Riv. Mat. Univ. Parma (4) 9 (1983), 433-445

LUIS A. CORDERO and MANUEL DE LEON (*)

Tensor fields and connections on cross-sections in the frame bundle of a parallelizable manifold (**)

Introduction

Let M be an n-dimensional differentiable manifold of class C^∞, TM its tangent bundle and $\mathscr{F}M$ its frame bundle. The differential geometry of TM has been studied by many authors and a survey of their results can be found in Yano and Ishihara [**5**]. On the other hand, the differential geometry of $\mathscr{F}M$ can be investigated by developing a theory of lifts of tensor fields and linear connections from M to $\mathscr{F}M$ similar to that for TM; this has been firstly done by Mok $[\mathbf{3}]_{1,2}$, who introduced the complete lifts of tensor fields of type $(1, s)$, $s \geqslant 0$, and of linear connections on M; more recently, in $[\mathbf{1}]_{1,2}$, we extended Mok's definitions to tensor fields of type $(0, s)$, $s \geqslant 0$, and introduced the horizontal and diagonal lifts of tensor fields as well as the horizontal lift of linear connections.

When a field of global frames is given in a parallelizable manifold M, it defines a cross-section $\sigma: M \to \mathscr{F}M$ in the frame bundle. In this paper, we study the behaviour on this cross-section of lifts of tensor fields and linear connections on M.

After a brief summary of definitions and results which are needed later, in **2** the complete lifts of tensor fields along the n-dimensional submanifold $\sigma(M)$ of $\mathscr{F}M$ are considered. In **3**, we study the particular case of almost complex structures, Riemannian metrics or symplectic forms on M. Finally, in **4**, the linear connection ∇' induced on $\sigma(M)$ from the complete lift ∇^C of a linear connection ∇ on M is studied.

(*) Indirizzo degli AA.: Departamento de Geometria y Topologia, Facultad de Matematicas, Universidad de Santiago de Compostela, Santiago de Compostela, Spain.
(**) Ricevuto: 14-VI-1982.

All results in this paper can be closely compared with those of the corresponding theory for cross-sections in the tangent bundle ([**4**], and [**5**], chapter III).

1 - Preliminaries

In this section, we shall fix our notations and recall, for later use, the definitions and some properties of the vertical and complete lifts of tensor fields and of the complete lift of linear connections to the frame bundle. Details can be found in Mok $[\mathbf{3}]_2$ and in our paper $[\mathbf{1}]_1$.

Manifolds, tensor fields and linear connections under consideration are all assumed to be differentiable and of class C^∞, and the manifolds to be connected.

1.1 – Indices $i, j, k, \ldots, \alpha, \beta, \gamma, \ldots$ have range in $\{1, 2, \ldots, n\}$. We put $h_\alpha = \alpha n + h$. Summation over repeated indices is always implied.

1.2 – Entries of matrices are written as A^i_j, A_{ij} or A^{ij}, and in all cases i is the row index while j is the column index.

1.3 – Let M be an n-dimensional manifold. Coordinate systems in M are denoted by (U, x^i), where U is the coordinate neighborhood and x^i are the coordinate functions. Components in (U, x^i) of geometric objects on M will be referred to simply as components in U, or just components. We denote the Lie derivative by $\mathscr{L}_X$, and by $T^r_s(M)$ the set of all the tensor fields on M of type (r, s).

Let T_xM be the tangent space at a point $x \in M$, $(X_\alpha) = (X_1, \ldots, X_n)$ a linear frame at x and $\mathscr{F}M$ the frame bundle over M, that is, the set of all frames at all points of M. Let $\pi\colon \mathscr{F}M \to M$ be the canonical projection of $\mathscr{F}M$ onto M; for the coordinate system (U, x^i) in M we put $\mathscr{F}U = \pi^{-1}(U)$. A frame (X_α) at x can be expressed uniquely in the form $X_\alpha = X^i_\alpha(\partial/\partial x^i)_x$. The induced coordinate system in $\mathscr{F}U$ is $\{\mathscr{F}U, (x^i, X^i_\alpha)\}$; we shall denote $\partial/\partial x^i$ by ∂_i and $\partial/\partial X^i_\alpha$ by ∂_{i_α}. The matrix $[X^i_\alpha]$ is non-singular and its inverse will be written as $[X^\alpha_i]$.

