Riv. Mat. Univ. Parma (4) 9 (1983), 361-372

EPIFANIO VIRGA (*)

On the interpretation of the Ising model as a Markov process (**)

1 - Introduction

As is known, the Ising model for ferromagnetism explains the spontaneous magnetization of ferromagnetic domains by postulating the existence of a short range non-magnetic interaction between the atoms of the cristal lattice. In 1971, working from rather general hypotheses, F. Spitzer [**11**] obtained the following result: *The representation of a crystal lattice in the Gibbs ensemble is equivalent to a Markov process, discrete and homogeneous in space, if only the pairs of nearest-neighbor lattice sites interact.*

This characterization follows from the one-to-one correspondence between the potential describing the interaction of the lattice sites in the Gibbs ensemble and the distribution of conditional probabilities, on which the representation of the lattice as a Markov process is based. Spitzer's theorem implies that the Ising model may be regarded as a Markov process where the events are ordered in space rather than in time. Accordingly, the Markovian interpretation of the Ising model is well founded, but this result does not suggest how to use the Markovian description in order to deduce the thermodynamic properties of the system. In this paper, starting from the Spitzer's result, we construct the canonical partition function of the one-dimensional Ising model, which in turn determines all the thermodynamic quantities. Moreover, we calculate the transition probabilities of the Markovian formulation in terms of the temperature

(*) Indirizzo: Istituto di Scienza delle Costruzioni, Facoltà di Ingegneria, Università, Via Diotisalvi 2, 56100 Pisa, Italy.

(**) Research partially supported by the «Gruppo Nazionale per la Fisica Matematica» of the C.N.R. (under contract GNFM-CRF). — Ricevuto: 1-III-1982.

and the external magnetic field. As is shown in what follows, both canonical partition function and transition probabilities are determined in an unambiguous way working only from the equality between Gibbsian and Markovian distributions of probabilities, based on Spitzer's theorem.

For an exhaustive account of main results concerning the Ising model, we refer to [**4**] and to the literature quoted therein; a special mention is due to the basic works by R. L. Dobruschin $[\mathbf{3}]_{1,\dots 4}$.

2 - Brief historical outline

In 1925, E. Ising [**6**] proposed a simple model for ferromagnetic substances which, unlike the existing Weiss theory (also known as the molecular field theory), was based on the hypothesis that the spontaneous magnetization of ferromagnetic domains (also called *Weiss domains*) may be explained by short-range interactions which are not due to atomic magnetic dipoles. Ising supposed that the interaction between the elementary atomic magnets (which causes the spontaneous magnetization of domains) decreases fairly quickly with distance so that it may be thought that only the nearest-neighbor atoms interact. He also accepted the hypotheses that W. Lenz [**9**] has put forward in 1920, to the effect that: (i) the magnetic dipoles of a three-dimensional lattice may take on only certain orientations allowed by the crystal structure; (ii) the configuration of the lattice with the lowest energy is the one in which all the elementary magnets have the same orientation.

Applying these hypotheses to the simple model of an ideal one-dimensional Weiss domain made up of n elementary magnetic moments, Ising worked out the mean magnetization within the thermodynamic limit, in which n becomes large. He showed that such a model possesses no ferromagnetic properties, that is, it does not give rise to spontaneous magnetization. Moreover, putting forward an intuitive argument, Ising extended the validity of this result to the similar models for two- and three-dimensional Weiss domains. Thus, he concluded that the hypothesis which limits the interaction between the elementary atomic magnets to nearest-neighbor pairs does not explains the spontaneous magnetization of domains.

Nevertheless, in 1952, C. N. Yang [**12**] showed that the spontaneous magnetization of the two-dimensional Ising model is different from zero at temperature less than a critical value T_c (see e.g. [**5**], p. 373). This contradicts Ising's conclusion and proves that the hypothesis of a short-range interaction between the elementary magnetic dipoles can lead to a complete understanding of the behaviour of ferromagnetics materials.

3 - The one-dimensional Ising model

The one-dimensional Ising model is made up of a chain of n equidistant magnetic dipoles lying in the same direction. Each of these is subject to an external constant magnetic field whose strength is B and whose direction is the same as that of the dipoles. Moreover, it is supposed that the magnetic dipoles are allowed only two orientations: the one, parallel to the external magnetic field and the other antiparallel. Therefore, the generic configuration of the whole system is specified by the vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ where $\sigma_i = 1$ if the i-th dipole points in the same direction as the external magnetic field, and $\sigma_i = -1$ otherwise. Finally, in accordance with Ising's hypotheses, it is assumed that each dipole of the lattice interacts with the pair of nearest-neighbors only.

