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Geodesic spheres and naturally reductive homogeneous spaces

Let (M, g) be an n-dimensional Riemannian manifold and denote by $G_m(r)$ the geodesic sphere of M with center $m \in M$ and radius r. We always suppose that $r < i(m)$ where $i(m)$ is the injectivity radius of (M, g) at m.

When M is a two-point homogeneous space the geodesic spheres are (reductive) *homogeneous* spaces. Recently W. Ziller proved [**8**] that all these geodesic spheres are *naturally reductive homogeneous spaces* except for the Cayley plane where none of them has this property.

The main purpose of this note is to give a new and independent proof of the natural reductivity by using the theorem of Ambrose and Singer [**1**].

1 - Preliminaries

Let (M, g) be a connected Riemannian manifold. Then (M, g) is said to be a *homogeneous Riemannian manifold* if there exists a group G of isometries of (M, g) acting transitively and effectively on M. Then M is diffeomorphic to G/K where K is the isotropy group of some point p in M.

Next let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K. Suppose $\mathfrak{m}$ is a vector space complement to $\mathfrak{k}$ in $\mathfrak{g}$ such that $\mathrm{Ad}\,(K)\,\mathfrak{m} \subseteq \mathfrak{m}$, *i.e.* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a *reductive* decomposition. Then we may identify $\mathfrak{m}$ with $T_p M$ by the map

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$X \mapsto X^*_p$, where X^* denotes the Killing vector field on (M, g) generated by the one-parameter subgroup $\{\exp(tX)\}$ acting on M. We denote by $\langle , \rangle$ the inner product on $\mathfrak{m}$ induced by the metric g.

Def. 1.1. The manifold (M, g) (or the metric g) is said to be *naturally reductive* if there exists a Lie group G and a subspace $\mathfrak{m}$ with the properties described above and such that

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0\,, \qquad X, Y, Z \in \mathfrak{m}\,, \tag{1}$$

where $[X, Y]_{\mathfrak{m}}$ denotes the projection of $[X, Y]$ on $\mathfrak{m}$.

It is clear that if we want to say that (M, g) is naturally reductive we first have to determine all transitive isometry groups G of M and then to consider all the complements of $\mathfrak{k}$ in $\mathfrak{g}$ which are invariant under $\mathrm{Ad}(K)$ and, in addition, satisfy (1). In many cases this is not an easy task but sometimes one can obtain a quick answer by using an infinitesimal characterization which we shall treat now.

As is well-known, E. Cartan proved that a connected, complete and simply connected Riemannian manifold is a symmetric space if and only if the curvature is constant under parallel translation. Ambrose and Singer extended this theory in order to be able to characterize Riemannian manifolds by a local condition which is to be satisfied at all points. More specifically they proved

Lemma 1.2 [**1**]. *Let (M, g) be a connected, complete and simply connected Riemannian manifold. Then (M, g) is homogeneous, i.e. there exists a transitive and effective group of isometries of M, if and only if there exists a tensor field T of type $(1, 2)$ such that, with $\tilde{\nabla} = \nabla - T$, we have*

$$\tilde{\nabla} g = 0\,, \qquad \tilde{\nabla} R = 0\,, \qquad \tilde{\nabla} T = 0\,. \tag{2}$$

Here ∇ denotes the Levi Civita connection and R the Riemann curvature tensor of M.

Note that (2) is equivalent to the following conditions

$$g(T_X Y, Z) + g(Y, T_X Z) = 0\,, \qquad \nabla_X R = T_X \cdot R\,, \qquad \nabla_X T = T_X \cdot T\,, \tag{3}$$

for $X, Y, Z \in \mathfrak{X}(M)$ and where T_X acts as a derivation on the tensor algebra.

In [**6**], [**7**] we used this theorem to give a characterization of naturally reductive homogeneous spaces by means of the tensor T:

Lemma 1.3. *Let (M, g) be a connected, simply connected and complete Riemannian manifold. Then (M, g) is a naturally reductive homogeneous Rie-*

mannian space if and only if there exists a tensor field T of type (1, 2) *satisfying the conditions* (2) *and such that*

$$T_X X = 0 \tag{4}$$

for all $X \in \mathfrak{X}(M)$.

We refer to [6] for more details about the study of *homogeneous Riemannian structures* T on a Riemannian manifold.

3 - The second fundamental form of a geodesic sphere in a two-point homogeneous space

To prove the main results of this paper we shall need explicitly the second fundamental form of a geodesic sphere $G_m(r)$ with center m and radius r. We start by giving a brief description of an elegant and well-known method to obtain this form (see for example [2], [4]).

