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P-nearsteiner-systems and Frobenius groups (**)

0 - Introduction

We recall (cfr. [**2**]$_1$) that a permutation group (X, G) is 2^*-*transitive* if it is 2-transitive and for every $x, y \in X$ $(x \neq y)$ the stabilizer $G_{xy} \neq (1)$.

We also recall that by a *Frobenius group* G is meant a finite group with $G_x \neq (1)$ but $G_{xy} = (1)$ if $x \neq y$ (cfr. e.g. [**4**]).

In this paper we will show that, to any 3-transitive permutation group (X, Γ) and any subgroup D such that $\Gamma_{abc} \subset D \subset \Gamma_{ab}$ $(a, b, c \in X)$ we construct a geometric structure S which is obtained starting from the set X and an application f^0 mapping every triple of pairwise-distinct points into a point-set, called line. If $T \subseteq \Gamma$ is a group permutable with D and (X, T) is sharply 2-transitive, then $G = TD$ is 2^*-transitive, so that $S = X(G)$ (where $X(G)$ is the structure whose construction is obtained from the permutation group G (cfr. [**2**]$_1$, Theorem 6)). Moreover, from $X(G)$ we find (according to André [**1**]$_1$), the group space $X_w(G_w)$ (where $X_w = X - \{w\}$ and G_w is the stabilizer of w).

We also obtain that the group G_w is an imprimitive Frobenius group of rank $\varrho \neq 3$.

1 - Preludes

We give some preliminary definitions (cfr. [**2**]$_2$).

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Let X be a finite and not empty set, f an application $f: \Delta'(X) \to P(X)$ (where $\Delta'(X) = \{(x, y, z) \in X^3,\ x \neq y \neq z \neq x\}$).

Let $\mathcal{L} = \{f(x, y, z) \mid (x, y, z) \in \Delta'(X)\}$.

We call *nearsteiner-system* the structure $(X, \mathcal{L})$ such that

(A1) $x, y, z \in f(x, y, z)$;
(A2) $w \in f(x, y, z) - \{x, y\}$ iff $f(x, y, z) = f(x, y, w)$;
(A3) $f(x, y, z) = f(x, z, y) = f(x, y, r)$ implies $f(x, y, r) = f(x, r, y)$.

We call *points* the elements of X, *line of base* (x, y) each element $f(x, y, z)$ of $\mathcal{L}$.

We have not necessarily $f(x, y, z) = f(y, x, z)$.

Moreover, if $\|$ is an equivalence relation on $\mathcal{L}$, then we call *nearsteiner-system with parallelism* (*P-nearsteiner-system*) the structure $S = (X, \mathcal{L}, \|)$, where $S^0 = (X, \mathcal{L})$ is a nearsteiner-system and

(PO) for every $x, y \in X$ $(x \neq y)$ and for every $L' \in \mathcal{L}$, there exists one and only one line L parallel to L' and with base (x, y) (denoted by $(x, y)\|L'$).

Let now (X, G) be a 2^*-transitive permutation group. Let $f: \Delta'(X) \to P(X)$ be an application which carries the triple (x, y, z) into the set $f(x, y, z) = \{x, y\} \cup G_{xy}(z)$. Moreover, let us say two lines L, L' be parallel if there exists a $g \in G$ such that $L' = g(L)$.

Then the structure $X(G) = (X, \mathcal{L}, \|)$ is a P-nearsteiner-system (cfr. [**2**]$_1$). In this case we have $f(x, y, z) = f(x, z, y)$.

If G_{xy} is fixed-point-free (*fpf*) on $X - \{x, y\}$ (for all $x, y \in X$, $x \neq y$), then the lines are equipotent (cfr. [2]$_1$). We give the further

Def. 1.1. A $(3, q)$-*Steiner system with parallelism* is a P-nearsteiner-system $(X, \mathcal{L}, \|)$ such that

(i) each line contains q points exactly;
(ii) given the triple $(x, y, z) \in \Delta'(X)$ there exists one and only one line L which contains x, y, z.

