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Fixed points of holomorphic maps

The purpose of this report is to briefly describe some recent advances in the study of the set of fixed points of holomorphic maps of bounded domains.

1 - Notations and preliminary results

Throughout this paper D will be a bounded domain in a complex Banach space E, whose open unit ball will be denoted by B; c_D and γ_D will be the Carathéodory distance and the Carathéodory differential metric on D (cf. e.g. [**3**]). If D is the open unit disc Δ in the complex plane $\boldsymbol{C}$, γ_Δ is the Poincaré differential metric

$$\gamma_\Delta(\zeta; 1) = \frac{1}{1 - |\zeta|^2} \qquad \zeta \in \Delta,$$

and c_Δ—which will be denoted also by ω— is the integrated form of the Poincaré metric

$$c_\Delta(\zeta_1, \zeta_2) = \omega(\zeta_1, \zeta_2) = \tfrac{1}{2} \log \frac{1 + |\zeta_1 - \zeta_2|\,|1 - \bar{\zeta}_1\zeta_2|^{-1}}{1 - |\zeta_1 - \zeta_2|\,|1 - \bar{\zeta}_1\zeta_2|^{-1}} \qquad \zeta_1, \zeta_2 \in \Delta .$$

The close relationship between c_D and γ_D when $D = \Delta$, becomes looser in higher dimension. In fact γ_D is the derivative of c_D (1); on the other hand c_D is not the integrated form of γ_D (cf. e.g. [**3**], p. 137).

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(1) Cf. [**15**] for definitions and bibliographical references. Contrary to what was stated at p. 218 of [**16**], the Kobayashi infinitesimal metric is *not* the derivative of the Kobayashi distance. A counterexample is provided by the domain of $\boldsymbol{C}^2$ constructed by W. Kaup at p. 20 of [**8**], as pointed out to the author by J. P. Vigué.

For a domain D_1 in a complex Banach space E_1, Hol (D, D_1) will denote the set of all holomorphic (i.e. Gateaux analytic and locally bounded) maps of D into D_1. In particular, Hol (D, D) will indicate the semi-group of all holomorphic maps of D into D; Aut (D) will stand for the subgroup of Hol (D, D) consisting of all holomorphic automorphisms of D (i.e. bijective biholomorphic maps of D onto D).

A set $K \subset D$ is said to be *completely interior* to D – in symbols $K \subset\subset D$– if the norm distance $d(\mathrm{K}, \mathsf{E} \setminus D)$ between K and $\mathsf{E} \setminus D$,

$$d(K, \mathsf{E} \setminus D) = \inf \{\|x - y\| : x \in K, y \notin D\} ,$$

is positive.

The topology of local uniform convergence on D is the topology of uniform convergence on finite unions of closed balls, completely interior to D. The latter topology is equivalent to the compact open topology if, and only if, $\dim_{\mathbb{C}} \mathsf{E} < \infty$. The domain D being bounded, the topology of local uniform convergence on D can be described, according to a theorem due to J. P. Vigué, in the following way (cf. e.g. [**3**], Proposition IV, 3.7, p. 104). Let C be a closed ball completely interior to D. For any $f_0 \in$ Hol (D, D), a fundamental system of neighborhoods of f_0 consists of the sets

$$\{f \in \mathrm{Hol}\,(D, D) : \sup \{\|f(x) - f_0(x)\| : x \in B\} < \varepsilon\}$$

when $\varepsilon > 0$ varies.

2 - Fixed points of holomorphic maps

For any $f \in$ Hol (D, D), Fix f will denote the set of all fixed points of f.

If $\overline{f(d)} \subset\subset D$, there is a constant k, with $0 < k < 1$, such that $c_D(f(x), f(y)) \leqslant k c_D(x, y)$ for all x, y in D. That implies the Earle-Hamilton fixed point theorem ([**1**], [**3**], Theorem V.5.2, p. 138)

If

$$\overline{f(D)} \subset\subset D , \tag{1}$$

f has a unique fixed point.