1.4 – Let S be a tensor field on M of type $(1, s)$, $s \geqslant 0$, and let $S^h_{j_1 \ldots j_s}$ be its local components in U; then the complete lift S^C of S to $\mathscr{F}M$ (see $[\mathbf{3}]_2$) is the tensor field of the same type given in $\mathscr{F}U$ by

$$(1.1) \qquad S^C = S^h_{j_1\ldots j_s}\partial_h \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} + (X^k_\gamma \partial_k S^h_{j_1 \ldots j_s})\partial_{h_\gamma} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} + \sum_{i=1}^{s} \delta^\beta_\gamma S^h_{j_1\ldots j_s} \partial_{h_\gamma} \otimes dx^{j_1} \otimes \ldots \otimes dX^{j_i}_\beta \otimes \ldots \otimes dx^{j_s}.$$

So, if $s=0$, that is, if X is a vector field on M with local components X^i, its complete lift X^C to $\mathscr{F}M$ is the vector field locally given by

$$X^C = X^i\partial_i + (X^k_\alpha \partial_k X^i)\partial_{i_\alpha}. \tag{1.2}$$

Let S be a tensor field on M of type $(0,s)$, $s\geq 1$, and let $S_{j_1\dots j_s}$ be its local components in U; then the complete lift S^C of S to $\mathscr{F}M$ (see [1]$_1$) is the tensor field of the same type given in $\mathscr{F}U$ by

$$S^C = \sum_{\alpha=1}^{n} \{(X^i_\alpha \partial_i S_{j_1\dots j_s})\,\mathrm{d}x^{j_1}\otimes\dots\otimes \mathrm{d}x^{j_s} + \sum_{k=1}^{s} S_{j_1\dots j_s}\mathrm{d}x^{j_1}\otimes\dots\otimes \mathrm{d}X^{j_k}_\alpha\otimes\dots\otimes \mathrm{d}x^{j_s}\}. \tag{1.3}$$

If φ is a differentiable function on M, then the complete lift φ^C of φ to $\mathscr{F}M$ is the differentiable function given by

$$\varphi^C = \sum_{\alpha=1}^{n} X^i_\alpha \partial_i \varphi. \tag{1.4}$$

Actually, the complete lift S^C of a tensor field S to $\mathscr{F}M$ of type (r,s), $r=0,1$, $s\geq 1$, is the unique tensor field on $\mathscr{F}M$ of the same type satisfying

$$S^C(X_1^C,\dots,X_s^C) = (S(X_1,\dots,X_s))^C, \tag{1.5}$$

for arbitrary vector fields $X_1,\dots,X_s$ on M.

The vertical lift S^V to $\mathscr{F}M$ of a tensor field S of type $(0,s)$, $s\geq 0$, on M is defined by setting $S^V=\pi^*S$.

1.5 – Let Γ be a linear connection on M with components Γ^h_{ji}. Its covariant differentiation will be denoted by ∇. The curvature tensor R and the torsion tensor T of Γ have components R^h_{kji} and T^h_{ji} respectively.

The complete lift Γ^C of Γ to $\mathscr{F}M$ is the unique linear connection on $\mathscr{F}M$ determined by the condition

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C,$$

where X, Y are arbitrary vector fields on M and ∇^C denotes the covariant differentiation with respect to Γ^C. The components of Γ^C are the following

$$\tilde{\Gamma}^h_{ji} = \Gamma^h_{ji}\,, \quad \tilde{\Gamma}^{h_\gamma}_{ji} = X^k_\gamma \partial_k \Gamma^h_{ji}\,, \quad \tilde{\Gamma}^{h_\gamma}_{ji_\alpha} = \delta^\gamma_\alpha \Gamma^h_{ji}\,, \quad \tilde{\Gamma}^{h_\gamma}_{j_\beta i} = \delta^\gamma_\beta \Gamma^h_{ji}\,,$$
$$\tilde{\Gamma}^h_{ji_\alpha} = 0\,, \quad \tilde{\Gamma}^h_{j_\beta i} = 0\,, \quad \tilde{\Gamma}^h_{j_\beta i_\alpha} = 0\,, \quad \tilde{\Gamma}^{h_\gamma}_{j_\beta i_\alpha} = 0\,. \tag{1.6}$$

Moreover, the curvature tensor and the torsion tensor of Γ^C are precisely R^C and T^C, respectively, that is, the complete lifts of the curvature tensor R and the torsion tensor T of Γ.

