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Bounded finitely additive set functions and integrability (**)

1 - Introduction

Suppose N is a positive integer. Functions of the following variety abound: f a function from $\boldsymbol{R}^N$ into $\boldsymbol{R}$, bounded on each bounded subset of $\boldsymbol{R}^N$, having the property that if U is a set, $\boldsymbol{F}$ is a field of subsets of U and $\{\xi_i\}_{i=1}^N$ is a sequence of real-valued finitely additive bounded functions with domain $\boldsymbol{F}$, then the integral (see **2**) $\int_U f(\xi_1(I), \dots, \xi_N(I))$ exists. Examples of such are $\sum_{i=1}^N |x_i|$, $\max\{x_1, \dots, x_N\}$, $\min\{x_1, \dots, x_N\}$, $\prod_{i=1}^N x_i$, and $\prod_{i=1}^N |x_i|\, p_i$ for $1 < N$, $0 < p_i$ for $i = 1, \dots, N$ and $\sum_{i=1}^N p_i = 1$. This paper is concerned with an integrability characterization theorem for such functions. We shall assume given a positive integer N and a function f from $\boldsymbol{R}^N$ into $\boldsymbol{R}$, bounded on each bounded subset of $\boldsymbol{R}^N$. Our principal result is the following, which we shall comment further upon after stating it

Theorem 5.1. (see **5**). *The following three statements are equivalent.*

(1) *If U is a set, $\boldsymbol{F}$ is a field of subsets of U and $\{\xi_i\}_{i=1}^N$ is a sequence of real-valued finitely additive bounded function with domain $\boldsymbol{F}$, then $\int_U f(\xi_1(I), \dots, \xi_N(I))$ exists.*

(2) *If $[r; s]$ is a number interval and $\{g_i\}_{i=1}^N$ is a sequence of real-valued functions with domain $[r; s]$ having bounded variation, then $\int_{[r;s]} f(dg_1, \dots, dg_N)$ exists* (see **2**).

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(3) *f satisfies the following conditions:*

(i) *if* $0 < c$, *then there is* $d > 0$ *such that if* $\{(x_1^{(j)}, \ldots, x_N^{(j)})\}_{j=1}^{m}$ *is a sequence of elements of* $\boldsymbol{R}^N$ *such that* $\sum_{j=1}^{m}\sum_{i=1}^{N}|x_i^{(j)}| < d$, *then* $\sum_{j=1}^{m}|f(x_1^{(j)}, \ldots, x_N^{(j)})| < c$;

(ii) *f is continuous*;

(iii) *suppose that* $\{D(w)\}_{w=1}^{\infty}$ *is a sequence of interval subdivisions* (see **2**) *of* $[0; 1]$ *and* $\{g_i\}_{i=1}^{N}$ *is a sequence of real-valued functions with domain* $Q = \{x\colon x \text{ in } \{p, q\},\ [p; q] \text{ in } D(w) \text{ for some } w\}$ *such that*

(a) $D(w+1) \ll D(w)$ *(see* **2***) for all* w,

(b) *for each* $[p, q]$ *such that* $\{p, q\} \subseteq Q$, $v[p; q] = \sup\{\sum_{i=1}^{N}\sum_{D(w)[p;q]}|g_i(s) - g_i(r)|\colon w \text{ a positive integer},\ \{p, q\} \subseteq \bigcup_{D(w)}\{r, s\},\ D(w)[p; q] = \{[r; s] \text{ in } D(w),\ p \leqslant r < s \leqslant q\}\} < \infty$,

(c) $norm\,(D(n)) \to 0$ *as* $n \to \infty$.

Then, if $0 < c$, *there is* $d > 0$ *and a positive integer* T *such that if* w *is a positive integer* $\geqslant T$ *and* $D(w)^* = \{I\colon I \text{ in } D(w),\ v(I) < d\}$, *then (see* **2***)*

$$\sum_{D(w)^*}|f(\Delta g_1, \ldots, \Delta g_N) - \sum_{D(w+1)(I)} f(\Delta' g_1, \ldots, \Delta' g_N)]| < c\,.$$

Now, at a superficial glance, the equivalence of statements (1) and (2) appears to be trivial; indeed the proof that (2) follows from (1) is so routine that we leave it to the reader. However, showing that (1) follows from (2) is quite another matter and requires some rather intricate considerations, notions expressed in statement (3). The argument for the equivalence of the statements thus runs as follows (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