Def. 1.2. The nearsteiner-systems $S^0 = (X, \mathcal{L})$ and $S^{0\prime} = (X', \mathcal{L}')$ are said to be *isomorphic* ($S^0 \cong S^{0\prime}$) if there exists a bijection $g: X \to X'$ such that (for $(x, y, z) \in \Delta'(X)$, $f': \Delta'(X') \to P(X')$)

$$g\big(f(x, y, z)\big) = f'\big(g(x), g(y), g(z)\big) . \tag{1.1}$$

Def. 1.3. The P-nearsteiner-systems $S = (X, \mathscr{L}, \|)$ and $S'=(X', \mathscr{L}', \|')$ are said *isomorphic*, if there exists an isomorphism $g: (X, \mathscr{L}) \to (X', \mathscr{L}')$ such that

$$(1.2) \qquad L\|M \text{ iff } g(L)\|'g(M) \quad \text{for } L, M \in \mathscr{L}.$$

An isomorphism $g: S \to S$ is called an *automorphism* of S.

Def. 1.4. An automorphism j of S is a *dilatation* if

$$(1.3) \qquad j(L)\|L \quad \text{for every } L \in \mathscr{L}.$$

Obviously the dilatations of S form a group J in the usual way. We write $J = \mathrm{Dil}(S)$.

An equivalence class of parallel lines is called a *direction*; we denote by R the set of all directions.

2 - Remarks on P-nearsteiner-systems

Let (X, Γ) be a 3-transitive permutation group, $(a, b, c) \in \Delta'(X)$, and D a not trivial group, with $\Gamma_{abc} \subset D \subset \Gamma_{ab}$.

We set

$$(2.1) \qquad f^0(a, b, c) = \{a, b\} \cup D(c) = \{a, b\} \cup \{d_1(c), \dots, d_k(c)\}, \qquad (d_1, \dots, d_k \in D).$$

As (X, Γ) is 3-transitive, given $x, y, z \in X$ pairwise distinct, there exists $j \in \Gamma$ such that

$$(2.2) \qquad j(a) = x, \quad j(b) = y, \quad j(c) = z.$$

We set

$$(2.3) \qquad f^0(x, y, z) = j\big(f^0(a, b, c)\big) = \{j(a), j(b)\} \cup \{jd_1(c), \dots, jd_k(c)\}.$$

This point-set is well defined. In fact let $j' \in \Gamma$ with $j'(a) = j(a) = x$, $j'(b) = j(b) = y$, $j'(c) = j(c) = z$; then $j^{-1}j'(a) = a$, $j^{-1}j'(b) = b$, $j^{-1}j'(c) = c$, hence $j^{-1}j' \in \Gamma_{abc} \subset D$. But we have $j'\big(f^0(a, b, c)\big) = \{j'(a), j'(b)\} \cup \{j'd_1(c), \dots, j'd_k(c)\}$, so that $j^{-1}j'\big(f^0(a, b, c)\big) = \{a, b\} \cup \{d_1'(c), \dots, d_k'(c)\} = f^0(a, b, c)$.

We call *points* the elements of X, and *lines* the subsets of X defined by (2.1) and (2.3).

We can easily prove that

(A1) $x, y, z \in f^0(x, y, z)$;
(A2) $w \in f^0(x, y, z) - \{x, y\}$ iff $f^0(x, y, z) = f^0(x, y, w)$.

In this way we obtain a structure $\tilde{S} = (X, \tilde{\mathscr{L}}) = X(\Gamma, a, b, c, D)$ associated to Γ, a, b, c, D.

Remark 2.1. We have $\Gamma \subseteq \mathrm{Aut}(\tilde{S})$.

Let $g \in \Gamma$, then $g(f^0(x, y, z)) = gj(f^0(a, b, c)) = f^0(g(x), g(y), g(z))$, (where $j(a) = x$, $j(b) = y$, $j(c) = z$, and $g(x) = gj(a)$, $g(y) = gj(b)$, $g(z) = gj(c)$).

