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Pseudo bases in Banach spaces (**)

Introduction

In what follows $\{x_n\}$ is a sequence of a Banach space B, span $\{x_n\} = \{\sum_{n=1}^{m} a_n x_n\}$, ω-span $\{x_n\} = \{\sum_{n=1}^{\infty} a_n x_n\}$ and $[x_n] = \overline{\text{span}}\,\{x_n\}$.

We say that $\{x_n\}$ is *ω-dipendent if* $x_m \in \omega\text{-span}\,\{x_n\}_{n\geqslant m+1}$ for every m, *l^1-dipendent* if $x_m \in \{\sum_{n=m+1}^{\infty} a_n x_n/\|x_n\|,\ \left\|\sum_{n=m+1}^{\infty} a_n x_n/\|x_n\|\right\| = \sum_{n=m+1}^{\infty} |a_n|\}$ for every m. Moreover we recall that $\{x_n\}$ is *pseudo basis* of B if $B = \omega\text{-span}\,\{x_n\}$, *basis* if $\{x_n\}$ is pseudo basis with $x_m \notin [x_n]_{n\neq m}$ for every m.

In what follows we call $S(x_n)$ the set of all the *complete subsequences* $\{x_{n_k}\}$ of $\{x_n\}$, i.e. such that $[x_{n_k}] = [x_n]$.

If $[x_n] = [x_n]_{n\geqslant m}$ for every m, this set has cardinality of continuum (prof. II of [**2**]$_1$). Since the bases are well known, the Note concerns the pseudo bases which are ω-dipendent, that we study by means of the set $S(x_n)$.

Suppose that B has a basis, then every $\{x_n\}$ which is dense in B is an ω-dipendent pseudo basis, however in this case $S(x_n)$ has all the possible types of sequences of B. We wish to know particular types of $S(x_n)$, precisely if it is possible that all the elements of $S(x_n)$ are pseudo bases.

In [**1**] we proved that every B has a sequence $\{x_n\}$ such that all the elements of $S(x_n)$ were ω-dipendent; moreover in [**2**]$_2$ we proved in particular B, for every $p \geqslant 1$, the existence of $\{x_n\}$ such that all the elements of $S(x_n)$ were l^p-dipendent, but not pseudo basic.

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Now we give an example of a sequence $\{x_n\}$, in a particular B, such that all the elements of $S(x_n)$ are pseudo bases of B and l^1-dipendent.

Example. Let $\{x_n\}$ be the naturale basis of l^1 and let us set

$$(1)_1 \qquad v_m = \frac{1}{2^m} \sum_{n=p_m}^{q_m} x_n\,, \quad \text{where } p_1 = 2,\ q_m = p_m + 2^m - 1 \text{ and } p_{m+1} = q_m + 1$$

for every m;

$$(1)_2 \qquad y_{2n} = \frac{x_n + v_n}{2} \quad \text{and} \quad y_{2n-1} = \frac{x_n - v_n}{2} \quad \text{for every } n\,;$$

$$(1)_3 \qquad Y = [y_{2n}] \quad \text{and} \quad Z = [z_n]\,, \quad \text{where } z_n = y_{2n-1} + Y \quad \text{for every } n\,.$$

Then $S(z_n)$ has cardinality of continuum and all its elements are pseudo bases of Z. Moreover, if $\{z_{n_k}\}$ is an element of $S(z_n)$, span $\{z_n\} \subset \{\sum_{k=m}^{\infty} a_k z_{n_k},\ \|\sum_{k=m}^{\infty} a_k z_{n_k}\| = \sum_{k=m}^{\infty} |a_k|\}$ for every m; hence all the elements of $S(z_n)$ are l^1-dipendent.

Remark. In (1) $Z \neq \{\sum_{n=1}^{\infty} a_n z_n,\ \|\sum_{n=1}^{\infty} a_n z_n\| = \sum_{n=1}^{\infty} |a_n|\}$.

2 - Proofs

In what follows when we use many indices we call n_1 the first, n_2 the 2nd and so on.

Proof of example. $Z = l^1/Y$ since by (1)

$$(2) \qquad z_n = (y_{2n-1} + y_{2n}) + Y = x_n + Y \quad \text{for every } n\,.$$

Firstly we prove that $\{z_n\}$ is l^1-dipendent.