3.1 – *The energy of the configurations*

For the energy of the pair of dipoles which occupy respectively, places i and $i+1$ in the lattice, the following expression holds true

$$E(\sigma_i, \sigma_{i+1}) = -J\sigma_i\sigma_{i+1} - \mu B(\sigma_i + \sigma_{i+1}) , \tag{3.1}$$

where J is the coupling constant between the dipoles, which is assumed to be positive ([1]), and μ is their magnetic moment. In equation (3.1) the first term of the right-hand side is due to the (non-magnetic) interaction between the dipoles, while the remainder describes the effect of the external magnetic field B. If J is assumed to be independent of the order parameter of the lattice (*hypothesis of homogeneity*) eq. (3.1) gives the following formula for the energy $E(\boldsymbol{\sigma}, B)$ of the generic configuration $\boldsymbol{\sigma}$

$$E(\boldsymbol{\sigma}, B) = -\sum_{\langle i,i+1\rangle} [J\sigma_i\sigma_{i+1} + \mu B(\sigma_i + \sigma_{i+1})] , \tag{3.2}$$

where the subscript $\langle i, i+1\rangle$ means that the summation is extended to all the pairs of nearest-neighbor dipoles. Moreover, if we make use of the periodic boundary condition

$$\sigma_{n+1} := \sigma_1 , \tag{3.3}$$

([1]) Indeed, Lenz's hypothesis, according to which the configuration with the lowest energy is the one where all the dipoles are parallel to the external magnetic field, requires the constant J to be positive.

which can be represented by placing the lattice dipoles in a circle lying on a plane orthogonal to the direction of the external magnetic field, equation (3.2) can be expressed in the following more suitable form

$$E(\boldsymbol{\sigma}, B) = -\sum_{i=1}^{n} [J\sigma_i\sigma_{i+1} + \tfrac{1}{2}\,\mu B(\sigma_i + \sigma_{i+1})]\,. \tag{3.4}$$

Hereafter, it is assumed throughout that the periodic boundary condition (3.3) allows us to know all the properties of the linear lattice, when we consider the thermodynamic limit $n \gg 1$.

3.2 – *The canonical (or Gibbs) ensemble*

The number of lattice dipoles is thought to be fixed, even if it is assumed to be arbitrarily large within the thermodynamic limit. Thus, the system may be described in the canonical (or Gibbs) ensemble, which for the configuration $\boldsymbol{\sigma}$ produces the probability

$$p(\boldsymbol{\sigma}) = \frac{\exp[-\beta E(\boldsymbol{\sigma}, B)]}{z(\beta, B)}\,, \tag{3.5}$$

where β is the ratio $1/KT$ (²) and z is the canonical partition function, defined by (³)

$$z(\beta, B) = \sum_{(\sigma)} \exp[-\beta E(\boldsymbol{\sigma}, B)]\,. \tag{3.6}$$

As is well-known, all the thermodynamic properties of the system can be expressed in terms of the partition function alone.

In order to work out $z(\beta, B)$, Ising makes use in [**6**] of a complicated calculation of a combinatorial kind, which is left out here to give preference to a procedure based on the Markovian interpretation of the model. It is convenient to define the dimensionless parameters θ and η given by

(3.7) $\qquad \theta := \beta J, \qquad\qquad$ (3.8) $\qquad \eta := \dfrac{\mu B}{J}\,.$

(²) As usual, K is the Boltzmann constant and T the absolute temperature.

(³) The symbol $\sum_{(\sigma)}$ means that the summation is extended to all the configurations of the system.

Moreover, if we set

$$\varepsilon(\boldsymbol{\sigma}, \eta) := \sum_{i=1}^{n} [\sigma_i \sigma_{i+1} + \tfrac{1}{2}\eta(\sigma_i + \sigma_{i+1})] , \tag{3.9}$$

the probability of the generic configuration $\boldsymbol{\sigma}$ becomes (see eq. (3.5))

$$p(\boldsymbol{\sigma}) = \frac{\exp[\theta\varepsilon(\boldsymbol{\sigma}, \eta)]}{Z(\theta, \eta)} , \tag{3.5$'$}$$

where Z is the partition function written in terms of θ and η

$$Z(\theta, \eta) = \sum_{(\sigma)} \exp[\theta\varepsilon(\boldsymbol{\sigma}, \eta)] . \tag{3.6$'$}$$

4 - The Markovian interpretation

The Spitzer's theorem, stated in **1**, insures that the one-dimensional Ising model can be regarded as a linear homogeneous Markov process; thus, the quantities σ_i make up a Markov chain ([4]) independent of the order variable i. This is the same as to think that the probability of the i-th dipole lying in the same direction as the external magnetic field or in the opposite one, is independent of the position held in the lattice, and that this probability is determined only by the actual orientation of the $(i-1)$-th dipole.

