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Product topologies on the space of distribution functions (**)

1 - Introduction

Let Δ_r be the set of r-dimensional distribution functions (briefly r-d. f.'s), i.e. the family of functions $H: \overline{\boldsymbol{R}}^r \to [0, 1]$, where $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\}$ such that (i) $H(x_1, x_2, \ldots, x_r) = 0$ if $x_i = -\infty$ for at least one index i, (ii) $H(+\infty, +\infty, \ldots, +\infty) = 1$, (iii) $V_H(R) \geqslant 0$ for every rectangle $R = \prod_{i=1}^{r}]a_i, b_i]$ with $a_i \leqslant b_i$ $(i = 1, 2, \ldots, r)$, where $V_H(R) := \sum \operatorname{sign}(c) H(c)$, the sum being taken over all the vertices c of R and where $\operatorname{sign}(c) = 1$ or -1 according as to whether $c_i = a_i$ for an even or an odd number of indeces.

In addition, a r-d. f. is usually assumed to be either left- or right-continuous in each variable. Since such a property will not be required in the sequel of the present note, we shall not make the choice just mentioned.

The (one-dimensional) *margins* (or marginal d. f.'s) of $H \in \Delta_r$ are defined by $F_i(x_i) := H(+\infty, +\infty, \ldots, +\infty, x_i, +\infty, \ldots, +\infty)$ $(i=1, 2, \ldots, r)$. Obviously one has $F_i \in \Delta$ $(i = 1, 2, \ldots, r)$ (Δ is the set of l-d. f.'s, see [**6**], [**8**], [**7**]). Thus a map $M_r: \Delta_r \to \Delta \times \Delta \times \ldots \times \Delta$ is defined by $M_r(H) = (F_1, F_2, \ldots, F_r)$.

We shall denote by Δ_r^0 the subset of Δ_r formed by the d. f.'s of those random vectors $(X_1, X_2, \ldots, X_r)$ the components of which are a.s. finite random variables, viz. $P[|X_i| = +\infty] = 0$ $(i = 1, 2, \ldots, r)$. If $H \in \Delta_r^0$, then proper-

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ties (i) and (ii) above are replaced respectively by

(i)′ $$\lim_{x_i \to -\infty} H(x_1, x_2, \dots, x_r) = 0 \qquad (i = 1, 2, \dots, r) \,,$$

(ii)′ $$\lim_{\min(x_1, x_2, \dots, x_r) \to +\infty} H(x_1, x_2, \dots, x_r) = 1 \,.$$

A sequence $\{H_n\colon n = 1, 2, \dots\} \subset \Delta_r$ is said to *converge weakly* to $H \in \Delta_r$ iff $\lim H_n(x_1, x_2, \dots, x_r) = H(x_1, x_2, \dots, x_r)$ at every point $(x_1, x_2, \dots, x_r)$ of continuity for H; one then writes $H_n \rightharpoonup H$. If $H \in \Delta_r$ let c_H^r denote the set of continuity points of H; if $r = 1$, i.e., if $F \in \Delta$, we shall simply write c_F. The properties of c_H^r are given, e.g., in [**10**] th. 2.2.3.

The set Δ can be metrized either by the metric d_S introduced by Sibley [**8**] and modified by Schweizer [**5**] or by the metric d_K introduced by the present author ([**7**]) after Kingman ([**2**], section 12.1). These metrics induce the same convergence in Δ, viz. $d_S(F_n, F) \to 0$ or $d_K(F_n, F) \to 0$ iff $F_n(x) \to F(x)$ at every $x \in c_F$. Both metric spaces (Δ, d_S) and (Δ, d_K) are compact and hence complete. The subset $\Delta^0 \subset \Delta$ is a metric space not only with respect to the restriction to Δ^0 of either d_S or d_K but also with respect to the Lévy metric d_L (see, e.g. [**3**] or [**4**]). The metric space (Δ^0, d_L) is complete [**4**] but not compact.

It is the aim of the present note to investigate the connexion between weak convergence in Δ_r and the product topology on $\Delta \times \Delta \times \dots \times \Delta$ induced by the topology of the metric d_S or d_K on Δ.