Let $\{z_{n'}\}$ be a complete subsequence of $\{z_n\}$, by (1) and (2) it follows

$$(3) \qquad \{z_{n'}\} \text{ is complete in } Z \Leftrightarrow \{y_{2n'-1}\} \cup \{y_{2n}\} \text{ is complete in } l^1\,.$$

Let $\{z_{n''}\}$ be the subsequence of $\{z_n\}$ which is complementary to $\{z_{n'}\}$, then we set

$$(4) \qquad \{z_n\}_{n=p_m}^{q_m} = \{z_{n'}\}_{n=p'_m}^{q'_m} \cup \{z_{n''}\}_{n=p''_m}^{q''_m} \quad \text{for every } m\,.$$

Let us fix m. We are going to get two sequences $\{a_{mi}\}_{i=1}^{\infty}$ and $\{v_{mi}\}_{i=1}^{\infty}$ such that

$$(5)\qquad D_m = \operatorname{dist}\left(x_m, [\{y_{2n'-1}\}_{n'>m} \cup \{y_{2n}\}]\right) = \lim_{i\to\infty} \|x_m - v_{mi}\| = 1 - \sum_{i=1}^{\infty} a_{mi}\,.$$

By (1) and (4) we have, for every k,

$$(6)\qquad 2y_{2k} - \frac{1}{2^k}\sum_{n=p_k'}^{q_k'} (y_{2n'-1} + y_{2n'})$$

$$= x_k + \frac{1}{2^k}\Big(\sum_{n=p_k}^{q_k} x_n - \sum_{n=p_k'}^{q_k'} x_{n'}\Big) = x_k + \frac{1}{2^k}\sum_{n=p_k''}^{q_k''} x_{n''} = x_k + \sum_{n_1=p_k''}^{q_k''} \frac{x_{n_1''}}{2^k}\,.$$

Hence by (1), (4) and (6) we can set

$$(7)_1\qquad v_{m1} = 2y_{2m} - \frac{1}{2^m}\sum_{n=p_m'}^{q_m'} (y_{2n'-1} + y_{2n'}) = x_m + \sum_{n_1=p_m''}^{q_m''} \frac{x_{n_1''}}{2^m}\,;$$

$$(7)_2\qquad v_{m2} = v_{m1} - \sum_{n_1=p_m''}^{q_m''} \frac{1}{2^m}\Big(2y_{2n_1''} - \frac{1}{2^{n_1''}}\sum_{n_2=p_{n_1''}'}^{q_{n_1''}'} (y_{2n_2'-1} + y_{2n_2'})\Big)$$

$$= x_m - \sum_{n_1=p_m''}^{q_m''}\ \sum_{n_2=p_{n_1''}''}^{q_{n_1''}''} \frac{x_{n_2''}}{2^{m+n_1''}}\,;$$

$$(7)_3\qquad v_{mi} = v_{m,i-1} + (-1)^{i+1}\sum_{n_1=p_m''}^{q_m''}\ \sum_{n_2=p_{n_1''}''}^{q_{n_1''}''} \cdots \sum_{n_{i-1}=p_{n_{i-}''}''}^{q_{n_{i-2}''}''} \frac{1}{2^{m+n_1''+\cdots+n_{i-2}''}}\Big(2y_{2n_{i-1}''}$$

$$- \frac{1}{2^{n_{i-1}''}}\sum_{n_i=p_{n_{i-1}''}'}^{q_{n_{i-1}''}'} (y_{2n_i'-1} + y_{2n_i'})\Big)$$

$$= x_m + (-1)^{i+1}\sum_{n_1=p_m''}^{q_m''}\ \sum_{n_2=p_{n_1''}''}^{q_{n_1''}''} \cdots \sum_{n_i=p_{n_{i-1}''}''}^{q_{n_{i-1}''}''} \frac{x_{n_i''}}{2^{m+n_1''+\cdots+n_{i-1}''}} \qquad \text{for } i \geqslant 3\,.$$

Moreover let us set

$$a_{m1}=\frac{q'_m-p'_m+1}{2^m},\qquad a_{m2}=\sum_{n_1=p''_m}^{q''_m}\frac{q'_{n''_1}-p'_{n''_1}+1}{2^{m+n''_1}},$$