4.1 – *The transition probabilities*

If $p(\sigma_{i+1}|\sigma_i)$ stands for the (conditional) probability that the orientation of the $(i+1)$-th dipole is expressed by σ_{i+1}, when the orientation of the i-th one is given by σ_i, and if

$$p(\sigma_{i+1}|\sigma_i) > 0 \quad \text{for every } \boldsymbol{\sigma} \text{ and } i , \tag{4.1}$$

the probability $\mathscr{P}(\boldsymbol{\sigma})$ of the configuration $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ turns out to be defined by the relation

$$\mathscr{P}(\boldsymbol{\sigma}) := p(\sigma_2|\sigma_1)p(\sigma_3|\sigma_2) \dots p(\sigma_{n+1}|\sigma_n) , \tag{4.2}$$

where the periodic boundary condition (3.3) is understood.

([4]) More details about Markov chains can be found, for example, in ref. **[1]**, **[2]**, **[10]**.

4.2 – *The Markov matrix*

Since the one-dimensional Ising model is represented by a Markov chain homogeneous in space, the transition probabilities may only take on four different values, which are the elements of the Markov matrix Q

$$Q := \begin{pmatrix} p_{++} & p_{-+} \\ p_{+-} & p_{--} \end{pmatrix} . \tag{4.3}$$

In fact, p_{++} is the probability that the $(i+1)$-th dipole is parallel to the external magnetic field, when the i-th one is also parallel, p_{-+} is the probability that the $(i+1)$-th dipole is antiparallel to the direction of the magnetic field, when, instead, the i-th one is parallel to this, and the remaining elements are defined in exactly the same way. Hence, the normalization condition of probability

$$\sum_{(\sigma)} \mathscr{P}(\boldsymbol{\sigma}) = 1 , \tag{4.4}$$

which, by virtue of eq. (4.2), is written as

$$\sum_{(\sigma)} p(\sigma_{n+1}|\sigma_n)\, p(\sigma_n|\sigma_{n-1}) \dots p(\sigma_2|\sigma_1) = 1 , \tag{4.4}$$

turns into the following property of matrix Q

$$\operatorname{tr}(Q^n) = 1 . \tag{4.6}$$

This shows that the matrix of the transition probabilities has to fulfill a constraint which depends on the number of dipoles in the lattice. Moreover, the normalization of conditional probabilities requires the elements of the Markov matrix to satisfy the following equations

$$p_{++} + p_{+-} = 1, \quad p_{-+} + p_{--} = 1 . \tag{4.7}$$

5 - Construction of the canonical partition function

Since the representation of the one-dimensional Ising model in the Gibbs ensemble and the formulation of it as a Markov process are equivalent, the canonical and Markovian distribuitions of probabilities have to be equal.

Therefore, if we assume that condition (4.1) is fulfilled, the following identity holds true

$$p(\boldsymbol{\sigma}) = \mathscr{P}(\boldsymbol{\sigma}) \quad \text{for every } \boldsymbol{\sigma}\,. \tag{5.1}$$

By applying eqs. (3.5) and (4.2), the identity (5.1) may be changed into an equation for the canonical partition function and the transition probabilities. We now proceed to show that, using only eq. (5.1), it is possible to determine within the thermodynamic limit these unknows, in such a manner that constraint (4.6) is fulfilled and hypothesis (4.1) verified.