2 - Preliminary theorems and definitions

For the notions of subdivision, refinement, $\sum$-boundedness and integral, we refer the reader to [**1**] as they apply to real number set-valued interval functions, and to $[\mathbf{1}]_{2,4}$ as they apply to real number set-valued set functions. Throughout this paper, when, in a given discussion, the context of set function vs. interval function is clear, we shall refer to such notions as « subdivision », « refinement », « integral » etc., without preamble and with at most minor notational changes. In either setting « $\ll$ » shall mean « refinement of ». If $E \ll D$ and I is in D, then $E(I)$ denotes $\{J\colon J \text{ in } E,\ J \subseteq I\}$. For real-valued functions defined on number sets we shall use the « Δ » notation in the standard way to denote differences; when the need arises to denote particular subdivisions from which differences arise, appropriate subscripts or superscripts will be attached. The reader is referred, respectively, to $[\mathbf{1}]_{1,2}$ for a statement of Kolmogoroff's differential equivalence theorem [2] and certain of its more immediate consequences, for interval functions and set functions.

Finally, if U is a set and $\boldsymbol{F}$ is a field of subsets of U, then p_{FAB} denotes the set of all real-valued bounded finitely additive functions defined on $\boldsymbol{F}$, and p^+_{FA} denotes the set of all nonnegative-valued elements of p_{FAB}:

3 - The subdivision norm infimum function

In this section we state a definition and theorem of $[\mathbf{1}]_3$ that we shall use in proving Theorem 4.1 and proving that in Theorem 5.1, (3) implies (1).

Suppose that U is a set, $\boldsymbol{F}$ is a field of subsets of U and μ is in p^+_{FA}:

Definition $[\mathbf{1}]_3$: μ^* is the function with domain $\boldsymbol{F}$ such that if V is in $\boldsymbol{F}$, then

$$\mu^*(V) = \inf \{\max \{\mu(J): J \text{ in } D\}: D \ll \{V\}\} .$$

Theorem 3.A.1 $[\mathbf{1}]_3$. $\int_U[\mu(I)^2 - \mu^*(I)^2] = 0$, so that if $0 < c$, then there is $D \ll \{U\}$ such that if $E \ll D$ and I is in E, then $\mu(I) - \mu^*(I) < c$.

4 - A $\sum$-boundedness characterization theorem

Theorem 4.1. *The following three statements are equivalent.*

(1) *If U is a set, $\boldsymbol{F}$ is a field of subsets of U and $\{\xi_i\}_{i=1}^N$ is a sequence of elements of p_{FAB}, then $f(\xi_1, \ldots, \xi_N)$ is $\sum$-bounded on U.*

(2) *If $0 < c$, then there is $d > 0$ such that if $\{(x_1^{(j)}, \ldots, x_N^{(j)})\}_{j=1}^m$ is a sequence elements of $\boldsymbol{R}^N$ such that $\sum_{j=1}^m \sum_{i=1}^N |x_i^{(j)}| < d$, then $\sum_{j=1}^m |f(x_1^{(j)}, \ldots, x_N^{(j)})| < c$.*

(3) *If $[r; s]$ is a number interval and $\{g_i\}_{i=1}^N$ is a sequence of real-valued functions with domain $[r; s]$ having bounded variation, then $f(\Delta g_1, \ldots, \Delta g_N)$ is $\sum$-bounded on $[r; s]$.*

Proof. We first show that (2) implies (1). Suppose (2) is true, U is a set, $\boldsymbol{F}$ is a field of subsets of U, and $\{\xi_i\}_{i=1}^N$ is a sequence of elements of p_{FAB}. There is $M \geqslant 0$ such that if V is in $\boldsymbol{F}$, then $|f(\xi_1(V), \ldots, \xi_N(V))| \leqslant M$. Let $\mu = \sum_{i=1}^N \int |\xi_i|$. There is $d > 0$ such that if $\{(x_1^{(j)}, \ldots, x_N^{(j)})\}_{j=1}^m$ is a sequence of elements of $\boldsymbol{R}^N$ such that $\sum_{j=1}^m \sum_{i=1}^N |x_i^{(j)}| < d$, then $\sum_{j=1}^m |f(x_1^{(j)}, \ldots, x_N^{(j)})| < 1$. There is $D \ll \{U\}$ such that if $E \ll D$ and I is in E, then $\mu(I) - \mu^*(I) < d$. Let $Q =$ the number of elements of D. Suppose $E \ll D$. For each I in D, there is $J(I)$ in $E(I)$ such that $\mu(J(I)) = \max\{\mu(J): J \text{ in } E(I)\}$, so that