(8)

$$a_{mi}=\sum_{n_1=p''_m}^{q''_m}\ \sum_{n_2=p''_{n''_1}}^{q''_{n''_1}}\cdots\sum_{n_{i-1}=p''_{n''_{i-2}}}^{q''_{n''_{i-2}}}\frac{q'_{n''_{i-1}}-p'_{n''_{i-1}}+1}{2^{m+n''_1+\cdots+n''_{i-1}}}\quad\text{for } i\geqslant 3\,.$$

By (1), (4) and (8) it follows

$$1-a_{m1}=\frac{(q_m-p_m+1)-(q'_m-p'_m+1)}{2^m}=\frac{q''_m-p''_m+1}{2^m},$$

$$1-a_{m1}-a_{m2}=\frac{q''_m-p''_m+1}{2^m}-\sum_{n_1=p''_m}^{q''_m}\frac{q'_{n''_1}-p'_{n''_1}+1}{2^{m+n''_1}}$$

$$=\sum_{n_1=p''_m}^{q''_m}\frac{1}{2^m}\Big(1-\frac{q'_{n''_1}-p'_{n''_1}+1}{2^{n''_1}}\Big)=\sum_{n_1=p''_m}^{q''_m}\frac{q''_{n''_1}-p''_{n''_1}+1}{2^{m+n''_1}};$$

moreover by induction it follows

$$1-\sum_{j=1}^{i}a_{mj}=\sum_{n_1=p''_m}^{q''_m}\ \sum_{n_2=p''_{n''_1}}^{q''_{n''_1}}\cdots\sum_{n_{i-1}=p''_{n''_{i-2}}}^{q''_{n''_{i-2}}}\frac{q''_{n''_{i-1}}-p''_{n''_{i-1}}+1}{2^{m+n''_1+\cdots+n''_{i-1}}}\quad\text{for } i\geqslant 3\,. \tag{9}$$

By (9) $0\leqslant 1-\sum_{j=1}^{i}a_{mj}=(1-\sum_{j=1}^{i-1}a_{mj})-a_{mi}$, that is $a_{mi}\leqslant 1-\sum_{j=1}^{i-1}a_{mj}$ for $i\geqslant 3$; hence by (8) it follows $a_{mi}\geqslant 0$ for every i and $0\leqslant\sum_{i=1}^{\infty}a_{mi}\leqslant 1$. Let us set

$$\{y_{(m,1)j}\}_{j=1}^{L_{m1}}=\{y_{2n'_1-1}\}_{n_1=p'_m}^{q'_m}, \tag{10$_1$}$$

$$\{y_{(m,2)j}\}_{j=1}^{L_{m2}}=\{y_{(m,1)j}\}_{j=1}^{L_{m1}}\cup\Big\{\bigcup_{n_1=p''_m}^{q''_m}\{y_{2n'_2-1}\}_{n_2=p'_{n''_1}}^{q'_{n''_1}}\Big\},$$

$$\{y_{(m,i)j}\}_{j=1}^{L_{mi}}=\{y_{(m,i-1)j}\}_{j=1}^{L_{m,i-1}}\cup\Big\{\bigcup_{n_1=p''_m}^{q''_m}\ \bigcup_{n_2=p''_{n''_1}}^{q''_{n''_1}}\cdots\bigcup_{n_{i-1}=p''_{n''_{i-2}}}^{q''_{n''_{i-2}}}\{y_{2n'_i-1}\}_{n_i=p'_{n''_{i-1}}}^{q'_{n''_{i-1}}}\Big\}$$

for $i \geqslant 3$,

$$(10)_2 \qquad \{y_{(m)_j}\}_{j=1}^{\infty} = \bigcup_{i=1}^{\infty} \{y_{(m,i)_j}\}_{j=1}^{L_{mi}}.$$

Since $\{x_n\}$ is the natural basis of l^1, by (1), (7), (9) and (10) it follows

$$\text{dist}\,(x_m, [\{y_{(m,i)_j}\}_{j=1}^{L_{mi}} \cup \{y_{2n}\}]) = \|x_m - v_{mi}\| = 1 - \sum_{j=1}^{i} a_{mj}, \qquad \text{for } i \geqslant 3.$$

On the other hand, by (1), (4), (5) and (10), $D_m = \text{dist}\,(x_m, [\{y_{(m)_j}\}_{j=1}^{\infty} \cup \{y_{2n}\}])$; therefore (5) is proved.