2 - Lifts of tensor fields on a cross-section

Let σ be a cross-section of the frame bundle $\mathcal{F}M$ of M, that is $\sigma\colon M \to \mathcal{F}M$ a mapping of class C^∞ such that $\pi\cdot\sigma = \text{identity}$. Then σ defines a field of global frames on M, that is, at each point $x \in M$, $\sigma(x) = (\sigma_1(x), \ldots, \sigma_n(x))$ is a linear frame at x. If we put $\sigma = (\sigma_1, \ldots, \sigma_n)$ then each σ_α is a vector field globally defined on M. Assume that σ_α has local components $\sigma^h_\alpha(x)$ with respect to a coordinate system (U, x^i) in M, that is $\sigma_\alpha = \sigma^h_\alpha \partial_h$ in U. Then $\sigma(M)$, which will be called the *cross-section* determined by σ, is the n-dimensional submanifold of $\mathcal{F}M$ locally expressed in $\mathcal{F}U$ by $x^h = x^h$, $X^h_\alpha = \sigma^h_\alpha$, and from these equations we easily see that the n vector fields given with respect to the induced coordinates in $\mathcal{F}M$ by

$$B_i = \partial_i + (\partial_i \sigma^h_\alpha)\partial_{h_\alpha} \tag{2.1}$$

are tangent to $\sigma(M)$.

For a vector field X on M with local components X^h, we shall denote by BX the vector field on $\mathcal{F}M$ given in $\mathcal{F}U$ by

$$BX = X^h \partial_h + X^i(\partial_i \sigma^h_\alpha)\partial_{h_\alpha}\,. \tag{2.2}$$

Obviously, BX is tangent to $\sigma(M)$ and the correspondence $X \to BX$ determines a mapping $B\colon T^1_0(M) \to T^1_0(\sigma(M))$ which is in fact the differential of $\sigma\colon M \to \mathcal{F}M$ and so an isomorphism of $T^1_0(M)$ onto $T^1_0(\sigma(M))$.

From (2.2), we easily obtain, for any $X, Y \in T^1_0(M)$,

$$[BX, BY] = B[X, Y]\,. \tag{2.3}$$

Let U be a coordinate neighborhood in M; then, the $n + n^2$ local vector fields $B_i = B(\partial_i)$, $C_{j_\alpha} = \partial_{j_\alpha}$, form a local family of frames along $\sigma(M)$ which will be called the *adapted frame* of $\sigma(M)$ in $\mathscr{F}U$.

Hereafter, we shall assume $\sigma = (\sigma_1, \ldots, \sigma_n)$ and hence $\sigma(M)$ to be fixed, and shall study some properties of the cross-section $\sigma(M)$ with respect to the adapted frame above.

Let X be a vector field on M and X^C its complete lift to $\mathscr{F}M$, which is locally given by (1.2); then, since

$$\mathscr{L}_{\sigma_\alpha} X = (\sigma^k_\alpha \partial_k X^j - X^k \partial_k \sigma^j_\alpha)\partial_j ,$$

we have along $\sigma(M)$

$$X^C = BX + (\mathscr{L}_{\sigma_\alpha} X)^j C_{j_\alpha} . \tag{2.4}$$

Therefore

Proposition 2.1. *A necessary and sufficient condition for the complete lift X^C of a vector field X on M to $\mathscr{F}M$ to be tangent to the cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \ldots, \sigma_n)$ is that the Lie derivative of X with respect to each σ_α vanishes*, i.e. $\mathscr{L}_{\sigma_\alpha} X = 0$, $1 \leqslant \alpha \leqslant n$.

The *adapted coframe* of $\sigma(M)$ in $\mathscr{F}U$ dual to the adapted frame $\{B_i, C_{j_\alpha}\}$ is easily shown to be given along $\sigma(M)$ by

$$\eta^i = \mathrm{d}x^i , \qquad \eta^{i_\alpha} = -(\partial_j \sigma^i_\alpha)\,\mathrm{d}x^j + \mathrm{d}X^i_\alpha . \tag{2.5}$$

Then taking into account (1.3)-(1.5) and (2.4), if τ is a differentiable 1-form on M with local components τ_i, its complete lift τ^C along $\sigma(M)$ is locally expressed in terms of the adapted coframe by

$$\tau^C = \sum_{\alpha=1}^{n} \{(\mathscr{L}_{\sigma_\alpha} \tau)_i \eta^i + \delta_{\alpha\beta} \tau_i \eta^{i_\beta})\} .$$

Therefore

Proposition 2.2. *A necessary and sufficient condition for the complete lift τ^C of a 1-form τ on M to $\mathscr{F}M$ to be zero for all vector fields tangent to the cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \ldots, \sigma_n)$ is that the Lie derivative of τ with respect to the vector field $\bar{\sigma} = \sum_{\alpha=1}^{n} \sigma_\alpha$ vanishes*, i.e. $\mathscr{L}_{\bar{\sigma}} \tau = 0$.