$$\sum_{E(I)-\{J(I)\}} \sum_{i=1}^{N} |\xi_i(J)| \leqslant \sum_{E(I)-\{J(I)\}} \mu(J) = \mu(I) - \mu(J(I)) \leqslant \mu(I) - \mu^*(I) < d\,,$$

so that

$$\sum_E |f(\xi_1(J), \dots, \xi_N(J))| = \sum_D \left[\left(\sum_{E(I)-\{J(I)\}} |f(\xi_1(J), \dots, \xi_N(J))|\right) + |f(\xi_1(J(I)), \dots, \xi_N(J(I)))|\right] \leqslant \sum_D [(1) + M] = Q(1+M)\,.$$

Therefore $f(\xi_1, \dots, \xi_N)$ is $\sum$-bounded on U. Therefore (2) implies (1).

The proof that (1) implies (3) is quite routine and we leave the details to the reader.

Now suppose that (3) is true but that (2) is not true, so that there is $c > 0$ such that if $d > 0$, then there is a sequence $\{(x_1^{(j)}, \dots, x_N^{(j)})\}_{j=1}^{m}$ of elements of $\boldsymbol{R}^N$ such that $\sum_{j=1}^{m} \sum_{i=1}^{N} |x_i^{(j)}| < d$, but $\sum_{j=1}^{m} |f(x_1^{(j)}, \dots, x_N^{(j)})| \geqslant c$. Simple considerations tell us that for each positive integer n there is a sequence $\{(x_1^{(j)}(n), \dots, x_N^{(j)}(n))\}_{j=1}^{m(n)}$ of elements of $\boldsymbol{R}^N$ such that $\sum_{j=1}^{m(n)} \sum_{i=1}^{N} |x_1^{(j)}(n)| < 2^{-n}$, but $\sum_{j=1}^{m(n)} |f(x_1^{(j)}(n), \dots, x_N^{(j)}(n))| \geqslant c/2$, and such that either for all n, all of $\{f(x_1^{(j)}(n), \dots, x_N^{(j)}(n))\}_{j=1}^{m(n)}$ are nonnegative, or for all n, all of $\{f(x_1^{(j)}(n), \dots, x_N^{(j)}(n))\}_{j=1}^{m(n)}$ are nonpositive. For each $i = 1, \dots, N$, let $z_i^{(1)}, z_i^{(2)}, \dots = x_i^{(1)}(1), \dots, x_i^{(m(1))}(1), x_i^{(1)}(2), \dots, x_i^{(m(2))}(2), \dots$. For each $i = 1, \dots, N$, let g_i denote the function on $[0; 1]$ such that $g_i(0) = 0$, and if m is a positive integer and $1/(m+1) < x \leqslant 1/m$, then $g_i(x) = \sum_{j=m}^{\infty} z_i^{(j)}$. Clearly, $\{g_i\}_{i=1}^{N}$ is a sequence of real-valued functions with domain $[0; 1]$ having bounded variation. Now, $\{\sum_{j=1}^{w} f(z_1^{(j)}, \dots, z_N^{(j)})\}_{w=1}^{\infty}$ is an unbounded monotonic sequence. There is a number M such that if I is a subinterval of $[0; 1]$, then $|f(\Delta g_1, \dots, \Delta g_N)| \leqslant M$. Now suppose that D is an interval subdivision of $[0; 1]$ and $0 < K$. There is a positive integer t such that for some $[0; p]$ in D, $1/t < p$. There is a positive integer $w > t$ such that

$$\left|\sum_{j=t}^{w} f(z_1^{(j)}, \dots, z_N^{(j)})\right| > K + M + |f(g_1(p) - g_1(1/t), \dots, g_N(p) - g_N(1/t))| + \sum_{D-\{[0;p]\}} |f(\Delta g_1, \dots, \Delta g_N)|\,.$$