Theorem 2.2. *Let (X, Γ) be a 3-transitive permutation group and T a subgroup of Γ sharply 2-transitive on X. Moreover let $a, b, c \in X$ pairwise-distinct and D a not trivial group such that*

(i) $\Gamma_{abc} \subset D \subset \Gamma_{ab}$,
(ii) *D is permutable with T.*

Then $G = TD$ is 2^*-transitive on X and $X(G) = (X, \mathscr{L}, \|)$ is a P-nearsteiner-system; moreover $\tilde{S} = (X, \mathscr{L})$.

Since D is permutable with T, we have that $TD = DT = G$ is a group; moreover G is 2-transitive on X, because (X, T) is 2-transitive.

We have $D_{ab} = D \subseteq G_{ab}$; now we can prove that $G_{ab} \subseteq D$. In fact, if $g \in G_{ab}$, then $g = td$ with $t \in T$, $d \in D = D_{ab}$; from $a = g(a) = td(a)$, $b = g(b) = td(b)$ follows $a = d(a) = t^{-1}(a)$, $b = d(b) = t^{-1}(b)$, so that $t^{-1} = 1$; therefore $g \in D$, hence $G_{ab} = D \neq (1)$. Since (X, G) is 2-transitive, the stabilizers G_{ab} and G_{xy} $(x \neq y)$ are equipotent, therefore (X, G) is 2^*-transitive, then $X(G) = (X, \mathscr{L}, \|)$ is a P-nearsteiner-system (cfr. [**2**]$_1$).

Now we prove that $\Gamma = T\Gamma_{ab}$.

Let $g \in \Gamma$, we put $g(a) = a'$ and $g(b) = b'$; for (X, T) is 2-transitive there exists $t \in T$ such that $t(a) = a'$, $t(b) = b'$. Then $g(a) = t(a)$, $g(b) = t(b)$ and therefore $t^{-1}g(a) = a$ and $t^{-1}g(b) = b$. Hence $t^{-1}g \in \Gamma_{ab}$, then it is $g = tj'$ (with $j' \in \Gamma_{ab}$). Therefore for every $g \in \Gamma$, we have $g = tj'$ in one and only one way, because $T \cap \Gamma_{ab} = (1)$ (note that, if there exists $g \in \Gamma_{ab} \cap T$, then $g(a) = a$ and $g(b) = b$; but $g \in T$, hence $g = 1$). Let $x, y, z \in X$, then there exists $j \in \Gamma$ such that $j(a) = x$, $j(b) = y$, $j(c) = z$; then

$$f^0(x, y, z) = j(f^0(a, b, c)) = tj^0(\{a, b\} \cup D(c)) = tj^0(\{a, b\} \cup G_{ab}(c))$$

$$= \{x, y\} \cup tj^0 G_{ab}(j^{0-1}t^{-1}(z)) = \{x, y\} \cup G_{xy}(z)$$

(where $tj^0 = j$). Therefore $f^0(x, y, z) \in \mathscr{L}$ and conversely, so that $\tilde{S} = (X, \mathscr{L})$.

3 - On a Frobenius group

At first we recall some definitions from Andrè (cfr. [**1**]$_1$). Let $S' = (X', f', \|')$ be a structure consisting in a non-void point-set X', a mapping $f'\colon X'^2 - \{(x, x) \mid x \in X'\} \to P(X')$ which carries the pair (x, y) $(x \neq y)$ into the set $\overline{xy}$ (whose elements are called *lines*), and an equivalence relation $\|'$ on the set $\mathscr{L}' = \{\overline{xy} \mid x, y \in X'$ and $x \neq y\}$ of all lines, called *parallelism*. We write $(X', \mathscr{L}', \|')$ instead of $(X', f', \|')$.