2 - The case $r = 2$

Theorem 1. *Let $H_n \in \Delta_2$ $(n = 1, 2, \dots, \infty)$ be continuous on $\overline{\mathbf{R}}^2$ (in particular, hence, $H_n \in \Delta_2^0$, $n = 1, 2, \dots, \infty$). If $H_n \to H_\infty$, then $M_2(H_n) \to M_2(H_\infty)$ in the sense of the convergence induced by the product topology on $\Delta \times \Delta$,* i.e. *if $M_2(H_n) = (F_1^{(n)}, F_2^{(n)})$ $(n = 1, 2, \dots, \infty)$, then $F_1^{(n)} \to F_1^{(\infty)}$ and $F_1^{(n)} \to F_2^{(\infty)}$.*

Proof. Since all the d. f.'s in question are continuous, weak convergence means pointwise convergence on $\overline{\mathbf{R}}^2$. Thus $F_1^{(n)}(x) = H_n(x, +\infty) \to H_\infty(x, +\infty) = F_1^{(\infty)}(x)$ for $x \in \overline{\mathbf{R}}$. Likewise for $F_2^{(n)}$.

One cannot eliminate the requirement that H_n be continuous as is shown by the following

Example 1. Let $\varepsilon_a \in \Delta$ be defined, for $a \in \overline{\boldsymbol{R}}$, by

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > a\,, \end{cases}$$

the value of ε_a at $x = a$ being fixed by the choice of left- or right-continuity; $\varepsilon_\infty(x) = 0$ for $x \in \boldsymbol{R} \cup \{-\infty\}$, $\varepsilon_\infty(+\infty) = 1$.

Let $F^{(n)} = \varepsilon_n$ $(n = 1, 2, \ldots)$, $F^{(\infty)} = \varepsilon_0/2 + \varepsilon_\infty/2$, $G^{(n)} = \varepsilon_n/2 + \varepsilon_\infty/2$ $(n = 1, 2, \ldots)$, $G^{(\infty)} = \varepsilon_\infty$; obviously $F^{(n)} \in \Delta$ and $G^{(n)} \in \Delta$ $(n = 1, 2, \ldots, \infty)$. Define the sequence $\{H_n\colon n = 1, 2, \ldots, \infty\} \subset \Delta_2$ by $H_n(x, y) = F^{(n)}(x)G^{(n)}(y)$ $(n = 1, 2, \ldots, \infty)$. H_∞ is continuous on $\overline{\boldsymbol{R}}^2$, since $H_\infty(x, y) = 0$ for $(x, y) \in \boldsymbol{R}^2$, but not on $\boldsymbol{R}^2$, for $F_1^{(\infty)}(x) = H_\infty(x, +\infty) = F^{(\infty)}(x)$ and $F_2^{(\infty)}(y) = H_\infty(+\infty, y) = G^{(\infty)}(y)$. At every point $(x, y) \in \boldsymbol{R}^2$ one has $H_n(x, y) = \varepsilon_n(x)\varepsilon_n(y)/2$ so that $H_n \to 0$ on $\overline{\boldsymbol{R}}^2$ as n goes to $+\infty$. Thus H_n converges weakly to H_∞. However, since $M_2(H_n) = (F^{(n)}, G^{(n)})$, $F^{(n)}$ does not converge weakly to $F^{(\infty)}$. Indeed $F^{(\infty)}$ is continuous at every real point $x > 0$, where $F^{(\infty)}(x) = 1/2$, but $F^{(n)}(x) = \varepsilon_n(x)$ that goes to zero as n tends to $+\infty$.

The example above also shows that if $H \in \Delta_2$ and $M_2(H) = (F, G)$ then the inclusion $c_F \times c_G \subset c_H^2$ may be strict. To see this take $H = H_\infty$. Then every point $(0, y)$ with $y \in \boldsymbol{R}$ belongs to $c_{H\infty}^2$ but not to $c_F(\infty) \times c_G(\infty)$.

In the other direction, let (F, G) be given in $\Delta \times \Delta$. As is well-known (see, e.g., [**1**]), F and G determine a class $\Gamma(F, G) \subset \Delta_2$ of 2-d. f.'s of which they are the margins. $\Gamma(F, G) = M_2^{-1}(F, G)$ is called the *Fréchet class of F and G*. The following results will be needed (see [**9**]$_2$ or [**11**]).