Let $E = \{[r; s]: [r; s] = [0; 1/(w+1)], [1/(w+1); 1/w], \dots, [1/(t+1); 1/t], [1/t; p]$, or $[r; s]$ is in $D - \{[0; p]\}\}$. Thus

$$\begin{aligned}\left|\sum_E f(\Delta g_1, \dots, \Delta g_N)\right| &= |f(g_1(1/(w+1)), \dots, g_N(1/(w+1))) \\ &+ \sum_{j=t}^{w} f(z_1^{(j)}, \dots, z_N^{(j)}) + f(g_1(p) - g_1(1/t), \dots, g_N(p) - g_N(1/t)) \\ &+ \sum_{D-\{[0;p]\}} f(\Delta g_1, \dots, \Delta g_N)| \geqslant - |f(g_1(1/(w+1)), \dots, g_N(1/(w+1)))| \\ &+ \left|\sum_{j=t}^{w} f(z_1^{(j)}, \dots, z_N^{(j)})\right| - |f(g_1(p) - g_1(1/t), \dots, g_N(p) - g_N(1/t))| \\ &- \sum_{D-\{[0;p]\}} |f(\Delta g_1, \dots, \Delta g_N)| \geqslant -M + K + M = K\,.\end{aligned}$$

Therefore $f(\Delta g_1, \ldots, \Delta g_N)$ is not $\sum$-bounded on $[0; 1]$, a contradiction.

Therefore (3) implies (2). Therefore (1), (2) and (3) are equivalent.

5 - The integrability characterization theorem

In this section we prove Theorem 5.1, as stated previously.

Proof of Theorem 5.1. As mentioned in the introduction, we leave showing that (1) implies (2) to the reader.

We now show that (2) implies (3). Suppose that (2) is true. First, we see that (i) if (3) immediately follows by Theorem 4.1.

We now show that (ii) if (3) holds. Suppose not. Then there is $c > 0$, $(z_1, \ldots, z_N)$ in $\boldsymbol{R}^N$ and a sequence $\{(z_1^{(j)}, \ldots, z_N^{(j)})\}_{j=1}^{\infty}$ of elements of $\boldsymbol{R}^N$ such that $\sum_{j=1}^{\infty} \sum_{i=1}^{N} |z_i - z_i^{(j)}| < \infty$, $\quad |f(z_1, \ldots, z_N) - f(z_1^{(j)}, \ldots, z_N^{(j)})| \geqslant c \quad$ for all j. Clearly, now, for each $i = 1, \ldots, N$, the real-valued function, g_i, with domain $[0; 1]$ such that $g_i(0) = 0$, $g_i(1/j) = z_i^{(j)}$ for each positive integer j, and $g_i(x) = z_i$ otherwise, has bounded variation.

Now, suppose that D is an interval subdivision of $[0; 1]$. There is $p > 0$ such that $[0; p]$ is in D. There is $d > 0$ such that if $\{(x_1^{(v)}, \ldots, x_N^{(v)})\}_{v=1}^{m}$ is a sequence of elements of $\boldsymbol{R}^N$ such that $\sum_{v=1}^{m} \sum_{i=1}^{N} |x_i^{(v)}| < d$, then $\sum_{v=1}^{m} |f(x_1^{(v)}, \ldots, x_N^{(v)})| < c/4$. There is a positive integer t such that $1/t < p$ and $\sum_{j=t}^{\infty} \sum_{i=1}^{N} |z_i - z_i^{(j)}| < d$.

Consider the interval subdivisions $E_1 = \{[r; s] : [r; s] = [0; 1/t], [1/t; p]$, or $[r; s]$ in $D - \{[0; p]\}\}$ and $E_2 = \{[r; s] : [r; s] = [0; (1/2)(1/(t+1) + 1/t)], [(1/2) \cdot (1/(t+1) + 1/t); 1/t], [1/t; p]$, or $[r; s]$ in $D - \{[0; p]\}\}$. Now

$$\sum_{E_2} f(\Delta g_1, \ldots, \Delta g_N) - \sum_{E_1} f(\Delta g_1, \ldots, \Delta g_N)|$$

$$= |f(z_1, \ldots, z_N) + f(z_1^{(t)} - z_1, \ldots, z_N^{(t)} - z_N) + f(g_1(p) - z_1^{(t)}, \ldots, g_N(p) - z_N^{(t)})$$