Then the structure S' is called an *LP-space* if the following axioms hold:

(L1) $x, y \in \overline{xy}$;
(L2) $z \in \overline{xy} - \{x\}$ implies $\overline{xy} = \overline{xz}$;
(L3) $\overline{xy} = \overline{yx} = \overline{xr}$ implies $\overline{xr} = \overline{rx}$;
(P1) for every $x \in X'$ and for every $l \in \mathscr{L}'$ there exists one and only one line l' with $x \in l' \|' l$, (which we denote by $\{x\} \|' l$);
(P2) every line parallel to a straight line (which is a line $\overline{xy} = \overline{yx}$) is a straight line;
(P3) from $\overline{xy} \|' \overline{x'y'}$ follows $\overline{yx} \|' \overline{y'x'}$.

Let (X', G') be a transitive permutation group; if $x, y \in X'$ $(x \neq y)$, we set $\overline{xy} = \{x\} \cup G'_x(y)$; we obtain the structure $X'_{G'}$ (cfr. [**1**]$_1$). Moreover (cfr. [**1**]$_1$, Definition 2.6), the transitive permutation group (X', G') is *normally transitive* if it satisfies one of the following conditions:

(a) every line of $X'_{G'}$ contains at least three points;
(b) $N(G'_x) = G'_x$, for every $x \in X'$;
(c) if $x \neq y$, then $G'_x \neq G'_y$.

Now, let (X, G) be a 2^*-transitive permutation group, then the structure $X(G) = (X, \mathscr{L}, \|)$ is a P-nearsteiner-system. Let, moreover, $N(G_{xy}) = G_{xy}$ (for all $x, y \in X$, $x \neq y$); then every line of $X(G)$ has at least four points (cfr. [**2**]$_1$, Remark 2). Let $f(x, y, z) = f(x, z, y)$ (for $(x, y, z) \in \Delta'(X)$).

From $X(G)$ we derive a new incidence structure (for every $w \in X$) $(X_w, \mathscr{L}_w, \|_w)$, where $X_w = X - \{w\}$, $\mathscr{L}_w = \{l' \,|\, l' \cup \{w\} = f(w, x, y) \in \mathscr{L}\}$.

More exactly $l' \in \mathscr{L}_w$ iff

$$l' \cup \{w\} = f(w, x, y) = \{w, x\} \cup G_{wx}(y) = \{w\} \cup \{x\} \cup G'_x(y)$$

(with $G' = G_w$). Hence $l' = \{x\} \cup G'_x(y) = \overline{xy}$.

If $l, l' \in \mathscr{L}_w$, we call (cfr. $[\mathbf{1}]_1$), $l \|_w l'$ iff there exists a $g \in G'$ such that $l' = g(l)$. We know (cfr. $[\mathbf{1}]_1$) that $\|_w$ is an equivalence relation on $\mathscr{L}_w$. Then we have the following

Remark 3.1. The structure $(X_w, \mathscr{L}_w, \|_w)$ is an *LP*-space.

Since (X, G) is 2^*-transitive (hence 2-transitive), (X_w, G_w) is transitive, moreover it is normally transitive, because $N(G_{xy}) = G_{xy}$ and so every line of $\mathscr{L}_w$ has at least three points; we have (cfr. $[\mathbf{1}]_1$, Satz 4.2) that $(X_w, \mathscr{L}_w, \|_w)$ is an *LP*-space.

We can easily prove that if $l \|_w l'$, then $l \cup \{w\} \| l' \cup \{w\}$.

In fact, let $l = \overline{xy}$ and $l' = \overline{x'y'}$; if $l \|_w l'$ there exists $g \in G_w$ such that $g(l) = l'$, that is $g(\overline{xy}) = g(\{x\} \cup G'_x(y)) = \{x'\} \cup G'_{x'}(y')$.

Then we have $l \cup \{w\} = \{x, w\} \cup G_{xw}(y)$ and $l' \cup \{w\} = \{x', w\} \cup G_{wx'}(y') = g(l \cup \{w\})$. Then $l \cup \{w\} \| l' \cup \{w\}$.