Let ξ be a unit vector of $T_m M$ and denote by $\gamma(r)$ the geodesic tangent to ξ, i.e. $\gamma(r) = \exp_m(r\xi)$. The Jacobi field equation along γ is

$$Y'' + R_{\gamma' Y}\gamma' = 0 .$$

Let $\{e_i,\ i = 1, \ldots, n$ and $e_1 = \xi\}$ be an orthonormal basis at m and denote by $\{E_i,\ i = 1, \ldots, n\}$ the orthonormal basis along γ obtained by parallel translation of $\{e_i\}$ along γ. Next we consider the $n-1$ Jacobi vector fields Y_a, $a = 2, \ldots, n$, along γ with initial conditions

$$Y_a(0) = 0 , \qquad Y'_a(0) = e_a .$$

Put

$$Y_a(r) = (A_a^b E_b)(r) .$$

This gives rise to the endomorphism-valued function $r \mapsto A(r)$ and the endomorphism-valued equation

$$A'' + R \circ A = 0 , \tag{5}$$

with initial conditions

$$A(0) = 0 , \qquad A'(0) = I , \tag{6}$$

where I is the identity and R the symmetric endomorphism of $\{\gamma'(r)\}^{\perp} \subset T_{\gamma(r)}M$ given by $R(r)X = R_{\gamma'(r)X}\gamma'(r)$, $X \in \{\gamma'(r)\}^{\perp}$.

Since $\gamma'(r)$ is a unit normal vector of $G_m(r)$ at $p = \exp_m(r\xi)$, the *shape operator* S of $G_m(r)$ at p is given by $SX = \nabla_X \gamma'(r)$, $X \in T_pG_m(r)$, and it is well-known that $S_p = (A'A^{-1})(r)$.

Now we derive *explicit* formulas for S when M is a two-point homogeneous space. In this case we can always choose a basis $\{e_i\}$ at m which diagonalizes $R(0)$ and, since M is a symmetric space, $\{E_i\}$ diagonalizes R at each point $\gamma(r)$.

For Euclidean space E^n we have $A = rI$ and hence

$$S = \frac{1}{r} I . \tag{7}$$

Next, for a space of constant curvature μ we obtain $A = \alpha I$ where

$$\alpha = \frac{\sin\sqrt{\mu}\, r}{\sqrt{\mu}} \quad \text{for } \mu > 0, \qquad \alpha = \frac{\sinh\sqrt{|\mu|}\, r}{\sqrt{|\mu|}} \quad \text{for } \mu < 0.$$

In this case we have

$$S = \beta I \tag{8}$$

with $\beta = \sqrt{\mu}\cot\sqrt{\mu}\, r$ for $\mu > 0$, $\beta = \sqrt{|\mu|}\coth\sqrt{|\mu|}\, r$ for $\mu < 0$.

Finally we consider the case $\boldsymbol{CP}^n, \boldsymbol{HP}^n$ and Cay P^2 or their noncompact duals. In this case there are only two eigenvalues for the endomorphism R. More specifically we have

$$R = \begin{pmatrix} \alpha I_p & 0 \\ 0 & \frac{\alpha}{4} I_q \end{pmatrix} \tag{9}$$

where $p + q = n - 1$ (see for example [3]). In what follows we consider the case $\alpha > 0$. The formulas for $\alpha < 0$ can be obtained by replacing the trigonometric functions by hyperbolic functions. From (5), (6) and (7) we obtain

$$A(r) = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} \sin\sqrt{\alpha}\, r\, I_p & 0 \\ 0 & \frac{2}{\sqrt{\alpha}} \sin\frac{\sqrt{\alpha}}{2} r I_q \end{pmatrix}$$

and hence

$$S(r) = \begin{pmatrix} \sqrt{\alpha}\cot\sqrt{\alpha}r\, I_p & 0 \\ 0 & \frac{\sqrt{\alpha}}{2}\cot\frac{\sqrt{\alpha}}{2}r\, I_q \end{pmatrix}. \tag{10}$$

From this it follows at once that the *second fundamental form* σ can be written as

$$\sigma(X, Y) = g(SX, Y) = ag(X, Y) + bR_{\gamma' X \gamma' Y}, \qquad X, Y \in T_p G_m(r), \tag{11}$$

where

$$\begin{aligned} a &= \frac{1}{3}\{4\frac{\sqrt{\alpha}}{2}\cot\frac{\sqrt{\alpha}}{2}r - \sqrt{\alpha}\cot\sqrt{\alpha}r\}, \\ b &= \frac{4}{3\alpha}\{-\frac{\sqrt{\alpha}}{2}\cot\frac{\sqrt{\alpha}}{2}r + \sqrt{\alpha}\cot\sqrt{\alpha}r\}. \end{aligned} \tag{12}$$

Note that in all these cases the eigenvalues of the shape operator are constant on each geodesic sphere, i.e. are radial functions.