Theorem 2. (Sklar) *Let $H \in \Delta_2$ and let F and G be its margins. Then there is a (generally non-unique) function $C\colon [0, 1] \times [0, 1] \to [0, 1]$ called (2-)copula such that $H(x, y) = C[F(x), G(x)]$ $((x, y) \in \overline{\boldsymbol{R}}^2)$. The function C has, among others, the following properties*

(1) $\qquad C(s, 1) = s\,, \quad C(1, t) = t \qquad (s, t \in [0, 1])\,;$

(2) $\qquad C(s, 0) = C(0, t) = 0 \qquad (s, t \in [0, 1])\,;$

(3) $\qquad |C(s_1, t_1) - C(s_2, t_2)| \leq |s_1 - s_2| + |t_1 - t_2| \qquad (s_1, s_2, t_1, t_2 \in [0, 1])$

and therefore is uniformly continuous on $[0, 1] \times [0, 1]$;

(4) $\qquad \max(s + t - 1, 0) \leq C(s, t) \leq \min(s, t) \qquad (s, t \in [0, 1])\,,$

and moreover the functions C' and C'', with $C'(s, t) := \max(s + t - 1, 0)$ *and* $C''(s, t) = \min(s, t)$ *are themselves copulae.*

Copulae were introduced by Sklar ([**9**]) in 1959. For an exhaustive list of their properties one should consult [**9**]$_2$. Theorem 2 presents only those results that will be needed for the purpose of the work here reported.

One can now prove

Theorem 3. *Let $F_n \rightharpoonup F_\infty$ and $G_n \rightharpoonup G_\infty$ with $F_n, G_n \in \Delta$ $(n = 1, 2, \ldots, \infty)$,* i.e. *$d(F_n, F_\infty) \to 0$ and $d(G_n, G_\infty) \to 0$ where either $d = d_S$ or $d = d_K$. Then, for every copula C, one has $C(F_n, G_n) \to C(F_\infty, G_\infty)$ in the sense of weak convergence on Δ_2.*

Proof. $C(F_n, G_n)$ converges pointwise to $C(F_\infty, G_\infty)$ on $c_{F_\infty} \times c_{G_\infty}$; indeed one has, in view of (3)

$$|C[F_n(x), G_n(y)] - C[F_\infty(x), G_\infty(y)]| \leqslant |F_n(x) - F_\infty(x)| + |G_n(y) - G_\infty(y)|.$$

But $\overline{\boldsymbol{R}}^2 \supset \overline{c_{F_\infty} \times c_{G_\infty}} = \bar{c}_{F_\infty} \times \bar{c}_{G_\infty} = \overline{\boldsymbol{R}}^2$, so that $c_{F_\infty} \times c_{G_\infty}$ is dense in $\overline{\boldsymbol{R}}^2$; the assertion is now a consequence of the equivalence of weak convergence with pointwise convergence on a dense set (see, e.g., [**10**]).

If a sequence $\{(F_n, G_n)\colon n = 1, 2, \ldots, \infty)\} \subset \Delta \times \Delta$ is given, then choosing a copula C implies assigning a sequence $\{H_n\colon n = 1, 2, \ldots, \infty\} \subset \Delta_2$ such that $H_n = C(F_n, G_n)$ $(n = 1, 2, \ldots, \infty)$. Theorem 3 thus states that for the sequence $\{H_n\}$ in Δ_2 that corresponds to the given sequence $\{(F_n, G_n)\}$ in $\Delta \times \Delta$ and to the copula C chosen, one has $H_n \rightharpoonup H_\infty$ and that this holds for every possible choice of C.