Viceversa, if $L = f(w, x, y)$ and $L' = f(w, x', y')$ are parallel in $X(G)$, we can find in the set $\{g \in G \,|\, g(L) = L'\}$ an element of G_w. In fact, given $x, x' \in X'$, there exists a $g \in G_w$ such that $g(x) = x'$ (because (X_w, G_w) is transitive). Therefore $g(L) = f(w, x', g(y))$ is a line parallel to L and with base (w, x') then (by P0) $g(L) = L'$. Hence the lines $l = L - \{w\} = \{x\} \cup G'_x(y)$ and $l' = L' - \{w\} = \{x'\} \cup G'_{x'}(y') = \overline{g(x)\,g(y)}$ are $\|_w$ (where $G' = G_w$). Then $l \|_w l'$ iff $l \cup \{w\} \| l' \cup \{w\}$.

Theorem 3.2. *Let $S = (X, \mathscr{L}, \|)$ be a $(3, q)$-Steiner system with parallelism. Let $G = \mathrm{Dil}\,(S)$ such that, for every $r \in R$, (r, G) is a transitive permutation group* (1), *and let $u, v \in X$ $(u \neq v)$ such that $G_{uv} \neq (1)$; then (for $w \in X$) $(X - \{w\}, G_w)$ is an imprimitive Frobenius group.*

(1) G is here considered as a permutation group on the lines belonging to r, induced by (X, G).

In fact, under the above conditions we have $S = X(G)$ (cfr. [**2**]$_2$, Theorem 2.3); moreover G_w is a Frobenius group on $X - \{w\}$ (cfr. [**2**]$_1$, Remark 7), then the structure $(X_w, \mathscr{L}_w, \|_w)$ is an *LP*-space (cfr. [**1**]$_2$); moreover the elements of $\mathscr{L}_w$ are equipotent (cfr. [**1**]$_2$, Satz 3.2), therefore G_w is imprimitive (cfr. [**1**]$_2$, Satz 3.4). We give the following

Def. 3.1. By an *incidence structure with parallelism* (*P-incidence structure*) of kind t, is meant a triple $(X, \mathscr{L}, \|)$, where X is a finite and not empty set of elements (called *points*), $\mathscr{L}$ is a system of subsets of X (called *lines*) such that $\forall x_1, \dots, x_{t+1} \in X$ (pairwise-distinct) there exists at least an element $f(x_1, \dots, x_{t+1})$ of $\mathscr{L}$ incident with them; and $\|$ an equivalence relation on $\mathscr{L}$ such that

(P4) $\forall x_1, \dots, x_t \in X$ (pairwise-distinct) and for every $l \in \mathscr{L}$, there is exactly a line l' parallel to l and incident with the given points. It will be denoted by $\{x_1, \dots, x_t\}\|l$.

Clearly a *P*-nearsteiner-system (and hence a $(3, q)$-Steiner system with parallelism) is a *P*-incidence structure of kind 2, and an *LP*-space is a *P*-incidence structure of kind 1.

Def. 3.2. A *subspace* of a *P*-incidence structure of kind t $(X, \mathscr{L}, \|)$ is a subset U of X such that

(1) if $x_1, \dots, x_{t+1} \in U$ (pairwise-distinct), then $f(x_1, \dots, x_{t+1}) \subseteq U$;

(2) $\forall x_1, \dots, x_t \in U$ (pairwise-distinct) and for every line $l \subseteq U$, $l' = \{x_1, \dots, x_t\}\|l \subseteq U$.

For $t = 1, 2$ we find again the Definition 6.2 of [**1**]$_1$, and Definition 3.1 of [**2**]$_2$ respectively.

Therefore, if U is a subspace of the *P*-incidence structure $(X, \mathscr{L}, \|)$, $(U, \mathscr{L}_U, \|_U)$ is a subsystem of $(X, \mathscr{L}, \|)$, where $\mathscr{L}_U = \mathscr{L} \cap P(U)$, and $\|_U$ is the equivalence relation induced on $\mathscr{L}_U$ by $\|$.