The following theorem weakens slightly the hypothesis of, and adds some information to the Earle-Hamilton theorem

Theorem I. *Let* $f \in \text{Hol}(D, D)$. *If the closure of the set of iterates* $\{f, f^2, f^3, \ldots\}$ *for the topology of local uniform convergence on* D *contains a map* $g \in \text{Hol}(D, D)$ *such that* $\overline{g(D)} \subset\subset D$, *then* f *has a unique fixed point*, x_0, *and the sequence* $\{f^n\}_{n=1,2,\ldots}$ *converges to the constant map* $x \mapsto x_0$ *for the topology of local uniform convergence on* D.

Weaker conditions than (1) for the existence of fixed points have been established for the open unit ball B of a complex Hilbert space H. In this case c_B can be explicitly computed (cf. e.g. [**3**], p. 156) and turns out to be an inner distance. Given any two points x and y in B there is a unique geodesic (cf. [**12**], p. 141 for the definition) $\lambda\colon [0, 1] \to B$ for c_B, such that $\lambda(0) = x$, $\lambda(1) = y$. The ball B being homogeneous, the curve λ—which will be called the *c_B-segment* joining x and y $(y \neq x)$—can be obtained in the following way. Let $g \in \text{Aut}\, B$ be such that $g(x) = 0$. Then

$$\lambda(t) = g^{-1}(tg(y)) \qquad t \in [0, 1] .$$

Necessary and sufficient conditions for the existence of fixed points in the Hilbert ball $B \subset \mathsf{H}$ have been established. Besides the results reviewed in [**4**], some more recent advances will now be indicated.

A subset $S \subset B$ is said to be *c_B star-shaped* at $x \in S$ if the c_B-segment λ, joining x and any $s \in S$, is such that $\lambda([0, 1]) \subset S$.

Proposition 2.1 [**10**]. *The map* $f \in \text{Hol}(B, B)$ *has a fixed point if, and only if, there exists a non-empty* c_B *star-shaped subset* $S \subset B$ *such that* $f(S) \subset S$ *and* $\overline{f(S)} \subset B$.

The fact that $\text{Fix}\, f$ is the intersection of B with a closed affine subspace of f weakens condition (1).

Corollary 2.2 [**10**]. *If* $\overline{f(B)} \subset B$ *then* $\text{Fix}\, f$ *consists of one point.*

Theorem II [**11**]. *The set* $\text{Fix}\, f$ *is non empty if, and only if, there exists a non-empty convex subset* $X \subset B$ *such that* $f(X) \subset X$ *and* $\overline{f(X)} \subset B$.

The following result by K. Goebel [**5**] concerns the case where $\text{Fix}\, f = \emptyset$ for $f \in \text{Hol}(B, B)$.

Proposition 2.3. *If* $\text{Fix}\, f = \emptyset$, *there exists a point* $u \in \partial B$ *such that for all real* k, *with* $0 < k < 1$, *the ellipsoid*

$$E_k(u) = \{x \in B : \frac{|1-(x, u)|^2}{1-\|x\|^2} < \frac{k}{1-k}\}$$

is invariant: $(f(E_k(u)) \subset E_k(u)$.

Consider now the *bi-disc* $B \times B$ of $\mathsf{H} \times \mathsf{H}$.

Theorem III [**11**]. *Let* $f \in \mathrm{Hol}(B\times B, B\times B)$. *Then* $\mathrm{Fix}\, f \neq \emptyset$ *if, and if, there exists a non-empty convex subset* $X \subset B\times B$ *such that* $f(X) \subset X$ *and the weak closure of* $f(X)$ *is contained in* $B\times B$.

If the weak closure of $f(B\times B)$ is contained in $B\times B$, then Fix f consists of one point.

If $f \in \mathrm{Aut}(B)$, complete results are available (cf. [**3**] and the relevant bibliography therein). For a discussion of the existence of fixed points of holomorphic automorphisms of the unit ball of a Banach M-lattice, cf. [**13**].