$$- \{f(z_1^{(t)}, \ldots, z_N^{(t)}) + f(g_1(p) - z_1^{(t)}, \ldots, g_N(p) - z_N^{(t)})\} |$$

$$= |f(z_1, \ldots, z_N) + f(z_1^{(t)} - z_1, \ldots, z_N^{(t)} - z_N) - f(z_1^{(t)}, \ldots, z_N^{(t)})|$$

$$\geqslant |f(z_1, \ldots, z_N) - f(z_1^{(t)}, \ldots, z_N^{(t)})| - |f(z_1^{(t)} - z_1, \ldots, z_N^{(t)} - z_N)| \geqslant c - c/4 = 3c/4 .$$

Therefore $\int_{[0;1]} f(dg_1, \ldots, dg_N)$ does not exist, a contradiction. Therefore (ii) holds.

We now show that (iii) holds. Suppose not. Then there is $\{D(w)\}_{w=1}^{\infty}$ and $\{g_i\}_{i=1}^{N}$ satisfying conditions (a), (b) and (c), but such that for some $c > 0$, if $0 < d$ and T is a positive integer, then there is a positive integer $w \geqslant T$ such that for some I in $D(w)$, $v(I) < d$, and for $D^*(w) = \{I: I \text{ in } D(w),\ v(I) < d\}$,

$$\sum_{D^*(w)} |[f(\Delta g_1, \ldots, \Delta g_N) - \sum_{D(w+1)} f(\Delta' g_1, \ldots, \Delta' g_N)]| \geqslant c.$$

Suppose that $i = 1, \ldots, N$. Let the real-valued function h_i with domain $[0; 1]$ be defined as follows. If x is in Q, $h_i(x) = g_i(x)$; if x is not in Q, $h_i(x) = \lim g_i(y)$, y in Q, $y \to x-$, this limit existing by an argument almost identical to one of the well-known arguments showing that a real-valued function having bounded variation on an interval is quasi-continuous on that interval. Clearly, $g_i \subseteq h_i$. We now show that h_i has bounded variation on $[0; 1]$. Suppose that D is an interval subdivision of $[0; 1]$. There is a function, u, with domain $\bigcup_{[p;q] \text{ in } D} \{p, q\}$, range $\subseteq Q$, increasing, such that if x is in $Q \cap [\bigcup_{[p;q] \text{ in } D} \{p, q\}]$, then $u(x) = x$, such that for each $[p; q]$ in D, $p < u(q) \leqslant q$, and such that if x is in $\bigcup_{[p;q] \text{ in } D}\{p, q\}$, then $|h_i(x) - g_i(u(x))| < 1/2m$, where m is the number of intervals of D, so that

$$\sum_D |h_i(q) - h_i(p)| = \sum_D |h_i(q) - g_i(u(q)) + g_i(u(q))$$

$$- [h_i(p) - g_i(u(p))] - g_i(u(p))| \leqslant \sum_D [2/2m + |g_i(u(q)) - g_i(u(p))|]$$

$$= 1 + \sum_D | g_i(u(q)) - g_i(u(p))| \leqslant 1 + v([0; 1]).$$

Therefore h_i has bounded variation on $[0; 1]$.

Therefore $\int_{[0;1]} f(dh_1, \ldots, dh_N)$ exists. However, suppose D is a subdivision of $[0; 1]$. There is $d > 0$ such that if $\{(x_1^{(j)}, \ldots, x_N^{(j)})\}_{j=1}^{v}$ is a sequence of elements of $\boldsymbol{R}^N$ such that $\sum_{j=1}^{v} \sum_{i=1}^{N} |x_i^{(j)}| < d$, then $\sum_{j=1}^{v} |f(x_1^{(j)}, \ldots, x_N^{(j)})| < c/8m$, where m is the number of intervals in D. There is a positive integer w such that for some I in $D(w)$, $v(I) < d$, and for $D^*(w) = \{I: I \text{ in } D(w), v(I) < d\}$,

$$\sum_{D^*(w)} |f(\Delta g_1, \ldots, \Delta g_N) - \sum_{D(w+1)(I)} f(\Delta' g_1, \ldots, \Delta' g_N)| \geqslant c.$$