5.1 – *The modified Markov matrix*

By virtue of eq. (3.9), (3.5)′ becomes

$$p(\boldsymbol{\sigma}) = \frac{1}{Z(\theta,\eta)} \prod_{i=1}^{n} \exp\{\theta[\sigma_i\sigma_{i+1} + \tfrac{1}{2}\eta(\sigma_i + \sigma_{i+1})]\}\,. \tag{5.2}$$

Moreover, inserting eqs. (4.2) and (5.2) into (5.1), we have

$$\frac{1}{Z(\theta,\eta)} = \prod_{i=1}^{n} \exp\{-\theta[\sigma_i\sigma_{i+1} + \tfrac{1}{2}\eta(\sigma_i + \sigma_{i+1})]\}\, p(\sigma_{i+1}|\sigma_i)\,. \tag{5.3}$$

Adding up both members of equation (5.3) over all the 2^n configurations, we obtain

$$\frac{2^n}{Z(\theta,\eta)} = \operatorname{tr}(R^n)\,, \tag{5.4}$$

where R is the *modified Markov matrix* (⁵) defined by

$$R := \begin{pmatrix} p_{++}\exp[-\theta(1+\eta)] & p_{-+}\exp[\theta] \\ p_{+-}\exp[\theta] & p_{--}\exp[-\theta(1-\eta)] \end{pmatrix}. \tag{5.5}$$

Notice that the elements R_{ij} of R not necessarily fulfill the normalization conditions of a Markov matrix $R_{1i} + R_{2i} = 1$, $(i = 1, 2)$.

We now intend to show that, by means of eq. (5.1), the eigenvalues of Q and R can be expressed in terms of the canonical partition function.

(⁵) A different matrix of this same kind was introduced by H. A. Kramers and G. H. Wannier in their matrix formulation of the one-dimensional Ising model (cfr. [7], Sect. 2).

5.2 – *The eigenvalues of Q*

By putting $p := p_{++}$, $q := p_{--}$, and using eqs. (4.7), the characteristic equation of the Markov matrix may be written in the following form

$$\lambda^2 - (p+q)\lambda + p + q - 1 = 0\,,$$

whose solutions are [6] $\lambda_1 = 1$, $\lambda_2 = p + q - 1$.

It is easy to express the transition probabilities p and q, and therefore the eigenvalue λ_2, in terms of the partition function $Z(\theta, \eta)$. Indeed, if we consider the particular configuration $\boldsymbol{\sigma}^+$ described by $\sigma_i^+ := 1$ for every i, we obtain (see eq. (5.3)) $p^n = \big(1/Z(\theta, \eta)\big) \exp[n\theta(1+\eta)]$, from which, by defining

$$\zeta(\theta, \eta) := [Z(\theta, \eta)]^{1/n}\,, \tag{5.6}$$

it follows that

$$p = \frac{1}{\zeta} \exp[\theta(1+\eta)]\,. \tag{5.7}$$

In a similar manner, by considering the particular configuration $\boldsymbol{\sigma}^-$ defined by $\sigma_i^- := -1$ for every i, we find

$$q = \frac{1}{\zeta} \exp[\theta(1-\eta)]\,. \tag{5.8}$$

Thus, the eigenvalue λ_2 of Q becomes

$$\lambda_2 = \frac{2}{\zeta} \exp[\theta] \cosh \theta\eta - 1\,. \tag{5.9}$$

Notice that this equation gives

$$\lambda_2 > -1\,. \tag{5.10}$$

5.3 – *The eigenvalues of R*

Let λ_+ and $\lambda_-(\lambda_+ > \lambda_-)$ be the eigenvalues of the modified Markov matrix.

---

(6) As is well-known (see e.g. [1], Sect. 2.2), a Markov matrix has at least one eigenvalue equal to 1.

Then eq. (5.4) becomes

$$\frac{2^n}{Z(\theta,\eta)} = \lambda_+^n + \lambda_-^n\,, \tag{5.11}$$

and, by introducing eqs. (5.7) and (5.8) into (5.5), we have

$$R = \frac{1}{\zeta}\begin{pmatrix} 1 & \exp[\theta]\{\zeta - \exp[\theta(1-\eta)]\} \\ \exp[\theta]\{\zeta - \exp[\theta(1+\eta)]\} & 1 \end{pmatrix}.$$

Thus, equation (5.11) can be written in the form

$$\mu_+^n + \mu_-^n = 2^n\,, \tag{5.12}$$

where μ_+ and $\mu_-(\mu_+ > \mu_-)$ are the eigenvalues of the matrix ζR. As is easy to check, μ_+ and μ_- have the following expressions

$$\mu_\pm = 1 \pm \exp[\theta]\{\zeta^2 - 2\exp[\theta]\zeta\cosh\theta\eta + \exp[2\theta]\}^{\frac{1}{2}}\,. \tag{5.13}$$

It is not difficult to show that the square root in the right-hand side of eq. (5.13) is real for every value of θ and η.

