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A non-linear abstract differential equation with almost-periodic solution (**)

Introduction

In this paper we complete Theorem 1.1 (Chapter 10) of [**3**] to a simple quasi-linear situation, considering mild instead of regular solutions.

1 - Let X be a Banach space, and $S(t)$, $t\geqslant 0 \to \mathscr{L}(X)$ be a Co-operator semi-group, verifying an estimate $\|S(t)\|\leqslant M \exp[\beta t]$ $\forall t\geqslant 0$, where β *is a negative number*. Let also A be its infinitesimal generator.

Next consider a function $f(x,t)$, $X\times \boldsymbol{R}\to X$, which is continuous with respect to t for any $x\in X$, and verifies an uniform Lipschitz condition

$$\|f(x_1,t)-f(x_2,t)\|\leqslant N\|x_1-x_2\| \qquad \forall x_1,x_2\in X\,, \quad \forall t\in \boldsymbol{R}\,.$$

Remark that as a consequence, the function $f(\varphi(t),t)$, $\boldsymbol{R}\to X$ is (strongly) continuous, if $\varphi(t)$, $\boldsymbol{R}\to X$ is continuous.

We give the following

Def. The continuous function $u(t)$, $\boldsymbol{R}\to X$, is *a mild solution over* $\boldsymbol{R}$ *of the differential equation*

$$u'(t) = Au(t) + f\big(u(t),t\big)\,, \tag{1}$$

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if the functional relation

$$u(t) = S_{t-a}u(a) + \int_a^t S_{t-\sigma} f(u(\sigma), \sigma)\, d\sigma \tag{2}$$

is satisfied $\forall a \in \boldsymbol{R}$, $\forall t \geqslant a$.

Our aim here is to establish the following

Theorem. *Let us assume that the continuous function* $f(x, t)$, $X \times \boldsymbol{R} \to X$ *is almost-periodic in* t, *uniformly for* x *in compact subsets of* X (1), *and has a sufficiently small Lipschitz constant. Then there exists one and only one almost-periodic mild solution over* $\boldsymbol{R}$, $u(t)$, *of the differential equation* (1).

2 – We need a preliminary Lemma before we shall be able to apply the contraction mapping principle.

Lemma. *Given any almost-periodic function* $g(t)$, $\boldsymbol{R} \to X$, *there exists one and only one almost-periodic function* $v(t)$, *verifying the relation*

$$v(t) = S_{t-a}\, v(a) + \int_a^t S_{t-\sigma}\, g(\sigma)\, d\sigma \qquad \forall a \in \boldsymbol{R}\,, \quad \forall t \geqslant a\,. \tag{3}$$

The uniqueness follows in the following way. If $\omega(t)$, $\boldsymbol{R} \to X$ is a *bounded* over $\boldsymbol{R}$ mild solution of $v' = Av$ (that is $\omega(t) = S_{t-a}\omega(a)$, $\forall a \in \boldsymbol{R}$, $t \geqslant a$), then $\omega(t) = \theta$ $\forall t \in \boldsymbol{R}$. In fact, take a sequence of real numbers $t_n \downarrow -\infty$. For a given $t \in \boldsymbol{R}$ we have $\omega(t) = S_{t-t_n}\omega(t_n)$ (as soon as $t_n < t$). Hence

$$\|\omega(t)\| \leqslant M \operatorname*{Sup}_{t \in \boldsymbol{R}} \|\omega(t)\| \exp[\beta(t - t_n)] \to 0 \qquad \text{as } n \to \infty \quad \text{and } \omega(t) = \theta\,.$$

The existence of the almost-periodic mild solution. Consider (as in [**3**]) the function $v(t)$ defined by the (improper) integral

$$v(t) = \lim_{R \downarrow -\infty} \int_R^t S(t - \sigma) g(\sigma)\, d\sigma\,.$$

(1) This means that, if K is any compact in X, then, $\forall \varepsilon > 0$, the set $\bigcap_{x \in K} J(\varepsilon, f(x, t))$ is relatively dense on $\boldsymbol{R}$ (as in [**2**], p. 7).

It is proved in [**3**] (pp. 123-124) that $v(t)$ is a continuous almost-periodic function, $\boldsymbol{R} \to X$, verifying the estimate

$$\|v(t)\| \leqslant \frac{M}{|\beta|} \sup_{\sigma \in \boldsymbol{R}} \|g(\sigma)\| \qquad t \in \boldsymbol{R} .$$

Hence, it remains to show that $v(t)$ is a mild solution over $\boldsymbol{R}$. Take therefore any $a \in \boldsymbol{R}$ and $t \geqslant a$. From

$$v(t) = \int_{-\infty}^{t} S(t-\sigma) g(\sigma)\, d\sigma \quad \text{we derive that} \quad v(a) = \int_{-\infty}^{a} S(a-\sigma) g(\sigma)\, d\sigma$$

which implies that $S(t-a)v(a) = \int_{-\infty}^{a} S(t-\sigma) g(\sigma)\, d\sigma$ and accordingly

$$S(t-a)v(a) + \int_{a}^{t} S(t-\sigma) g(\sigma)\, d\sigma = \int_{-\infty}^{t} S(t-\sigma) g(\sigma) = v(t) .$$

Thus the Lemma is established.