Suppose that there is some element of $D^*(w)$ included in no interval of D and let $E^*(w) = \{I: I \text{ in } D^*(w),\ I \text{ included on no interval of } D\}$. Clearly, $E^*(w)$ contains no more than m elements. Since, for each I in $E^*(w)$, $\max\{\sum_{i=1}^{N} |\Delta_I g_i|, \sum_{i=1}^{N} \sum_{D(w+1)(I)} |\Delta_J g_i|\} \leqslant v(I) < d$, it follows that

$$\sum_{E^*(w)} |f(\Delta g_1, \ldots, \Delta g_N) - \sum_{D(w+1)(I)} f(\Delta' g_1, \ldots, \Delta' g_N)|$$

$$\leqslant \sum_{E^*(w)} |f(\Delta g_1, \ldots, \Delta g_N)| + \sum_{E^*(w)} \sum_{D(w+1)(I)} |f(\Delta' g_1, \ldots, \Delta' g_N)| < 2mc/8m = 4/c.$$

Therefore $E^*(w) \neq D^*(w)$, $D^*(w) - E^*(w)$ is a subset of a refinement of D, and

$$\sum_{D^*(w)-E^*(w)} |f(\Delta h_1, \ldots, \Delta h_N) - \sum_{D(w+1)(I)} f(\Delta' h_1, \ldots, \Delta' h_N)|$$

$$= \sum_{D^*(w)-E^*(w)} |f(\Delta g_1, \ldots, \Delta g_N) - \sum_{D(w+1)(I)} f(\Delta' g_1, \ldots, \Delta' g_N)|$$

$$\geqslant \sum_{D^*(w)} |f(\Delta g_1, \ldots, \Delta g_N) - \sum_{D(w+1)(I)} f(\Delta' g_1, \ldots, \Delta' g_N)$$

$$- \left(\sum_{E^*(w)} |f(\Delta g_1, \ldots, \Delta g_N)| + \sum_{E^*(w)} \sum_{D(w+1)(I)} |f(\Delta' g_1, \ldots, \Delta' g_N)|\right)$$

$$\geqslant c - c/4 = 3c/4.$$

Therefore, by differential equivalence, $\int_U f(dh_1, \ldots, dh_N)$ does not exist, a contradiction.

Therefore (iii) holds. Therefore (2) implies (3).

We now show that (3) implies (1). Suppose that (3) is true, the hypothesis of (1), is satisfied, but that $\int_U f(\xi_1(I), \ldots, \xi_N(I))$ does not exist. There is $c > 0$ such that if $P' \ll P \ll \{U\}$, then there is $H \ll P'$ such that

$$c \leqslant |\sum_P f(\xi_1(1), \ldots, \xi_N(I)) - \sum_H f(\xi_1(J), \ldots, \xi_N(J))| .$$

Let $\mu = \sum_{i=1}^N \int |\xi_i|$. It follows, inductively, that there are sequences, $\{P(n)\}_{n=1}^\infty$ and $\{E(n)\}_{n=1}^\infty$ such that for each n, $P[n+1) \ll E(n) \ll P(n) \ll \{U\}$,

$$c \leqslant |\sum_{P(n)} f(\xi_1(I), \ldots, \xi_N(I)) - \sum_{P(n+1)} f(\xi_1(J), \ldots, \xi_N(J)) ,$$

and if $E \ll E(n)$, then, for each I in E, $\mu(I) - \mu^*(I) < 1/n$.

Clearly, now, there are sequences $\{D(w)\}_{w=1}^\infty$ and $\{g_i\}_{i=1}^N$ satisfying the following conditions.

(1) For each positive integer, n, $D(n)$ is an interval subdivision of $[0; 1]$.

(2) For each $i = 1, \ldots, N$, g_i is a function from $Q = \{x : x \text{ in } (p, q), [p; q] \text{ in } D(n) \text{ for some } n\}$.

(3) For each positive integer m, $D(m+1) \ll D(m)$.

(4) Norm $(D(n)) \to 0$ as $n \to \infty$.

(5) For each positive integer n, there is a reversible function $B(n)$ from $P(n)$ into $D(n)$ such that:

(a) if m is a positive integer, V is in $P(m)$ and I is in $P(m+1)(V)$, then $B(m+1)(I) \subseteq B(m)(V)$;

(b) if m is a positive integer, V is in $P(m)$ and $i = 1, \ldots, N$, then $\xi_i(V) = \Delta_I g_i$, where $I = B(m)(V)$, and if J is in $D(m)$ and is not in the range of $B(m)$, then $\Delta_J g_i = 0$.