Let $F \in T^1_1(M)$; then, taking into account (1.1), (2.1) and (2.5), if F^h_j are the local components of F, its complete lift F^C to $\mathscr{F}M$ along $\sigma(M)$ is locally given in terms of the adapted frame by

$$F^C = F_j^h B_h \otimes \eta^j + \delta^{\alpha\beta}(\mathscr{L}_{\sigma_\alpha} F)_j^h C_{h_\beta} \otimes \eta^j + \delta_\beta^\alpha F_j^h C_{h_\alpha} \otimes \eta^{j_\beta}. \tag{2.6}$$

Similarly, let $G \in T_2^0(M)$ with local components G_{ji}; then, the complete lift G^C of G to $\mathscr{F}M$ along $\sigma(M)$ is locally given in terms of the adapted coframe by

$$G^C = (\mathscr{L}_{\bar\sigma} G)_{ji} \eta^j \otimes \eta^i + \sum_{\alpha=1}^n \{G_{ji} \eta^{j_\alpha} \otimes \eta^i + G_{ji} \eta^j \otimes \eta^{i_\alpha}\}, \qquad \bar\sigma = \sum_{\alpha=1}^n \sigma_\alpha. \tag{2.7}$$

And, analogously, the vertical lift G^V of G to $\mathscr{F}M$ along $\sigma(M)$ is locally given by

$$G^V = G_{ji} \eta^j \otimes \eta^i. \tag{2.8}$$

3 - Lifts of tensor fields of type (1, 1) and of type (0, 2) on a cross-section

3.1 – *Lifts of tensor fields of type* (1, 1).

Let $F \in T_1^1(M)$ with local components F_j^h. Then, from (2.2) and (2.6), we have along the cross-section determined by $\sigma = (\sigma_1, \ldots, \sigma_n)$ that

$$F^C(BX) = F_j^h X^j B_h + \delta^{\alpha\beta}(\mathscr{L}_{\sigma_\alpha} F)_j^h X^j C_{h_\beta},$$

for any vector field X on M with components X^h. Since $B(FX) = F_j^h X^j B_h$, we have

$$F^C(BX) = B(FX) + \delta^{\alpha\beta}(\mathscr{L}_{\sigma_\alpha} F)_j^h X^j C_{h_\beta}. \tag{3.1}$$

When $F^C(BX)$ is tangent to $\sigma(M)$ for any vector field X on M, F^C is said to leave $\sigma(M)$ invariant. Thus we have from (3.1)

Proposition 3.1. *The complete lift F^C of an element F of $T_1^1(M)$ leaves the cross-section $\sigma(M)$ invariant if and only if $\mathscr{L}_{\sigma_\alpha} F = 0$ for every $\alpha = 1, 2, \ldots, n$.*

Now, assume that F^C leaves $\sigma(M)$ invariant. Then we can define an element $F^{C\#} \in T_1^1(\sigma(M))$ by

$$F^{C\#}(BX) = F^C(BX) = B(FX), \tag{3.2}$$

for arbitrary $X \in T_0^1(M)$; $F^{c\#}$ is called the *tensor field induced* on $\sigma(M)$ from F^c. We now see from (2.2) that $F^{c\#} = \sigma^* F$, where σ^* denotes the differential of $\sigma: M \to \mathscr{F}M$.

Let us now recall from [**3**]$_2$ that if F defines on M a polynomial structure of rank r and structural polynomial $P(t) = 0$ (i.e. rank $F = r$ and $P(F) = 0$), then its complete lift F^c to $\mathscr{F}M$ defines on $\mathscr{F}M$ a polynomial structure with the same structural polynomial and with rank $F^c = (n+1)r$. Moreover, if N_F, N_{F^c} denote the Nijenhuis tensors of F and of F^c respectively, then $(N_F)^c = N_{F^c}$.