Proof of the theorem. From our hypothesis it follows (see Appendix), that $\forall \varphi(t)$ which is almost-periodic, $\boldsymbol{R} \to X$, the composite function $f(\varphi(t), t)$ has the same property. It makes therefore sense to consider the mapping from the (Banach) space $AP(X)$ (of almost-periodic functions, $\boldsymbol{R} \to X$ endowed with the uniform norm over $\boldsymbol{R}$) into itself defined as follows: $\varphi \in AP(X) \to T\varphi = u$ which is the unique mild almost-periodic solution over $\boldsymbol{R}$ of the above considered differential equation $u' = Au + f(\varphi(t), t)$. We show now that T is a strict contraction on $AP(X)$ (when the Lipschitz constant N is small enough).

Therefore, take $\varphi_1, \varphi_2 \in AP(X)$ and let $u_i = T\varphi_i$, $i = 1, 2$, so that

$$u_i(t) = \int_{-\infty}^{t} S(t-\sigma) f(\varphi_i(\sigma), \sigma)\, d\sigma \qquad (i = 1, 2) .$$

We get $u_1(t) - u_2(t) = \int_{-\infty}^{t} S(t-\sigma)[f(\varphi_1(\sigma), \sigma) - f(\varphi_2(\sigma), \sigma)]\, d\sigma$ and

$$\begin{aligned} \|u_1(t) - u_2(t)\| &\leqslant \int_{-\infty}^{t} \|S(t-\sigma)\|_{\mathscr{L}(X)} N \|\varphi_1(\sigma) - \varphi_2(\sigma)\|\, d\sigma \\ &\leqslant M \cdot N \Big(\int_{-\infty}^{t} \exp[\beta(t-\sigma]\, d\sigma \Big) \sup_{\sigma \in \boldsymbol{R}} \|\varphi_1(\sigma) - \varphi_2(\sigma)\| ; \end{aligned}$$

therefore
$$\|T\varphi_1 - T\varphi_2\|_{AP(X)} \leqslant \frac{M \cdot N}{|\beta|} \|\varphi_1 - \varphi_2\|_{AP(X)} ,$$

which is a strict contraction when $N < |\beta|/M$.

Let $u(t) \in AP(X)$ be a fixed point of T; therefore

$$u(t) = \int_{-\infty}^{t} S(t-\sigma) f(u(\sigma), \sigma)\, d\sigma, \qquad \text{then} \quad u(a) = \int_{-\infty}^{a} S(a-\sigma) f(u(\sigma), \sigma)\, d\sigma ,$$

and for $t \geqslant a$

$$S(t-a)u(a) + \int_{a}^{t} S(t-\sigma) f(u(\sigma), \sigma)\, d\sigma$$
$$= \int_{-\infty}^{a} S(t-\sigma) f(u(\sigma), \sigma)\, d\sigma + \int_{a}^{t} S(t-\sigma) f(u(\sigma), \sigma)\, d\sigma = \int_{-\infty}^{t} S(t-\sigma) f(u(\sigma), \sigma)\, d\sigma ,$$

which means that $u(t)$ is a mild solution (almost-periodic).

It only remains to prove the following (compare [**1**], Theorem 2.10 and Theorem 2.11).

Appendix. *Let the continuous function $f(x, t)$, $X \times \boldsymbol{R} \to X$, be almost-periodic, $\boldsymbol{R} \to X$, $\forall x \in X$, and uniformly for $x \in K$-any compact subset of X. Then, if $\varphi(t) \in AP(X)$, the composite function $f(\varphi(t), t)$, $\boldsymbol{R} \to X$ is also almost-periodic.*

Proof. If K is any compact set in X, the family of almost-periodic functions $\{f(x, t)\}_{x \in K}$ is a relatively compact family in $C_b(\boldsymbol{R}; X)$-space of continuous bounded functions over $\boldsymbol{R}$. This follows from Lyusternik's theorem ([**2**], p. 7) in the following way:

(i) Fix $t_0 \in \boldsymbol{R}$; the set $\{f(x, t_0)\}_{x \in K}$ is the continuous image in X of the compact set K, hence it is compact in X.