Now suppose that $p < q$, $\{p, q\} \subseteq Q$, $i = 1, \ldots, N$, w is a positive integer such that $\{p, q\} \subseteq \bigcup_{D(w)}\{r, s\}$, and $D(w)[p; q] = \{[r; s]:[r; s] \text{ in } D(w),\ p \leqslant r < s \leqslant q\}$. Consider $W = \{V: V \text{ in } P(w), B(w)(V) \text{ in } D(w)[p; q]\}$. Clearly

$$\sum_{D(w)[p;q]} |\Delta g_i| = \sum_W |\xi_i(V)| \leqslant \sum_W \int_V |\xi_i(I)| .$$

Thus $\{D(w)\}_{w=1}^{\infty}$ and $\{g_i\}_{i=1}^{N}$ satisfy the hypothesis of (iii). Let v be defined as in (iii). There is $d > 0$ and positive integer T such that if w is a positive integer $\geqslant T$ and $D(w)^* = \{I: I \text{ in } D(w),\ v(I) < d\}$, then

$$\sum_{D(w)^*} |f(\Delta g_1, \ldots, \Delta g_N) - \sum_{D(w+1)(I)} f(\Delta' g_1, \ldots, \Delta' g_N)| < c/8 .$$

Before proceeding, we make the following inclusion observation and leave the inductive argument to the reader. If each of m and w is a positive integer, $m < w$, V is in $P(m)$, Y is in $P(w)$ and $B(w)(Y) \subseteq B(m)(V)$, then $Y \subseteq V$.

There is a positive integer Z such that $Zd > \mu(U)$. There is a positive integer $m^* > T$ such that if each of $(y_1, \ldots, y_N)$ and $(x_1, \ldots, x_N)$ is in $\boldsymbol{R}^N$, and for $i = 1, \ldots, N$, $|y_i - x_i| < 1/(m^* - 1)$ and $-1 - \mu(U) \leqslant \min\{y_i, x_i\} \leqslant \max\{y_i, x_i\} \leqslant 1 + \mu(U)$, then $|f(y_1, \ldots, y_N) - f(x_1, \ldots, x_N)| < c/8Z$, and such that if $\{(z_1^{(j)}, \ldots, z_N^{(j)})\}_{j=1}^{m}$ is a sequence of elements of $\boldsymbol{R}^N$ such that $\sum_{j=1}^{m}\sum_{i=1}^{N} |z_i^{(j)}| < 1/(m^* - 1)$, then $\sum_{j=1}^{m} |f(z_1^{(j)}, \ldots, z_N^{(j)})| < c/8Z$. Now

$$\begin{aligned} c &\leqslant |\sum_{P(m^*)} f(\xi_1(V), \ldots, \xi_N(V)) - \sum_{P(m^*+1)} f(\xi_1(J), \ldots, \xi_N(J))| \\ &\leqslant |\sum_{P(m^*)} [f(\xi_1(V), \ldots, \xi_N(V)) - \sum_{P(m^*+1)(V)} f(\xi_1(J), \ldots, \xi_N(J))]| \\ &\leqslant |\sum_{P(m^*)^*} [f(\xi_1(V), \ldots, \xi_N(V)) - \sum_{P(m^*+1)(V)} f(\xi_1(J), \ldots, \xi_N(J))]| \\ &\qquad + |\sum_{P(m^*)-P(m^*)^*} [f(\xi_1(V), \ldots, \xi_N(V)) - \sum_{P(m^*+1)(V)} f(\xi_1(J), \ldots, \xi_N(J))]| , \end{aligned}$$

where $P(m^*)^* = \{V: V \text{ in } P(m^*),\ \mu(V) < d\}$ (if any).