We shall also need the *Gauss equation* for the geodesic sphere $G_m(r)$

$$R'_{XYZW} = R_{XYZW} + \sigma(X, Z)\sigma(Y, W) - \sigma(X, W)\sigma(Y, Z), \tag{13}$$

where R' is the Riemann curvature tensor of $G_m(r)$ and $X, Y, Z, W \in \mathscr{X}(G_m(r))$.

4 - Geodesic spheres and naturally reductive homogeneous spaces

In this section we prove our main results. First we consider the trivial case.

Theorem 4.1. *The geodesic spheres in E^n or in a space of constant curvature are naturally reductive homogeneous spaces.*

Proof. It follows easily from (7), (8), (13) and the fact that

$$R_{XYZW} = \alpha\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}$$

that $T = 0$ satisfies the equations (2) of Ambrose and Singer. So the result follows at once from Lemma 1.3. (Note that $T = 0$ implies that $G_m(r)$ is a symmetric space.)

The case of CP^n, HP^n or their noncompact duals is a bit more complicated.

Theorem 4.2. *Let M be the complex projective space $CP^n(\alpha)$ of constant holomorphic sectional curvature α or its noncompact dual. Then the geodesic spheres are naturally reductive homogeneous spaces.*

Proof. We consider the case $\alpha > 0$. The case $\alpha < 0$ can be obtained by replacing the trigonometric functions by hyperbolic functions.

Let J denote the almost complex structure on M and denote by F the Kähler form on M, i.e. $F(X, Y) = g(X, JY)$ where $X, Y \in \mathfrak{X}(M)$. The curvature tensor of M is given by

$$
\begin{aligned}
R_{XYZW} = \frac{\alpha}{4}\{&g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + F(X, Z)F(Y, W) \\
&- F(X, W)F(Y, Z) + 2F(X, Y)F(Z, W)\} .
\end{aligned} \tag{14}
$$

Further, let $G_m(r)$ be a geodesic sphere and put $p = \exp_m(r\xi)$. Denote by η the 1-form on $G_m(r)$ defined by

$$\eta(X) = g(X, J\gamma'(r)) , \quad X \in T_pG_m(r) , \quad p \in G_m(r) . \tag{15}$$

It follows from (10) (with $p = 1$ and $q = 2n - 2$) or (11) and (14) that

$$\sigma(X, Y) = \lambda g(X, Y) + \mu\eta(X)\eta(Y) , \quad X, Y \in T_pG_m(r) , \tag{16}$$

where λ and μ are radial functions.

Next we put

$$T = 3\lambda\eta \wedge F \tag{17}$$

on the geodesic sphere $G_m(r)$. Let ∇' denote the Riemannian connection on $G_m(r)$. Then it follows easily from (15), (16) and (17)

$$(\nabla'_X\eta)(Y) = -\lambda F(X, Y) , \quad (T_X\cdot\eta)(Y) = -\lambda F(X, Y) , \qquad X, Y \in T_pG_m(r) . \tag{18}$$

Hence, with $\tilde\nabla = \nabla' - T$, we have

$$\tilde\nabla\eta = 0 . \tag{19}$$

Similarly we obtain for $X, Y, Z \in T_p G_m(r)$

$$(20) \qquad (\nabla'_X F)(Y, Z) = (T_X \cdot F)(Y, Z) = \lambda[g(X, Y)\eta(Z) - g(X, Z)\eta(Y)]\,,$$

and so

$$(21) \qquad \tilde{\nabla} F = 0\,.$$

Next, since $\nabla g = 0$, we get from (16) and (19)

$$(22) \qquad \tilde{\nabla} S = 0$$

and hence, (13), (14), (21) and (22) imply $\tilde{\nabla} T = \tilde{\nabla} R' = 0$.

This means that $G_m(r)$ is a homogeneous space. Moreover, since T is a 3-form, the condition (4) is fulfilled and the geodesic sphere is naturally reductive.

Note that the expression (17) for the 3-form T can be obtained by solving the system of equations (2) explicitly. On the other hand it is very natural that the 3-form T is expressed by means of the 1-form η and the Kähler form F which are the natural forms related to the geometrical situation.