(ii) The set $\{f(x, t)\}_{x \in K}$ is uniformly almost-periodic, by assumption.

(iii) The set $\{f(x, t)\}_{x \in K}$ is equi-uniformly continuous over $\boldsymbol{R}$, that is $\|f(x, t') - f(x, t'')\|_X < \varepsilon$ *if* $|t' - t''| < \delta_K(\varepsilon)$, $\forall x \in K$.

In fact, from (ii) it follows that $f(x, t)$ is uniformly continuous with respect to x on K, uniformly on $\boldsymbol{R}$, which is now proved: f is uniformly continuous on $K \times [0, L]$, $\forall L > 0$. We know that $\bigcap_{x \in K} J(\varepsilon/9, f(x, t)) = T$ is relatively dense. Let L be an inclusion length. Given $t \in \boldsymbol{R}$, $\exists \zeta \in [-t, -t + L] \cap T$, so that $0 \leqslant t + \zeta \leqslant L$. Next, $\exists \delta > 0$, so that $x, y \in K$ and $\|x - y\| < \delta \Rightarrow \|f(x, t) - f(y, t)\| < \varepsilon/9$ for $0 \leqslant t \leqslant L$. Then, for any $t \in \boldsymbol{R}$, and $x, y \in K$, $\|x - y\| < \delta$, we obtain

$$\|f(x, t) - f(y, t)\| \leqslant \|f(x, t) - f(x, t + \zeta)\| + \|f(x, t + \zeta) - f(y, t + \zeta)\|$$
$$+ \|f(y, t + \zeta) - f(y, t)\| < \varepsilon/3 .$$

Now, by compactness of K, $\exists$ a finite subset $\{x_1, x_2, \ldots, x_n\} \subset K$, such that, $\forall y \in K$, $\|y - x_i\| < \delta$ for some i. The finite family of almost-periodic-hence uniformly continuous over $\boldsymbol{R}$-functions, $\{f(x_1, t), \ldots, f(x_n, t)\}$ is obviously equi-uniformly continuous so that $\exists \varrho(\varepsilon/3)$ with property

$$|t' - t''| < \varrho => \|f(x_i, t') - f(x_i, t'')\| < \frac{\varepsilon}{3} \quad \forall i = 1, 2, \ldots, n .$$

Given now $y \in K$, take x_i so that $\|x_i - y\| < \delta$. It follows, for $|t' - t''| < \varrho(\varepsilon/3)$, the inequality

$$\|f(y, t') - f(y, t'')\| \leqslant \|f(y, t') - f(x_i, t')\| + \|f(x_i, t') - f(x_i, t'')\| + \|f(x_i, t'') - f(y, t'')\| < \varepsilon .$$

Thus, the family $\{f(x, t)\}_{x \in K}$ is relatively compact in $C_b(\boldsymbol{R}; X)$. If now φ is almost-periodic, $\boldsymbol{R} \to X$, the closure of its range, $K = \overline{\operatorname{Ran} \varphi}$ is a compact set in X so that the family of functions $\{f(x, t)\}_{x \in K}$ is relatively compact in $C_b(\boldsymbol{R}; X)$. Adding one element (the function $\varphi(t)$) maintains relative compactness of the family of almost-periodic functions, $\boldsymbol{R} \to X$, $\{f(x, t)\}_{x \in \overline{\operatorname{Ran} \varphi}} \cup \{\varphi(t)\}$. Applying again Lyusternik's theorem we find that the set of ε' common almost-periods $T_1 = \bigcap_{x \in K} J(\varepsilon', f(x, t)) \cap J(\varepsilon', \varphi)$ is relatively dense on the real line. Now, as was proved above, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in \overline{\operatorname{Ran} \varphi}$ and $\|x - y\| < \delta$ implies $\|f(x, t) - f(y, t)\| < \varepsilon/3$ $\forall t \in \boldsymbol{R}$. If we now take $\varepsilon' = \inf(\varepsilon/3, \delta)$ we can find a $\zeta \in \bigcap_{x \in K} J(\varepsilon/3, f(x, t)) \cap J(\delta, \varphi)$ in any interval of length $L_{\varepsilon'}$ on the real line. For any such ζ it is

$$\|f(\varphi(t + \zeta), t + \zeta) - f(\varphi(t), t)\| \leqslant \|f(\varphi(t + \zeta), t + \zeta) - f(\varphi(t), t + \zeta)\| + \|f(\varphi(t), t + \zeta) - f(\varphi(t), t)\| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon ,$$

because $\|\varphi(t + \zeta) - \varphi(t)\| < \delta$ $\forall t \in \boldsymbol{R}$.

This proves the almost-periodicity of the composite function $f(\varphi(t), t)$.

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