Now we are going to prove $D_m = 0$. We can proceed as for proof of prop. I of [**2**]$_2$: for every $x_{n_1''}$, with $p_m'' \leqslant n_1 \leqslant q_m''$, let $D_{n_1''} = \text{dist}\,(x_{n_1''}, [\{y_{2n'-1}\}_{n'>n_1''} \cup \{y_{2n}\}])$, let $\{y_{(n_1'')_j}\}_{j=1}^{\infty}$ be the subsequence of $\{y_{2n'-1}\}$ which is associated to $x_{n_1''}$ as $\{y_{(m)_j}\}_{j=1}^{\infty}$ of (10) for x_m, then we find $D_m = \sum_{n_1=p_m''}^{q_m''} D_{n_1''}/2^m$.

Moreover for a fixed $x_{\tilde{n}_1''}$, if $\{y_{(*,\tilde{n}_1'')_j}\}_{j=1}^{\infty}$ is the subsequence of $\{y_{2n'-1}\}$ which is complementary to $\{y_{(\tilde{n}_1'')_j}\}_{j=1}^{\infty}$, by (1) we have

$$\begin{aligned}\text{dist}\,(x_{\tilde{n}_1''}, [\{y_{2n'-1}\} \cup \{y_{2n}\}]) &= \min\{D_{\tilde{n}_1''};\ \text{dist}\,(x_{\tilde{n}_1''}, [\{y_{(*,\tilde{n}_1'')_j}\}_{j=1}^{\infty} \cup \{y_{2j}\}])\} \\ &\geqslant \min\{D_{\tilde{n}_1''};\ \sum_{\substack{n_1=p_m'' \\ n_1 \neq \tilde{n}_1}}^{q_m''} D_{n_1''}\}.\end{aligned}$$

Therefore by (3), since $\{z_{n'}\}$ is complete in Z, either $D_{\tilde{n}_1''} = 0$ or $D_{n_1''} = 0$ for $p_m'' \leqslant n_1 (\neq \tilde{n}_1) \leqslant q_m''$; that is there exists $\bar{n}_1'' \in \{n_1''\}_{n_1=p_m''}^{q_m''}$ such that $D_m = (1/2^m) D_{\bar{n}_1''}$, hence $D_m \leqslant 1/2^m$, and so on as in proof of prop. I of [**2**]$_2$.

So proceeding we find $D_m = 0$, hence by (5) and (7)

$$x_m = 2y_{2m} - \sum_{n_1=p_m'}^{q_m'} \frac{y_{2n_1'-1} + y_{2n_1'}}{2^m} - \sum_{n_1=p_m''}^{q_m''} \frac{1}{2^m} \Big(2y_{2n_1''} - \sum_{n_2=p'_{n_1''}}^{q'_{n_1''}} \frac{y_{2n_2'-1} + y_{2n_2'}}{2^{n_1''}}\Big)$$

$$+ \sum_{i=3}^{\infty} (-1)^{i+1} \Big(\sum_{n_1=p_m''}^{q_m''} \sum_{n_2=p''_{n_1''}}^{q''_{n_1''}} \cdots \sum_{n_{i-1}=p''_{n''_{i-2}}}^{q''_{n''_{i-2}}} \frac{1}{2^{m+n_1''+\cdots+n_{i-2}''}} \Big(2y_{2n_{i-1}''} - \sum_{n_i=p'_{n''_{i-1}}}^{q'_{n''_{i-1}}} \frac{y_{2n_i'-1} + y_{2n_i'}}{2^{n''_{i-1}}}\Big)\Big).$$