Theorem 4.3. *Let M be the quaternionic projective space $\boldsymbol{H}P^n(\alpha)$ of maximal sectional curvature $\alpha > 0$ or its noncompact dual. Then the geodesic spheres are naturally reductive homogeneous spaces.*

Proof. We do the case $\alpha > 0$. There exist locally three almost complex structures J_i, $i = 1, 2, 3$, such that

$$J_1 J_2 = J_3\,, \qquad J_2 J_3 = J_1\,, \qquad J_3 J_1 = J_2\,.$$

Denote by F_i, $i = 1, 2, 3$, the associated 2-forms, i.e. $F_i(X, Y) = g(X, J_i Y)$, $X, Y \in \mathfrak{X}(M)$. Then the curvature of M is given by

$$(23) \qquad \begin{aligned} R_{XYZW} = \frac{\alpha}{4}\{&g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ &+ \sum_{i=1}^{3} (F_i(X, Z)F_i(Y, W) - F_i(X, W)F(Y, Z) + 2F_i(X, Y)F_i(Z, W))\}\,. \end{aligned}$$

Next, let $G_m(r)$ be a geodesic sphere and put $p = \exp_m(r\xi)$. Define the three 1-forms η_i, $i = 1, 2, 3$, on $G_m(r)$ by

$$(24) \qquad \eta_i(X) = g(X, J_i\gamma'(r))\,, \qquad X \in T_p G_m(r)\,, \qquad p \in G_m(r)\,.$$

Then, it follows from (10) (with $p=3$, $q=4n-4$) or (11) and (23) that

$$\sigma(X, Y) = \lambda g(X, Y) + \mu \sum_{i=1}^{3} \eta_i(X)\eta_i(Y) , \qquad X, Y \in T_p G_m(r) , \tag{25}$$

where λ and μ are radial functions.

Note that we can always choose the almost complex structures J_i, $i=1, 2, 3$, such that $\nabla_X J_i|_p = 0$ for any fixed X, (see for example [**5**]).

Now we shall prove that the 3-form

$$T = 3\lambda \sum_{i=1}^{3} \eta_i \wedge F_i - 6\mu\, \eta_1 \wedge \eta_2 \wedge \eta_3 \tag{26}$$

gives the required tensor field. Therefore we need several formulas which are easily verified. Let (i, l, m) be a cyclic permutation of $(1, 2, 3)$. Then we have

$$\nabla'_X \eta_i = -\frac{\lambda}{2} i_X F_i + \mu i_X(\eta_l \wedge \eta_m) , \qquad i = 1, 2, 3 , \tag{27}$$

where ∇' denotes the Riemannian connection on $G_m(r)$ and i_X is the interior product with respect to X. With $\theta_X(Y) = g(X, Y)$ we also have

$$\nabla'_X F_i = 2\lambda \theta_X \wedge \eta_i + 2\mu \sum_{k=1}^{3} \eta_k(X)\, \eta_k \wedge \eta_i , \qquad i = 1, 2, 3 . \tag{28}$$

Further

$$T_X \cdot \eta_i = -\frac{\lambda}{2} i_X F_i + (2\lambda + \mu)\, i_X(\eta_l \wedge \eta_m) , \tag{29}$$

$$T_X \cdot F_i = 2\lambda(\eta_l(X) F_m - \eta_m(X) F_l) + 2\lambda\, (\theta_X \wedge \eta_i) + 2\mu \sum_{k=1}^{3} \eta_k(X)(\eta_k \wedge \eta_i), \tag{30}$$

for $i = 1, 2, 3$. Hence (27)-(30) imply, with $\tilde{\nabla} = \nabla - T$,

$$\tilde{\nabla}_X \eta_i = -2\lambda i_X(\eta_l \wedge \eta_m) , \qquad \tilde{\nabla}_X F_i = -2\lambda(\eta_l(X) F_m - \eta_m(X) F_l) , \tag{31}$$

for $i = 1, 2, 3$. From (31) and (26) we obtain $\tilde{\nabla} T = 0$, and, finally, from (25), (31) and $\tilde{\nabla} g = 0$ we get

$$\tilde{\nabla} S = 0 . \tag{32}$$

The result follows now at once since (31), (32), the expression (23) for R' and the Gauss equation (13) imply $\tilde{\nabla} R' = 0$.

5 - Remark

The Cayley plane is much more difficult to handle and we have been unable to obtain the result of Ziller by a method which is similar to that used for the other two-point homogeneous spaces. But we believe that it must be possible to use the result of Ambrose and Singer to obtain the nonexistence of a naturally reductive homogeneous structure. We hope to come back on this problem in another paper.

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