So, if F defines on M a polynomial structure of rank r and $P(F) = 0$, and if F^c leaves $\sigma(M)$ invariant, then $F^{c\#}$ verifies $P(F^{c\#}) = 0$ and rank $F^{c\#} = r$, and, hence, $F^{c\#}$ defines on $\sigma(M)$ a polynomial structure of the same type.

Taking into account (1.1) and (2.1), one obtains

$$(N_F)^c(BX, BY) = B(N_F(X, Y)) + \sum_{\alpha=1}^{n} (\mathscr{L}_{\sigma_\alpha} N_F)_{ji}^h X^j Y^i C_{h_\alpha}, \tag{3.3}$$

along $\sigma(M)$, for any X, $Y \in T_0^1(M)$ with local components X^j, Y^i respectively. Thus

Proposition 3.2. *Let N_F and N_{F^c} be respectively the Nijenhuis tensor of $F \in T_1^1(M)$ and of its complete lift F^c to $\mathscr{F}M$. Then, in order that $N_{F^c}(BX, BY)$ be tangent to the cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \ldots, \sigma_n)$, for any vector fields X, Y on M, it is necessary and sufficient that $\mathscr{L}_{\sigma_\alpha} N_F = 0$ for every $\alpha = 1, 2, .., n$.*

We now assume that F^c leaves $\sigma(M)$ invariant. Then, from (2.3) and (3.2), we have

$$\begin{aligned} N_{F^c}(BX, BY) &= [F^c(BX), F^c(BY)] - F^c[F^c(BX), BY] \\ &\qquad - F^c[BX, F^c(BY)] + (F^c)^2[BX, BY] \\ &= [F^{c\#}(BX), F^{c\#}(BY)] - F^{c\#}[F^{c\#}(BX), BY] - F^{c\#}[BX, F^{c\#}(BY)] + (F^{c\#})^2[BX, BY], \end{aligned}$$

that is $N_{F^c}(BX, BY) = N_{F^{c\#}}(BX, BY)$, for arbitrary vector fields X, Y on M. Then, since $\mathscr{L}_{\sigma_\alpha} F = 0$ implies $\mathscr{L}_{\sigma_\alpha} N_F = 0$, from (3.3) we have

Proposition 3.3. *Suppose that the complete lift F^c of $F \in T_1^1(M)$ leaves $\sigma(M)$ invariant. Then $N_{F^{c\#}} = 0$ if and only if $N_F = 0$.*

Next, let us suppose that $F \in T_1^1(M)$ defines an almost complex structure on M, i.e. $F^2 = -1$; then, F^C defines an almost complex structure on $\mathscr{F}M$. Recall that a submanifold in an almost complex manifold with structure F is said to be *invariant* or *almost analytic* when F leaves the submanifold invariant. Thus, from the previous propositions, we deduce

Proposition 3.4. *Let be $\sigma = (\sigma_1, \ldots, \sigma_n)$ a global frame field on a parallelizable manifold M with an almost complex structure F. A necessary and sufficient condition for the cross-section $\sigma(M)$ in $\mathscr{F}M$ to be almost analytic in the almost complex manifold $\mathscr{F}M$ with structure F^C is that each vector field σ_α be almost analytic, that is, $\mathscr{L}_{\sigma_\alpha} F = 0$. In this case, the cross-section $\sigma(M)$ is an almost complex manifold with structure tensor $F^{C\#}$ which is induced on $\sigma(M)$ from F^C; moreover, $N_{F^{C\#}} = 0$, that is $F^{C\#}$ is complex analytic, if and only if F is complex analytic in M, that is, $N_F = 0$.*

Let X be a vector field on M and $F \in T_1^1(M)$ such that F^C leaves $\sigma(M)$ invariant. Then, $(\mathscr{L}_{BX} F^{C\#})(BY) = B((\mathscr{L}_X F)(Y))$ for any $Y \in T_0^1(M)$; therefore $\mathscr{L}_{BX} F^{C\#} = 0$ *if and only if* $\mathscr{L}_X F = 0$ and hence

Proposition 3.5. *Let F be an almost complex structure in M such that F^C leaves the cross-section $\sigma(M)$ invariant. Then, for any vector field X on M, BX is almost analytic in $\sigma(M)$ if and only if X is almost analytic in M.*

3.2 – *Lifts of tensor fields of type* $(0, 2)$.