5.4 – *The thermodynamic limit*

Equation (5.12) can also be expressed as follows

$$\mu_+^n\left[1 + \left(\frac{\mu_-}{\mu_+}\right)^n\right] = 2^n\,,$$

which, since $\mu_-/\mu_+ < 1$, in the limit $n \gg 1$ leads to $\mu_+ = 2$. Therefore, in the thermodynamic limit, ζ solves the equation (see eq. (5.13))

$$\zeta^2 - 2\exp[\theta]\cosh(\theta\eta)\zeta + 2\sinh 2\theta = 0\,. \tag{5.14}$$

This algebraic equation in ζ has the roots

$$\zeta_\pm = \zeta_\pm(\theta,\eta) = \exp[\theta](\cosh\theta\eta \pm \{\sinh^2\theta\eta + \exp[-4\theta]\}^{\frac{1}{2}})\,. \tag{5.15}$$

By calculating the right-hand side of eq. (3.6)′ at $\theta = 0$, we have $Z(0, u) = 2^n$ and, by virtue of definition (5.6) $\zeta(0,\eta) = 2$, for every η. On the other hand, from eq. (5.15) it follows that $\zeta_+(0,\eta) = 2$, $\zeta_-(0,\eta) = 0$. Thus, the solution

ζ_- is not acceptable and ζ can be given in the form

$$(5.16)\qquad \zeta(\theta, \eta) = \zeta_+(\theta, \eta) = \exp[\theta](\cosh\theta\eta + \{\sinh^2\theta\eta + \exp[-4\theta]\}^{\frac{1}{2}}) .$$

Therefore, in the thermodynamic limit, we obtain

$$(5.17)\qquad Z(\theta, \eta) = \exp[n\theta](\cosh\theta\eta + \{\sinh^2\theta\eta + \exp[-4\theta]\}^{\frac{1}{2}})^n ,$$

and applying eqs. (5.7), (5.8) and (4.7), the transition probabilities read

$$(5.18)_1\qquad p_{++} = p_{++}(\theta, \eta) = \exp[\theta\eta](\cosh\theta\eta + \{\sinh^2\theta\eta + \exp[-4\theta]\}^{\frac{1}{2}})^{-1} ,$$

$$(5.18)_2\qquad p_{--} = p_{--}(\theta, \eta) = p_{++}(\theta, -\eta) ,$$

$$(5.18)_3\qquad p_{+-} = p_{+-}(\theta, \eta) = 1 - p_{++}(\theta, \eta) ,$$

$$(5.18)_4\qquad p_{-+} = p_{-+}(\theta, \eta) = 1 - p_{--}(\theta, \eta) .$$

Since $\zeta(\theta, \eta) > \exp[\theta]\cdot\cosh\theta\eta$ for every θ, η, the eigenvalue λ_2 of the matrix satisfies the inequality $\lambda_2 < 1$. This, together with (5.10), enables us to prove that eqs. (5.18) are consistent with condition (4.6). Indeed, this constraint also read $\lambda_1^n + \lambda_2^n = 1$, which, since $\lambda_1 = 1$, is satisfied within the thermodynamic limit, if $|\lambda_2| < 1$. Moreover, our basic hypothesis (4.1) is verified by solution (5.18). In fact, eq. (3.6)′ can be also put in the following form

$$Z(\theta, \eta) = \exp[n\theta(1 + \eta)] + \sum_{(\boldsymbol{\sigma}-\boldsymbol{\sigma}^+)} \exp[\theta\varepsilon(\boldsymbol{\sigma}, \eta)] ,$$

where the symbol $\sum_{(\boldsymbol{\sigma}-\boldsymbol{\sigma}^+)}$ means that the sum is extended to all the configurations except $\boldsymbol{\sigma}^+$. Thus,

$$\zeta(\theta, \eta) > \exp[\theta(1 + \eta)] \qquad \text{for every } \theta, \eta ,$$

which, by virtue of eqs. (5.7), (5.8) and (4.7), leads to prove that all the elements of the Markov matrix Q are strictly positive and less than one. Notice that this result is independent of the thermodynamic limit; thus, the Markovian formulation of the linear Ising model presented here is self-consistent, no matter how many the magnetic dipoles are.