Now, suppose that V is in $P(m^*)^*$. Consider $B(m^*)(V) = [p; q]$. Suppose that w is a positive integer $\geqslant m^*$ such that $\{p, q\} \subseteq \bigcup_{D(w)}\{r, s\}$. Letting $W = \{Y: Y \text{ in } P(w),\ B(w)(Y) \text{ in } D(w)[p; q]\}$, we see that $\bigcup_W Y \subseteq V$ and we have that

$$\begin{aligned} \sum_{i=1}^{N} \sum_{D(w)[p;q]} |\Delta_I g_i| &= \sum_{i=1}^{N} \sum_W |\xi_i(Y)| \leqslant \sum_{i=1}^{N} \sum_W \int_Y |\xi_i(J)| \\ &\leqslant \sum_{i=1}^{N} \int_V |\xi_i(J)| = \mu(V) < d . \end{aligned}$$

Thus $v[p; q] < d$, so that

$$|\sum_{P(m^*)^*} [f(\xi_1(V), \ldots, \xi_N(V)) - \sum_{P(m^*+1)(V)} f(\xi_1(J), \ldots, \xi_N(J))]|$$

$$\leqslant \sum_{B(m^*)(P(m^*)^*} |f(\Delta g_1, \ldots, \Delta g_N) - \sum_{B(m^*+1)(P(m^*+1)(V))} f(\Delta' g_1, \ldots, \Delta' g_N)| < c/8 .$$

Now, if V is in $P(m^*) - P(m^*)^*$, then $\mu(V) \geqslant d$, so that there are no more than Z elements in $P(m^*) - P(m^*)^*$. Suppose that V is in $P(m^*) - P(m^*)^*$. Then $\mu(V) - \mu^*(V) < 1/(m^* - 1)$, so that for J^* in $P(m^* + 1)(V)$ such that $\mu(J^*) = \max\{\mu(J): J \text{ in } P(m^* + 1)(V)\}$, we have that $\sum_{P(m^*+1)(V)-\{J^*\}} \mu(J) = \mu(V) - \mu(J^*) \leqslant \mu(V) - \mu^*(V) < 1/(m^* - 1)$, so that $\sum_{i=1}^{N} \sum_{P(m^*+1)(V)-\{J^*\}} |\xi_i(J)| < 1/(m^* - 1)$, and therefore also

$$\sum_{i=1}^{N} |\xi_i(V) - \xi_i(J^*)| = \sum_{i=1}^{N} |\sum_{P(m^*+1)(V)-\{J^*\}} \xi_i(J)|$$

$$\leqslant \sum_{i=1}^{N} \sum_{P(m^*+1)(V)-\{J^*\}} |\xi_i(J)| < 1/(m^* - 1). \quad \text{Therefore}$$

$$|\sum_{P(m^*)-P(m^*)^*} [f(\xi_1(V), \ldots, \xi_N(V)) - \sum_{P(m^*+1)(V)} f(\xi_1(J), \ldots, \xi_N(J))]|$$

$$\leqslant \sum_{P(m^*)-P(m^*)^*} |f(\xi_1(V), \ldots, \xi_N(V)) - f(\xi_1(J^*), \ldots, \xi_N(J^*))$$

$$- \sum_{P(m^*+1)(V)-\{J^*\}} f(\xi_1(J), \ldots, \xi_N(J))| \leqslant \sum_{P(m^*)-P(m^*)^*} |f(\xi_1(V), \ldots, \xi_N(V))$$

$$- f(\xi_1(J^*), \ldots, \xi_N(J^*))| + \sum_{P(m^*)-P(m^*)^*} \sum_{P(m^*+1)(V)-\{J^*\}} |f(\xi_1(J), \ldots, \xi_N(J))|$$

$$< Zc/8Z + Zc/8Z .$$

Therefore $c \leqslant c/8 + 2(Zc/8Z) = c/8 + c/4 = 3c/8$, a contradiction. Therefore $\int_U f(\xi_1(I), \ldots, \xi_N(I))$ exists.

Therefore (3) implies (1), so that, finally, (1), (2) and (3) are equivalent.

References

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Abstract

Suppose that N is a positive integer and f is a function from $\mathbf{R}^N$ into $\mathbf{R}$, bounded on each bounded subset of $\mathbf{R}^N$. Necessary and sufficient conditions are given in order that if U is a set, $\mathbf{F}$ is a field of subsets of U and $\{\xi_i\}_{i=1}^N$ is a sequence of real-valued finitely additive bounded functions with domain $\mathbf{F}$, then the integral, $\int_U f(\xi_1(I), \ldots, \xi_N(I))$, as a refinement-wise limit of sums, exist.

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