Let G be a tensor field of type $(0, 2)$ on M. Then, from (2.7), we have along the cross-section $\sigma(M)$,

$$G^C(BX, BY) = ((\mathscr{L}_{\bar\sigma} G)(X, Y))^V, \qquad \bar\sigma = \sum_{\alpha=1}^{n} \sigma_\alpha, \tag{3.4}$$

for any vector fields X, Y on M. Then, putting $G^{C\#}(BX, BY) = G^C(BX, BY)$, we have an element $G^{C\#}$ of $T_2^0(\sigma(M))$, and thus, from (3.4),

$$G^{C\#}(BX, BY) = ((\mathscr{L}_{\bar\sigma} G)(X, Y))^V, \qquad \bar\sigma = \sum_{\alpha=1}^{n} \sigma_\alpha,$$

for any vector fields X, Y on M. Therefore,

Proposition 3.6. $G^{C\#} = 0$ *if and only if* $\mathscr{L}_{\bar\sigma} G = 0$, *where* $\bar\sigma = \sum_{\alpha=1}^{n} \sigma_\alpha$.

In particular, if G is a Riemannian metric on M, then from Proposition 3.6 we get

Proposition 3.7. *A necessary and sufficient condition for the cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \dots, \sigma_n)$ in a Riemannian manifold M with metric G to be a null manifold with respect to the complete lift G^C of G,* i.e. $G^{C\#} = 0$, *is that* $\bar{\sigma} = \sum_{\alpha=1}^{n} \sigma_\alpha$ *be a Killing vector field in M,* i.e. $\mathscr{L}_{\bar{\sigma}} G = 0$.

On the other hand, from (2.8), we obtain along $\sigma(M)$

$$G^V(BX, BY) = (G(X, Y))^V$$

for any vector fields X, Y on M. Then we can define an element $G^{V\#} \in T_2^0(\sigma(M))$ by putting

$$G^{V\#}(BX, BY) = G^V(BX, BY) \tag{3.5}$$

for any vector fields X, Y on M. Thus we have $G^{V\#} = \sigma^* G$ where σ^* is the mapping induced from $\sigma\colon M \to \mathscr{F}M$. Hence

Proposition 3.8. *Let M be a Riemannian manifold with metric G. Then, the cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \dots, \sigma_n)$ is a Riemannian manifold with metric $G^{V\#}$ and the projection $\pi\colon \mathscr{F}M \to M$ is an isometry.*

Next, assume that $G \in T_2^0(M)$ is a 2-form; then, $G^{V\#}$ given by (3.5) is a 2-form on $\sigma(M)$, and a straightforward computation shows the identity

$$\mathrm{d}G^{V\#}(BX, BY, BZ) = (\mathrm{d}G(X, Y, Z))^V,$$

along $\sigma(M)$, for arbitrary vector fields X, Y, Z on M. Therefore

Proposition 3.9. *$G^{V\#}$ is closed along $\sigma(M)$ if and only if G is closed.*

Since rank $G^{V\#}$ along $\sigma(M)$ is equal to rank G on M, we easily deduce

Corollary 3.10. *The cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \dots, \sigma_n)$ is a symplectic manifold with respect to $G^{V\#}$,* i.e. $\mathrm{d}G^{V\#} = 0$ *and rank* $G^{V\#} = n$, *if and only if M is a symplectic manifold with respect to G,* i.e. $\mathrm{d}G = 0$ *and rank* $G = n$.

For an arbitrary $G \in T_2^0(M)$, we have along $\sigma(M)$

$$(\mathscr{L}_{BX} G^{V\#})(BY, BZ) = ((\mathscr{L}_X G)(Y, Z))^V,$$

for any vector fields X, Y, Z on M. Therefore

$$\mathscr{L}_{BX} G^{V\#} = 0 \quad \textit{if and only if} \quad \mathscr{L}_X G = 0 ,$$

and, hence

Corollary 3.11. (1) *Under the hypothesis of Proposition* 3.8, *a vector field X on M is Killing for the metric G on M if and only if BX is so for the metric $G^{V\#}$. on $\sigma(M)$.*

(2) *Under the hypothesis of Corollary* 3.10, *a vector field X on M is an infinitesimal symplectic automorphism with respect to G on M if and only if BX is so with respect to $G^{V\#}$ on $\sigma(M)$.*

4 - Linear connections induced on a cross-section

Let M be a manifold with a linear connection ∇. Then the frame bundle $\mathscr{F}M$ of M is a manifold with linear connection ∇^C, the complete lift of ∇. We now study the linear connection ∇', induced from ∇^C on the cross section $\sigma(M)$ determined by $\sigma = (\sigma_1, \ldots, \sigma_n)$ on M, with respect to the adapted frame of $\sigma(M)$.

