \cap J_{c/2}(\bar{w})$ the colour of the vertex of $J_k(\bar{v}) \cap I_{c/2}(\bar{w})$ from which they have distance less than c. Thus we obtain a c-colouring of the vertices of $J_k(\bar{v})$. Of course, to obtain a c-colouring of $V(T_n)$, the colours associated to the vertices $v_k \in J_{c/2}(\bar{w})$, must be different from those ones associated to the vertices $v_{\bar{k}} \in J_{c/2}(\bar{w})$, $\bar{k} < k$.

Let $a(k)$, $k > c/2$, be the minimum number of the colours, different from those ones assigned to the vertices of $J_{\bar{k}}(\bar{v})$, $c/2 \leqslant \bar{k} < k$, we introduce in this c-colouring of $V(T_n)$ in order to colour the vertices of $J_k(\bar{v})$.

To prove the other inequalities, we now determine an upper bound for $\sum_{k > c/2} a(k)$.

First we remark that the two c distant vertices have both even or odd distance from $\bar{v}$. Furthermore, for every vertex v_k, let $k_1 = \max \{0, k - c\}$ and $k_2 = \min \{k + c, \max d(v_k, v_{k'})\}$, where the v_k''s are the vertices such that $P(\bar{v}, v_k) \subset P(\bar{v}, v_{k'})$. Let v_{k_1} and v_{k_2} be the vertices such that $P(v, v_{k_1}) \subset P(v, v_k) \subset P(v, v_{k_2})$. Then the distance $\bar{k}$ from $\bar{v}$ of any vertex $v_{\bar{k}}$, having distance c from v_k, satisfies $k_1 \leqslant \bar{k} \leqslant k_2$. Thus we consider, without loss of generality, only the vertices v_k such that $k = c/2 + 2t$, $1 \leqslant t \leqslant c/2$. We call $a(t)$ the number $a(k)$.

If $c/2 \leqslant k \leqslant \min \{c', r'(T_n)\} = \bar{t}$ and $|I_{c/2}(\bar{v})| \geqslant 2$, consider the vertices v_{c-k} such that $P(\bar{v}, v_k) \cap P(\bar{v}, v_{c-k}) = \{\bar{v}\}$. The vertices v_k have colours different from the one of v_{c-k}. Furthermore, the number of the vertices $v_{\bar{k}}$, $c/2 \leqslant \bar{k} < k$, at distance c from v_k, which may have distinct colours is not greater than

$(k-c/2)/2 = t$, by lemma. Then we have

$$a(t) \leqslant |I_{c/2}(\overline{w})| - \big[\, |I_{c/2}(\overline{v})| + a(t-1) + a(t-2) + \ldots + a(1) - t \big].$$

On the other hand, $|I_{c/2}(\overline{w})| \leqslant |I_{c/2}(\overline{v})|$. Then, set $t' = \max\limits_{a(t)>0} t$, we have

$$a(t') + \sum_{t=1}^{t'-1} a(t) \leqslant t' - \sum_{t=1}^{t'-1} a(t) + \sum_{t=1}^{t'-1} a(t) \leqslant (\bar{t} - c/2)/2 .$$

Therefore, if $r'(T_n) \leqslant c'$, the theorem follows.

If $r(T_n) > c$, i.e. $\bar{t} = c'$, for every vertex v_k, $c < k \leqslant \min\{r'(T_n), 3c/2\} = \bar{\bar{t}}$ the number of the vertices $v_{\bar{k}}$, $k - c \leqslant \bar{k} < k$, at distance c from v_k, which may have distinct colours is not greater than $[k-(k-c)]/2 = c/2$, by lemma. Then, following the above procedure, we obtain

$$a(t'') + \sum_{t=1}^{t''-1} a(t) \leqslant c/2 - 1 - \sum_{t=1}^{t''-1} a(t) + \sum_{t=1}^{t''-1} a(t) = c/2 - 1 ,$$

where $t'' = (t - c/2)/2$. This proves the theorem.

These inequalities are the best possible. For instance, if we consider the trees such that $|I_{c/2}(\overline{v})| = \max |I_{c/2}(v)|$ and $r(T_n) = c/2$, the lower bound is attained. If we consider the trees such that $|I_{c/2}(\overline{v})| = \max |I_{c/2}(v)|$ and, for every $J_k(\overline{v})$, at least one vertex $\overline{w}$ exists such that $|I_{c/2}(\overline{w}) \cap J_k(\overline{v})| = |I_{c/2}(\overline{v})| - 1$ and for every $\bar{k}$, $k_1 \leqslant \bar{k} < k$, $J_{c/2}(\overline{w}) \cap J_{\bar{k}}(\overline{v}) \neq \emptyset$ results, one of the upper bounds is attained.

Corollary. *If* $c = 2$, *then* $\gamma_2(T_n) = \max\limits_{v} d(v)$.

Proof. Two vertices of T_n have distance c iff they are adjacent to the same vertex, i.e. $I_{c/2}(v)$ coincides with the set of vertices adjacent to v. If we assign different colours to vertices adjacent to the same vertex, we obtain a 2-colouring, thus $\gamma_2(T_n) \leqslant \max\limits_{v} d(v)$.

Theorem 4. *If* c *is odd, then* $\gamma_c(T_n) = 2$.

Proof. Since c is not greater than diameter of T_n, at least two vertices of T_n having distance c exist. Then $\gamma_c(T_n) \geqslant 2$. On the other hand, a tree is a special bipartite graph and two vertices have odd distance iff they belong to different set, X and Y. If we assign the same colour to vertices of X and another colour to vertices of Y, a c-colouring (c odd) results.

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Summary

The A-chromatic number, $\gamma_A(G)$, is determined for the complement of a cycle when A is either the set of even integers or the set of odd integers and for a cycle or its complement when A is a single integer not greater than their diameter. Moreover, when $A = \{c\}$ and G is a tree, lower and upper bounds for $\gamma_A(G)$ are determined.

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