From now on let $S = (X, \mathscr{L}, \|)$ be a $(3, q)$-Steiner system with parallelism, and let $G = \mathrm{Dil}(S)$ such that, for every $r \in R$, (r, G) is a transitive permutation group, and $G_{uv} \neq (1)$ (for some $u, v \in X$, with $u \neq v$).

Let $w \in X$ and let U_k be a subspace of S containing w; let $X_w = X - \{w\}$ and $U'_x = U_k \cap X_w$. Then we can prove the following

Theorem 3.3. *The set U'_x is a block of imprimitivity of G_w.*

In fact, if $U_k = \{w\}$, then $U'_k = \emptyset$ and $\emptyset$ is a block of imprimitivity of G_w; if $U_k = \{w, x\}$, then $U'_k = \{x\}$ and $\{x\}$ is a block of imprimitivity of G_w.

Otherwise, let $x, y \in U'_k$ $(x \neq y)$, then $f(w, x, y) \subseteq U_k$ and $\overline{xy} \subseteq X_w$, hence $\overline{xy} \subseteq U'_k$.

Moreover, let $x, l \subseteq U'_k$ with $l = \overline{ab} = \{a\} \cup G_{aw}(b)$; then $l \cup \{w\} = L = \{a, w\} \cup G_{aw}(b)$ is contained in U_k, hence $L' = \{x, w\} \| L \subseteq U_k$, then we have $L' = g(L)$ (with $g \in G_w$); so $l' = g(L) - \{w\} \|_w l$ and $l' \subseteq X_w \cap U_k$.

Then U'_k is a subspace of $S' = (X_w, \mathscr{L}_w, \|_w)$, hence it is a block of imprimitivity of G_w (cfr. [**1**]$_1$, Satz 6.3).

Besides we have (cfr. [**5**]) that X_w is the union of the members of a complete block system of G_w: $I = \{U'_k\}$.

Remark 3.4. If $U'_h \in I$, then there exists a subspace U_h of S, such that $U'_h = U_h \cap X_w$.

Let $U'_h \in I$ and U_k be a subspace of S containing w, then (cfr. [**5**]) the blocks U'_h and U'_k (with $U'_k = U_k \cap X_w$) are conjugate, hence there is a $g \in G_w$ such that $g(U'_k) = U'_h$; that is $U'_h = g(X_w \cap U_k) = X_w \cap g(U_k)$.

As U_k is a subspace of S and $g \in G_w \subseteq G$, $g(U_k)$ is also a subspace of S (cfr. [**2**]$_2$, Remark 3.1); moreover $w \in g(U_k)$; hence it is $U'_h = X_w \cap U_h$ (where $U_h = g(U_k)$).

We have finally the following

Theorem 3.5. *Let G be the dilatation group of the* $(3, q)$*-Steiner system with parallelism* $S = (X, \mathscr{L}, \|)$ *such that, for every* $r \in R$, (r, G) *is a transitive permutation group; and let* $u, v \in X$ $(u \neq v)$ *such that* $G_{uv} \neq (1)$. *Then, for* $w \in X$, (X_w, G_w) *is an imprimitive Frobenius group of rank* $\varrho \neq 3$.

We known (cfr. Theorem 3.2) that (X_w, G_w) is an imprimitive Frobenius group. Let us suppose that $G' = G_w$ have rank 3; then, for every $a \in X_w$, G'_a has exactly three orbits $\{a\}$, $\Delta(a)$, $\Upsilon(a)$. We set $k = |\Delta(a)|$, $l = |\Upsilon(a)|$, so we have $|X_w| = m = 1 + k + l$. In this case we have $k = l = q - 2$, and (cfr. [**3**], Lemma 4) the systems of imprimitivity of G' are the sets $\{a\} \cup \Delta(a)$ and hence $k < l$. But this is impossible, so that $\varrho \neq 3$.

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Riassunto

In questo articolo costruiamo particolari strutture geometriche « P-nearsteiner-systems » a partire da un gruppo di permutazioni 3-transitivo. Come casi particolari otteniamo le strutture di incidenza di $[\mathbf{2}]_1$.

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