Theorems 1 and 3 continue to hold if one replaces Δ_2 by Δ_2^0 and Δ by Δ^0; formally

Theorem 4. (a) *Let $H_n \in \Delta_2^0$ $(n = 1, 2, \ldots, \infty)$ be continuous on $\boldsymbol{R}^2$. If $H_n \rightharpoonup H_\infty$ then $M_2(H_n) \to M_2(H_\infty)$ in the sense of the product topology on $\Delta^0 \times \Delta^0$, viz. if $M_2(H_n) = (F_1^{(n)}, F_2^{(n)})$ $(n = 1, 2, \ldots, \infty)$ then $F_k^{(n)} \rightharpoonup F_k^{(\infty)}$ $(k = 1, 2)$.* (b) *If $F_n \rightharpoonup F_\infty$ and $G_n \rightharpoonup G_\infty$ with $F_n, G_n \in \Delta^0$ $(n = 1, 2, \ldots, \infty)$, then for every copula C one has $C(F_n, G_n) \to C(F_\infty, G_\infty)$ in the sense of weak convergence on Δ_2^0.*

Proof. The proof of (a) is identical with that of Theorem 1, whilst the proof of (b) is an immediate consequence of the following fact: If $(F, G) \in \Delta^0 \times \Delta^0$ then $C(F, G) \in \Delta_2^0$ for every copula C. This follows from the continuity of C and from (1) and (2). In fact, as $C(F, G) \in \Delta_2$, one has only to check that (i)′ and (ii)′ are verified, as indeed they are

$$\lim_{x\to-\infty} C[F(x), G(y)] = C[0, G(y)] = 0\,, \qquad \lim_{y\to-\infty} C[F(x), G(y)] = C[F(x), 0] = 0\,,$$

$$\lim_{\min(x,y)\to+\infty} C[F(x), G(y)] = C(1, 1) = 1\,.$$

Only partially do Theorems 2 and 4(b) answer the question of whether, for a given sequence $\{H_n\} \subset \Delta_2$, the weak convergence of both sequences of marginals implies the weak convergence of $\{H_n\}$. By theorems 2 and 4(b) this is clearly the case if $H_n = C(F_1^{(n)}, F_2^{(n)})$ with the *same* copula for $n=1, 2, \ldots, \infty$. As Example 2 further below will show, the answer is, in general, negative, viz. $\{F_1^{(n)}\}$ and $\{F_2^{(n)}\}$ may converge in Δ whilst $\{H_n = C_n(F_1^{(n)}, F_2^{(n)})\}$ does not converge in Δ_2. However, convergence of $\{H_n = C_n(F_1^{(n)}, F_2^{(n)})\}$ obtains under stronger assumptions on the limits $F_1^{(\infty)}$ and $F_2^{(\infty)}$.

Theorem 5. *Let* $\{H_n: n = 1, 2, \ldots, \infty\} \subset \Delta_2$ *and* $M_2(H_n) = (F_1^{(n)}, F_2^{(n)})$ $(n = 1, 2, \ldots, \infty)$. *If* $F_1^{(\infty)} = \varepsilon_a$ *and* $F_2^{(\infty)} = \varepsilon_b$ *for some* $a, b \in \mathbf{R}$, *then* $H_n \rightharpoonup H_\infty$.

Proof. There exist copulae C_n $(n=1, 2, \ldots, \infty)$ such that $H_n = C_n(F_1^{(n)}, F_2^{(n)})$ Then

$$\begin{aligned} |H_n(x, y) - H_\infty(x, y)| &= |C_n[F_1^{(n)}(x), F_2^{(n)}(y)] - C_\infty[\varepsilon_a(x), \varepsilon_b(y)]| \\ &\leqslant |C_n[F_1^{(n)}(x), F_2^{(n)}(y)] - C_\infty[F_1^{(n)}(x), F_2^{(n)}(y)]| \\ &\quad + |C_\infty[F_1^{(n)}(x), F_2^{(n)}(y)] - C_\infty[\varepsilon_a(x), \varepsilon_b(y)]|\,. \end{aligned}$$

Now, because of (3), one has

$$|C_\infty[F_1^{(n)}(x), F_2^{(n)}(y)] - C_\infty[\varepsilon_a(x), \varepsilon_b(y)]| \leqslant |F_1^{(n)}(x) - \varepsilon_a(x)| + |F_2^{(n)}(y) - \varepsilon_b(y)|$$

and this tends to zero as n tends to infinity if $x \neq a$ and $y \neq b$. Also, on account of (4),

$$\begin{aligned} &|C_n[F_1^{(n)}(x), F_2^{(n)}(y)] - C_\infty[F_1^{(n)}(x), F_2^{(n)}(y)]| \\ &\leqslant \min\{F_1^{(n)}(x), F_2^{(n)}(y)\} - \max\{F_1^{(n)}(x) + F_2^{(n)}(y) - 1, 0\}\,. \end{aligned}$$