Therefore, since by (1) $\|z_n\| \leqslant 1$ and since by (5) $\sum_{i=1}^{\infty} a_{mi} = 1$, by (2) and (8) it follows

$$(11)\qquad z_m = -\sum_{n_1=p'_m}^{q'_m} \frac{z_{n'_1}}{2^m} + \sum_{n_1=p''_m}^{q''_m} \sum_{n_2=p'_{n''_1}}^{q'_{n''_1}} \frac{z_{n'_2}}{2^{m+n''_1}} + \sum_{i=3}^{\infty} (-1)^i \sum_{n_1=p''_m}^{q''_m} \sum_{n_2=p''_{n''_1}}^{q''_{n''_1}} \cdots$$

$$\sum_{n_{i-1}=p''_{n''_{i-2}}}^{q''_{n''_{i-2}}} \sum_{n_i=p'_{n''_{i-1}}}^{q'_{n''_{i-1}}} \frac{z_{n'_i}}{2^{m+n''_1+\cdots+n''_{i-1}}} = \sum_{n=p'_m}^{\infty} b_{mn} z_{n'} ,$$

with
$$\sum_{n=p'_m}^{\infty} |b_{mn}| = \frac{q'_m - p'_m + 1}{2^m} + \sum_{n_1=p''_m}^{q''_m} \frac{q'_{n''_1} - p'_{n''_1} + 1}{2^{m+n''_1}}$$

$$+ \sum_{i=3}^{\infty} \Big\{ \sum_{n_1=p''_m}^{q''_m} \sum_{n_2=p''_{n''_1}}^{q''_{n''_1}} \cdots \sum_{n_{i-1}=p''_{n''_{i-2}}}^{q''_{n''_{i-2}}} \frac{q'_{n''_{i-1}} - p'_{n''_{i-1}} + 1}{2^{m+n''_1+\cdots+n''_{i-1}}} \Big\} = \sum_{i=1}^{\infty} a_{mi} = 1 .$$

On the other hand we can proceed for $z_{p'_m}$ as for z_m and we find $z_{p'_m}$

$$z_{p'_m} = \sum_{n=p'_{p'_m}}^{\infty} b_{p'_m,n} \cdot z_{n'} , \qquad \text{with} \quad \sum_{n=p'_{p'_m}}^{\infty} |b_{p'_m,n}| = 1 ;$$

but by (1) and (11) it is possible to check $b_{mn} \neq 0 \Rightarrow b_{p'_m,n} = 0$ and $b_{p'_m,n} \neq 0$ $\Rightarrow b_{mn}$ for every n; hence by (11) $z_m = \sum_{n=p'_m+1}^{\infty} c_{mn} z_{n'}$ with $\sum_{n=p'_m+1}^{\infty} |c_{mn}| = 1$ again.

Therefore for every m and for every i there exists $\{d_{ik}\}_{k=m}^{\infty}$ such that

$$(12)\qquad z_i = \sum_{n=m}^{\infty} d_{in} z_{n'} , \quad \text{with} \quad \sum_{n=m}^{\infty} |d_{in}| = 1 .$$

By (1) we can set

$$(13)\qquad q(1) = q_1 , \qquad q(i) = q_{q(i-1)} \quad \text{for } i > 1 .$$

By (1) and (13), for every m and for every $x \in \operatorname{span}\{x_n\}_{n=1}^{q(m)}$, it is easy to check

$$(14)\qquad \operatorname{dist}(x, Y) = \operatorname{dist}(x, [y_{2n}]_{n=1}^{q(m)}) .$$

Suppose now $z \in \operatorname{span}\{z_n\}$ and let us fix m.

By (2) there exists a natural number r such that $z = x + Y$ with $x \in \operatorname{span}\{x_i\}_{i=1}^{q(r)}$, on the other hand by (14) $\|z\| = \operatorname{dist}(x, [y_{2n}]_{n=1}^{q(r)})$, that is there exists $y \in [y_{2n}]_{n=1}^{q(r)}$ such that $\|x+y\| = \|z\|$, with $x + y = \sum_{i=1}^{q(r+1)} a_i x_i$ by (1) and (13), that is by (2) $z = \sum_{i=1}^{q(r+1)} a_i z_i$, with $\|z\| = \sum_{i=1}^{q(r+1)} |a_i|$.