3 - The structure of the set of fixed points

Complex geodesics for the Carathéodory distance c_D are a useful tool in the investigation of the geometric structure of the connected components of Fix f.

A holomorphic map $\varphi\colon \Delta \to D$ is called a *complex geodesic* for c_D at $\varphi(\zeta_0)$ (for some $\zeta_0 \in \Delta$), if

$$c_D(\varphi(\zeta_0), \varphi(\zeta)) = \omega(\zeta_0, \zeta) \tag{2}$$

for all $\zeta \in \Delta$. Such a map is injective and its range $\varphi(\Delta)$ is closed in D. It can be shown [**15**] that if (2) holds for *some* $\zeta \in \Delta \setminus \{\zeta_0\}$, then it holds for all $\zeta \in \Delta$, and therefore φ is a complex geodesic at $\varphi(\zeta_0)$ for c_D. As a consequence, if φ is a complex geodesic for c_D at $\varphi(\zeta_0)$ for some $\zeta_0 \in \Delta$, then φ is a complex geodesic for c_D at $\varphi(\zeta)$ for all $\zeta \in \Delta$.

Examples.

1. If D is a bounded, non simply-connected domain in $\mathbf{C}$, there are no complex geodesics for c_D.

2. If D is the open unit ball $B \subset \mathsf{E}$, for any $x \in B\setminus\{0\}$ there is a complex geodesic φ for c_B whose range $\varphi(\Delta)$ contains 0 and x. Such a geodesic is given by $\varphi(\zeta) = (\zeta/\|x\|)x$, and is the unique complex geodesic for c_B whose range contains both 0 and x if, and only if, $(1/\|x\|)x$ is a complex extreme point of $\bar{B}$ [**15**], [**16**].

Let $f \in \mathrm{Hol}(D, D)$ be such that $\mathrm{Fix}\, f \neq \emptyset$. If Fix f contains two distinct points, x_0 and x_1, and if there is a complex geodesic φ for c_D such that $\{x_0, x_1\} \subset \varphi(\Delta)$, then, being $c_D(x_0, x_1) = c_D(f(x_0), f(x_1))$, the holomorphic map $f\circ\varphi\colon \Delta \to D$ is a complex geodesic for c_D whose range contains x_0 and x_1. Hence, if there exists a unique complex geodesic for c_D whose range contains both x_0 and x_1, there exists a Moebius transformation σ of Δ such that $f\circ\varphi = \varphi\circ\sigma$ [**15**]. The fact that $\{x_0, x_1\} \subset \mathrm{Fix}\, f$ coupled with the injectivity of φ implies then that σ

has two distinct fixed points in Δ, and therefore is the identity, showing that $\varphi(\Delta) \subset \operatorname{Fix} f$. That proves [**16**].

Theorem IV. *If* $\operatorname{Fix} f$ *contains at least two points and for some* $x_0 \in \operatorname{Fix} f$ *and any* $x \in \operatorname{Fix} f$ *there exists a unique complex geodesic for* c_D *whose range contains* x_0 *and* x, *then* $\operatorname{Fix} f$ *is connected and* $\overline{\operatorname{Fix} f} \cap \partial D \neq \emptyset$.

Let B be the open unit ball in the complex Banach space E.

Let $f \in \operatorname{Hol}(B, B)$ be such that $0 \in \operatorname{Fix} f$. Let $\mathsf{F} = \{y \in \mathsf{E}: df(0)y = y\}$ and let $\mathsf{K} = \{y \in \mathsf{E} \setminus \{0\}: (1/\|y\|)y$ is a complex extreme point of $B\}$. Then

$$\mathsf{F} \cap \mathsf{K} \cap B = \operatorname{Fix} f \cap \mathsf{K}\,. \tag{3}$$

If $\mathsf{K} = \mathsf{E} \setminus \{0\}$, (3) reads $\mathsf{F} \cap B = \operatorname{Fix} f$.