From (1.6) and (2.1), through a direct computation we get along $\sigma(M)$

$$\nabla^C_{B_j} B_i = \Gamma^h_{ji} B_h + \sum_{\alpha=1}^{n} (\mathscr{L}_{\sigma_\alpha} \nabla)^h_{ji} C_{h_\alpha} ,$$

where Γ^h_{ji} are the components of ∇; therefore

$$\nabla'_{B_j} B_i = \Gamma^h_{ji} B_h$$

defines the induced linear connection ∇' on $\sigma(M)$, and

$$\nabla^C_{B_j} B_i = \nabla'_{B_j} B_i + \sum_{\alpha=1}^{n} (\mathscr{L}_{\sigma_\alpha} \nabla)^h_{ji} C_{h_\alpha}$$

is the Gauss formula for the cross-section $\sigma(M)$.

Proposition 4.1. *The cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \ldots, \sigma_n)$ is autoparallel with respect to ∇^C if and only if each σ_α, $1 \leqslant \alpha \leqslant n$, is an infinitesimal affine transformation in M,* i.e. *$\mathscr{L}_{\sigma_\alpha} \nabla = 0$ for any $\alpha = 1, 2, \ldots, n$.*

On the other hand, along $\sigma(M)$

$$\nabla^c_{B_j} C_{i_\alpha} = \Gamma^h_{ji} C_{h_\alpha},$$

which is the Weingarten formula for the cross-section $\sigma(M)$.

Let $\tilde{R}$ be the curvature tensor of ∇^c; since $[B_j, B_i] = 0$, we have

$$\tilde{R}(B_k, B_j) B_i = \nabla^c_{B_k} \nabla^c_{B_j} B_i - \nabla^c_{B_j} \nabla^c_{B_k} B_i .$$

Thus, by a straightforward computation we obtain

$$\text{(4.1)}\quad \tilde{R}(B_k, B_j) B_i = R^h_{ikj} B_h + \sum_{\alpha=1}^{n} \{\nabla_k (\mathscr{L}_{\sigma_\alpha}\nabla)^h_{ji} - \nabla_j (\mathscr{L}_{\sigma_\alpha}\nabla)^h_{ki} + T^l_{kj} (\mathscr{L}_{\sigma_\alpha}\nabla)^h_{li}\} C_{h_\alpha},$$

where R^h_{ikj} and T^h_{kj} are the components of the curvature and torsion tensors of ∇, respectively. Taking into account the well known formula

$$\nabla_k (\mathscr{L}_{\sigma_\alpha}\nabla)^h_{ji} - \nabla_j (\mathscr{L}_{\sigma_\alpha}\nabla)^h_{ki} = (\mathscr{L}_{\sigma_\alpha} R)^h_{ikj},$$

(4.1) reduces to

$$\text{(4.2)}\qquad \tilde{R}(B_k, B_j) B_i = R^h_{ikj} B_h + \sum_{\alpha=1}^{n} \{(\mathscr{L}_{\sigma_\alpha} R)^h_{ikj} + T^l_{kj} (\mathscr{L}_{\sigma_\alpha}\nabla)^h_{li}\} C_{h_\alpha}.$$

From (4.1) and (4.2), and since $\tilde{R} = R^c$, we deduce

Proposition 4.2. *Let R be the curvature tensor of a linear connection ∇ on M. In order that $R^c(\tilde{X}, \tilde{Y})\tilde{Z}$, evaluated for vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ tangent to the cross-section $\sigma(M)$, be always tangent to $\sigma(M)$ we have*

(1) *if ∇ is torsionfree, then a necessary and sufficient condition is the vanishing of the Lie derivatives $\mathscr{L}_{\sigma_\alpha} R$, for $\alpha = 1, 2, \dots, n$,*