But

$$\lim_{n\to\infty} \min\{F_1^{(n)}(x), F_2^{(n)}(y)\} = \min\{\varepsilon_a(x), \varepsilon_b(y)\} \qquad (x \neq a,\ y \neq b), \tag{5}$$

$$\lim_{n\to\infty} \max\{F_1^{(n)}(x)+F_2^{(n)}(y)-1, 0\}=\max\{\varepsilon_a(x)+\varepsilon_b(y)-1, 0\} \quad (x \neq a,\ y \neq b). \tag{6}$$

The limits (5) and (6) are equal, as is immediate to check directly, or by recourse to Theorem 1 (iii) in [**1**]. This proves the assertion.

Example 2. Let $\{\lambda_n: n = 1, 2, \dots\}$ be a real sequence which converges to $\lambda > 0$ (e.g., $\lambda_n = \lambda - 1/n$). Let $F_k^{(n)} \in \Delta$ ($k = 1, 2$; $n = 1, 2, \dots$) be defined by

$$F_1^{(n)}(t) = F_2^{(n)}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - \exp(-\lambda_n t) & \text{if } t > 0 . \end{cases}$$

Consider the 2-d. f. $H_n \in \Gamma(F_1^{(n)}, F_2^{(n)})$ defined by

$$H_{2k}(x, y) = \max\{F_1^{(2k)}(x) + F_2^{(2k)}(y) - 1, 0\} \qquad \text{if } n = 2k$$

and by
$$H_{2k+1}(x, y) = \min\{F_1^{(2k+1)}(x), F_2^{(2k+1)}(y)\} \qquad \text{if } n = 2k + 1.$$

The sequence $\{H_n\}$ does not converge weakly. In fact, for instance, $H_{2k}(x, y)=0$ on the square $[0, \ln 2/\lambda]\times[0, \ln 2/\lambda]$ on which $H_{2k+1}(x, y) \to 1 - \exp\{-\lambda \min (x, y)\}$ as k tends to infinity.

3 - Extensions and conclusion

With the obvious modifications to both statements and proofs, theorems 1, 3, 4 and 5, as well as Example 1 (but not Example 2) continue to hold for every $r > 2$. In Theorem 2, (1) and (2) read respectively

$$C(1, 1, \dots, 1, s, 1, \dots, 1) = s \qquad (i = 1, 2, \dots, r),$$

$$C(s_1, s_2, \dots, s_r) = 0 \qquad \text{if } s_i = 0 \text{ for at least an index } i .$$

The inequalities (4) become

$$\max(s_1 + s_2 + \dots + s_r - r + 1, 0) \leqslant C(s_1, s_2, \dots, s_r) \leqslant \min(s_1, s_2, \dots, s_r) ; \tag{7}$$

however the lower bound is *not* a r-copula if $r \geqslant 3$, although (7) provides the best possible lower bound ([**6**]). For the same reason one has to modify the sequence $\{H_{2k}\}$ in Example 2; it suffices to take

$$H_{2k}(x_1, x_2, \ldots, x_r) = \prod_{i=1}^{r} F_i^{(2k)}(x_i) \, .$$

The gist of theorems 1, 3, 4, 5 and their unstated analogues for $r > 2$ is that the concept of weak convergence in Δ_r, for $r > 1$, is slightly more general than the concept of convergence in the product topology in $\Delta \times \Delta \times \ldots \times \Delta$.

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Sommario

Si studiano i rapporti tra la convergenza debole in Δ_2, lo spazio delle funzioni di ripartizione doppie, e la convergenza nella topologia prodotto indotta in $\Delta \times \Delta$ dalla topologia della metrica nello spazio delle funzioni di ripartizione semplici Δ. Si mostra che il primo concetto è leggermente più generale del secondo.

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