Finally, by inserting eqs. (3.7) and (3.8) into (5.17) and (5.18), we find the canonical partition function and the transition probabilities in terms of the physical variables β and B

$$z(\beta, B) = \exp[n\beta J](\cosh \mu\beta B + \{\sinh^2 \mu\beta B + \exp[-4\beta J]\}^{\frac{1}{2}})^n , \tag{5.19}$$

$$p_{++} = \hat{p}_{++}(\beta, B) = \exp[\mu\beta B](\cosh \mu\beta B + \{\sinh^2 \mu\beta B + \exp[-4\beta J]\}^{\frac{1}{2}})^{-1}, \tag{5.20_1}$$

$$p_{--} = \hat{p}_{--}(\beta, B) = \hat{p}_{++}(\beta, -B) . \tag{5.20_2}$$

The equation (5.19) is the same as that found by Ising [**6**] by direct computation; however, here it has been deduced working only from the formulation of the Ising model as a Markov chain.

5.5 – *The magnetization of the lattice*

As is well-known (cfr. e.g. [**8**], n. 52), in the Gibbs ensemble the magnetic moment $\boldsymbol{M}(\beta, \boldsymbol{B})$ of a system subject to the external magnetic field $\boldsymbol{B}$ can be expressed in terms of the canonical partition function $z(\beta, \boldsymbol{B})$

$$\boldsymbol{M}(\beta, \boldsymbol{B}) = \frac{1}{\beta}\frac{\partial}{\partial \boldsymbol{B}} \ln z(\beta, \boldsymbol{B}) . \tag{5.21}$$

By inserting equation (5.19) into (5.21), we find that the magnetic moment per dipole of the one-dimensional Ising model is (see also ref. [**6**], p. 256)

$$\frac{1}{n} M(\beta, B) = \mu \sinh(\mu\beta B) \{\sinh^2(\mu\beta B) + \exp[-4\beta J]\}^{-1/2} . \tag{5.22}$$

In the limit $B = 0$, from equation (5.22) we obtain

$$\frac{1}{n} M(\beta, 0) = 0 \qquad \text{for every } \beta .$$

This same calculation led Ising (loc. cit.) to conclude that spontaneous magnetization is absent in the one-dimensional model and, therefore, it does not give rise to ferromagnetism.

Aknowledgment. The author wishes to thank dr. D. Bettoni for making him available the translation of Ising's paper [**6**].

References

[**1**] M. S. Bartlett, *An introduction to sthocastics processes*, Cambridge University Press, Cambridge 1978.

[2] A. T. Bharucha - Reid, *Elements of the theory of Markov processes and their applications*, McGraw-Hill, New York 1960.

[3] R. L. Dobruschin: [•]₁ *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theory Probab. Appl. **13** (1968), 197-224; [•]₂ *Gibbsian random fields for lattice systems with pairwise interactions*, Functional Anal. Appl. **2** (1968), 292-301; [•]₃ *The problem of uniqueness of a gibbsian random field and the problem of phase transitions*, Functional Anal. Appl. **2** (1968), 302-312; [•]₄ *Gibbsian random fields. The general case*, Functional Anal. Appl. **3**(1969), 22-28.

[4] G. Gallavotti, *Instabilities and phase transitions in the Ising model*, Riv. Nuovo Cimento **2** (1972), 133-169.

[5] K. Huang, *Statistical mechanics*, Wiley, New York 1963.

[6] E. Ising, *Beitrag zur theorie des ferromagnetismus*, Zeits. f. Physik **31** (1925), 253-258.

[7] H. A. Krames and G. H. Wannier, *Statistics of the two-dimensional ferromagnetism*, Phys. Rev. **60** (1941), 252-262.

[8] L. D. Landau e E. M. Lifsits, *Fisica statistica* (I), Editori Riuniti, Mir, Roma 1978.

[9] W. Lenz, *Beitrag zum verständnis der magnetischen erscheinungen in festen körpen*, Phys. ZS., **21** (1920), 613-615.

[10] P. Levy, *Processus stochastiques et mouvement browmien*, Gauthier-Villard, Paris 1966.

[11] F. Spitzer, *Markov random field and Gibbs ensembles*, Amer. Math. Montly **78** (1971), 142-154.

[12] C. N. Yang, *The spontaneous magnetization of a two-dimensional Ising model*, Phys. Rev. **85** (1952), 809-816.

Riassunto

Si costruisce la funzione di partizione canonica del modello unidimensionale di Ising nell'ambito della sua formulazione in termini di processo markoffiano discreto ed omogeneo nello spazio. Inoltre, le probabilità di transizione, che intervengono nella descrizione markoffiana, si esprimono in termini delle variabili macroscopiche del modello; cioè, la temperatura e l'intensità del campo magnetico esterno.

* * *