Consequently by (12) it follows

$$z = \sum_{i=1}^{q(r+1)} a_i \Big(\sum_{n=m}^{\infty} d_{in} z_{n'}\Big) = \sum_{n=m}^{\infty} b_n z_{n'}, \quad \text{with} \quad \sum_{n=m}^{\infty} |b_n| = \|z\|,$$

since $\|z\| \leqslant \sum_{n=m}^{\infty} |b_n| = \sum_{n=m}^{\infty} \Big| \sum_{i=1}^{q(r+1)} a_i d_{in} \Big| \leqslant \sum_{n=m}^{\infty} \sum_{i=1}^{q(r+1)} |a_i|\,|d_{in}| \leqslant \sum_{i=1}^{q(r+1)} |a_i| \Big(\sum_{n=m}^{\infty} |d_{in}|\Big) = \sum_{i=1}^{q(r+1)} |a_i| = \|z\|$.

Moreover by (2), (5), (1), (4) and (8) it follows that $\|z_m\| = \operatorname{dist}(x_m, [y_{2n}]) = 1$ for every m.

Hence $\operatorname{span}\{z_n\} \subset \{\sum_{n=m}^{\infty} a_n z_{n'}, \|\sum_{n=m}^{\infty} a_n z_{n'}\| = \sum_{n=m}^{\infty} |a_n|\}$; in particular $\{z_{n'}\}$ is l^1-dipendent. Let us fix m and $z' \in Z$.

By (1) and (2), $z' = x' + Y$, with $x' = \sum_{i=1}^{\infty} a_i' x_i$; hence $z' = \sum_{i=1}^{\infty} a_i' z_i$, with $\sum_{i=1}^{\infty} |a_i'| < +\infty$; therefore by (12)

$$z' = \sum_{i=1}^{\infty} a_i' \Big(\sum_{n=m}^{\infty} d_{in} z_{n'}\Big) = \sum_{n=m}^{\infty} c_n' z_{n'}, \qquad \text{with } c_n' = \sum_{i=1}^{\infty} a_i' d_{in} \quad \text{for every } n.$$

Therefore $\{z_{n'}\}_{n \geqslant m}$ is pseudo basis of Z for every m; which completes the proof of example.

Proof of remark. We shall not go much into details. Let us set

$$w_1 = \sum_{n_1=p_1+1}^{q_1} x_{n_1}, \qquad w_2 = \sum_{n_1=p_1+1}^{q_1} \frac{x_{p_{n_1}}}{2^{n_1}},$$

(15)

$$w_m = (-1)^m \sum_{n_1=p_1+1}^{q_1} \cdots \sum_{n_{m-1}=p_{n_{m-2}}+1}^{q_{n_{m-2}}} \frac{x_{p_{n_{m-1}}}}{2^{n_1+\ldots+n_{m-1}}} \quad \text{for every } m \geqslant 3,$$

$$\bar{x} = \sum_{n=1}^{\infty} w_n \quad \text{and} \quad \bar{z} = \bar{x} + Y.$$

We will prove

$$(16)\qquad \bar{z} = \sum_{n=1}^{\infty} c_n x_n + Y \quad \text{implies} \quad \sum_{n=1}^{\infty} |c_n| > \|\bar{z}\| .$$

By (15) $\|w_1\| = 1$, moreover by definitions of $\{p_n\}$ and $\{q_n\}$ of (1), since $n_{m-1} \geqslant m$ for $m \geqslant 2$, it follows that $\|w_m\| < 1/2^{m+1}$ for $m \geqslant 2$.

Let us set

$$(17)\qquad \begin{aligned} & u_1 = \sum_{n_1=p_1+1}^{q_1} x_{n_1}, \qquad u_2 = (-1) \sum_{n_1=p_1+1}^{q_1} \sum_{n_2=p_{n_1}+1}^{q_{n_1}} \frac{x_{n_2}}{2^{n_1}}, \\ & u_m = (-1)^{m+1} \sum_{n_1=p_1+1}^{q_1} \dots \sum_{n_m=p_{n_{m-1}}+1}^{q_{n_{m-1}}} \frac{x_{n_m}}{2^{n_1+\dots+n_{m-1}}} \qquad \text{for } m \geqslant 3 . \end{aligned}$$

It is possible to verify that

$$(18)\qquad \|u_m\| = \|u_{m-1}\| - \|w_m\| \quad \text{for } m \geqslant 2 \quad \text{and} \quad \lim_{m\to\infty} \|u_m\| > 1/2 .$$

Let us fix $m \geqslant 3$ and let $\sum_{n=1}^{\infty} d_n x_n \in Y$, since $\{y_{2n}\}$ is minimal by (1), (13), (14) and (17) it is possible to prove that

$$(19)\qquad \|u_m + \sum_{n=1}^{\infty} d_n x_n\| > \|u_m\| + \sum_{n=1}^{q(m-1)} a_n |d_n| ,$$

where $a_1 = 1/2$, $a_{n_1} = 1/2^{n_1}$ for $p_1 \leqslant n_1 \leqslant q_1$, $a_{n_i} = 1/2^{n_i}$ for $p_{n_{i-1}} \leqslant n_i \leqslant q_{n_{i-1}}$ and for $2 \leqslant i \leqslant m-1$.