If B is homogeneous (i.e., if $\operatorname{Aut}(B)$ acts transitively on B), condition $0 \in \operatorname{Fix} f$ can be dropped; cf. [**16**] for further details.

If $\mathsf{K} \subsetneq \mathsf{E} \setminus \{0\}$ there are examples of holomorphic maps $f: B \to B$ with $f(0) = 0$, such that $\operatorname{Fix} f \not\subset \mathsf{K} \cap \mathsf{F}$. Let B be the unit bi-disc of $\boldsymbol{C}^2$

$$B = \Delta \times \Delta = \{(z_1, z_2) \in \boldsymbol{C}^2: |z_1| < 1, |z_2| < 1\}\,.$$

Proposition 3.1 [**16**]. *For any* $f \in \operatorname{Hol}(\Delta \times \Delta, \Delta \times \Delta)$ *with* $\operatorname{Fix} f \neq \emptyset$ *one of the following cases necessarily occurs*:

$\operatorname{Fix} f$ *consists of one point*

$\operatorname{Fix} f$ *is the range of a complex geodesic for* c_D

f *is the identity map*.

Since $\mathsf{K} = \{(z_1, z_2): |z_1| = |z_2| \neq 0\}$, if $f(0, 0) = (0, 0)$ and $(z_1^0, z_2^0) \in \operatorname{Fix} f$, with $|z_1^0| \neq |z_2^0|$, then either f is the identity map or $\operatorname{Fix} f$ is the range of a complex geodesic for $c_{\Delta\times\Delta}$ and is not contained in K (cf. [**6**] for related results).

In [**17**] J. P. Vigué generalizes Proposition 3.1 proving

Theorem V. *Let* D *be a bounded convex taut domain in* $\boldsymbol{C}^n$. *Let* $f \in \operatorname{Hol}(D, D)$ *have two distinct fixed points* x, y *in* D. *If there is a complex geodesic for* c_D *whose range contains both* x *and* y, *then there exists a complex geodesic* φ *for* c_D *such that*

$$\{x, y\} \subset \varphi(\Delta) \subset \operatorname{Fix} f\,. \tag{4}$$

Recall ([**9**], p. 129) that a domain D of $\boldsymbol{C}^n$ (or a connected complex manifold) is *taut* if, for every sequence $\{\varphi_\nu\}$ in $\operatorname{Hol}(\Delta, D)$, one of the two following cases necessarily occurs:

(i) $\{\varphi_\nu\}$ contains a subsequence uniformly convergent in Hol $(\varDelta, D)$ on all compact subsets of $\varDelta$;

(ii) For all compact subsets $K \subset \varDelta$, $L \subset D$ there exists an index ν_0 such that $\varphi_\nu(K) \cap L = \emptyset$ for all $\nu > \nu_0$.

The hypotheses of theorem *V* are fulfilled if D is the open unit ball B of a finite dimensional normed complex vector space E, and $x = 0$, $y \in B \setminus \{0\}$. If, more in particular, B is homogeneous, the above conditions are satisfied for any choice of x, y $(x \neq y)$ in B.

The following questions arise naturally at this point.
Let $f \in$ Hol (D, D).

(*a*) Assume that f contains two distinct points x and y.

Does the existence of a complex geodesic for c_D, whose range contains x and y, imply that there is a complex geodesic φ for c_D satisfying (4)?

(*b*) Assume that every two points of D lie in the range of a complex geodesic for c_D.

Is Fix f *connected?*
If Fix f *contains more than one point, is* $\overline{\text{Fix}\, f} \cap \partial D \neq \emptyset$?

The results reviewed so far give partial answers to these questions. Let B be the open unit ball of a complex Hilbert space H and let $f \in$ Hol $(B \times B, B \times B)$ be such that Fix $f \neq \emptyset$. T. Kukzumow and A. Stachura **[11]** have shown that either Fix f consists of one point or $\overline{\text{Fix}\, f} \cap \partial(B \times B) \neq \emptyset$.

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