(2) *if ∇ has no vanishing torsion, then a sufficient condition is that $\sigma(M)$ be autoparallel with respect to ∇^c or, equivalently, that $\mathscr{L}_{\sigma_\alpha}\nabla = 0$ for every $\alpha = 1, 2, \dots, n$.*

Let $F \in T^1_1(M)$ be such that F^c leaves $\sigma(M)$ invariant or, equivalently, satisfying $\mathscr{L}_{\sigma_\alpha} F = 0$ for every $\alpha = 1, 2, \dots, n$. Then, it induces $F^{c\#} \in T^1_1(\sigma(M))$ and thus, along $\sigma(M)$, we obtain $\nabla'_{BX} F^{c\#}(BY) = B(\nabla_X F Y)$ for any vector fields X, Y on M. Therefore

Proposition 4.3. *Let $F \in T^1_1(M)$ be such that F^c leaves $\sigma(M)$ invariant. Then $\nabla' F^{c\#} = 0$ if and only if $\nabla F = 0$.*

Let be $G \in T^0_2(M)$ and let $G^{V\#}$ be the tensor field of type $(0, 2)$ induced from G on $\sigma(M)$; then, along $\sigma(M)$, we have

$$(\nabla'_{BX} G^{V\#})(BY, BZ) = \{(\nabla_X G)(Y, Z)\}^V ,$$

for any vector fields X, Y, Z on M. Therefore

$$\nabla' G^{V\#} = 0 \quad \textit{if and only if} \quad \nabla G = 0 ,$$

and thus, taking into account Propositions 3.8 and 3.9 and Corollary 3.10, we deduce

Proposition 4.4. (1) *Let G be a Riemannian metric on M and ∇ its Riemannian connection. Then, the connection ∇', induced on the cross-section $\sigma(M)$ from the complete lift ∇^C of ∇, is the Riemannian connection constructed from the metric $G^{V\#}$ induced on $\sigma(M)$ from G^V.*

(2) *Let G be an almost-symplectic* (resp. *symplectic*) *2-form on M and ∇ an adapted connection,* i.e. $\nabla G = 0$. *Then, the linear connection ∇', induced on the cross-section $\sigma(M)$ from the complete lift ∇^C of ∇, is adapted with respect to the almost-symplectic* (resp. *symplectic*) *form $G^{V\#}$ induced from G^V on $\sigma(M)$.*

Now, let $F \in T^1_1(M)$ and $G \in T^0_2(M)$ be such that F^C leaves $\sigma(M)$ invariant. Then, along $\sigma(M)$

$$G^{V\#}(F^{C\#}(BX), F^{C\#}(BY)) = G^{V\#}(B(FX), B(FY)) = \{G(FX, FY)\}^V ,$$

for any vector fields X, Y on M.

If a Riemannian metric G and a complex structure F on M satisfy the conditions $G(FX, FY) = G(X, Y)$, $\nabla_X F = 0$, for any vector fields X, Y, and ∇ being the Riemannian connection determined by G, then (F, G) is a Kählerian structure. Thus, taking into account the previous results, we have

Proposition 4.5. *Let (F, G) be a Kählerian structure on M such that F^C leaves the cross-section $\sigma(M)$ determined by $\sigma = (\sigma_1, \dots, \sigma_n)$ invariant. Then $(F^{C\#}, G^{V\#})$ is a Kählerian structure on the cross-section $\sigma(M)$.*

References

[1] L. A. CORDERO and M. DE LEON: [•]$_1$ *Lifts of tensor fields to the frame bundle* Rend. Circ. Mat. Palermo (2) **32** (1983), 236-271; [•]$_2$ *Horizontal lifts of connections to the frame bundle* Boll. Un. Mat. Ital. (to appear).

[2] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry* (I), (II), Interscience Publishers, New York 1963-69.

[3] K. P. MOK: [•]$_1$ *On the differential geometry of frame bundles of Riemannian manifolds,* J. Reine Angew Math. **302** (1978), 16-31; [•]$_2$ *Complete lift of tensor fields and connections to the frame bundle,* Proc. London Math. Soc. (3) **32** (1979), 72-88.

[4] K. YANO, *Tensor fields and connections on cross-sections in the tangent bundle, of a differentiable manifold,* Proc. Roy. Soc. Edinburgh, Sect. A **67** (1967), 277-288.

[5] K. YANO and S. ISHIHARA, *Tangent and cotangent bundles,* Differential Geometry, Marcel Dekker Inc., New York 1973.

* * *