By (1), (15), (17) and (19) it is possible to see that

$$(20)\qquad \sum_{n=1}^{m} w_n + Y = u_m + Y , \quad \text{with } \|u_m + Y\| = \|u_m\| \quad \text{for every } m \geqslant 1 .$$

Now we are going to prove (16).

By (15), (18) and (20) $\|\bar{z}\| = \lim_{m\to\infty} \|\sum_{m=1}^{m} w_n + Y\| = \lim_{m\to\infty} \|u_m\| > 1/2$; that is by

(16) $\sum_{n=1}^{\infty} |c_n| > 1/2$, hence by (19) there exists $\bar{m}$ such that

$$\sum_{n=1}^{q(\bar{m}-3)} a_n |c_n| > \sum_{n=\bar{m}+1}^{\infty} (1/2^n) . \tag{21}$$

By (15), (16) and (20) $\sum_{n=1}^{\infty} c_n x_n + Y = \sum_{n=1}^{\infty} w_n + Y = u_{\bar{m}} + \sum_{n=\bar{m}+1}^{\infty} w_n + Y$; therefore, since by (15) and (17) $u_{\bar{m}} + \sum_{n=\bar{m}+1}^{\infty} w_n \in [x_n]_{n>q(\bar{m}-3)}$, there exists $\{b'_n\}_{n=q(\bar{m}-1)+1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} c_n x_n = \sum_{n=1}^{q(\bar{m}-3)} c_n x_n + \sum_{n=q(\bar{m}-3)+1}^{\infty} b'_n x_n + u_{\bar{m}} + \sum_{n=\bar{m}+1}^{\infty} w_n ,$$

with
$$\sum_{n=1}^{q(\bar{m}-3)} c_n x_n + \sum_{n=q(\bar{m}-3)+1}^{\infty} b'_n x_n = \sum_{n=1}^{\infty} c_n x_n - (u_{\bar{m}} + \sum_{n=\bar{m}+1}^{\infty} w_n) \in Y.$$

Hence, since $\|w_n\| < 1/2^n$ for $n > 2$, by (15), (18), (19), (20) and (21) it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} |c_n| &= \Big\| \sum_{n=1}^{q(\bar{m}-3)} c_n x_n + \sum_{n=q(\bar{m}-3)+1}^{\infty} b'_n x_n + u_{\bar{m}} + \sum_{n=\bar{m}+1}^{\infty} w_n \Big\| \\ &\geqslant \Big\| \sum_{n=1}^{q(\bar{m}-3)} c_n x_n + \sum_{n=q(\bar{m}-3)+1}^{\infty} b'_n x_n + u_{\bar{m}} \Big\| - \Big\| \sum_{n=\bar{m}+1}^{\infty} w_n \Big\| \\ &> \sum_{n=1}^{q(\bar{m}-3)} a_n |c_n| + \|u_{\bar{m}}\| - \sum_{n=\bar{m}+1}^{\infty} (1/2^n) > \|u_{\bar{m}}\| > \lim_{m\to\infty} \|u_m\| = \|\bar{z}\| .\end{aligned}$$

Hence (16) is proved, which completes the proof of remark.

Bibliography

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Summary

Una successione $\{x_n\}$ *è pseudo base di uno spazio di Banach* B *se* $B=\{\sum_{n=1}^{\infty} a_n x_n\}$. *Tipi molto noti di pseudo basi sono le basi; la Nota considera gli altri tipi, di cui viene esaminato l'insieme di tutte le possibili sottosuccessioni complete. Viene dato un esempio in cui tale insieme ha la cardinalità del continuo ed è esclusivamente costituito da pseudo basi.*

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