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Homogenization estimates for quasi-variational inequalities of parabolic type (**)

1 - Introduction and results

Let $Y=\prod_{i=1}^{n}[0,y_i]\subseteq \boldsymbol{R}^n$, $(n>1)$, $\tau_0=[0,k_0]\subseteq \boldsymbol{R}$, $k_0>0$ fixed.

Let $\prod = Y\times\tau_0$; we consider functions $a_{ij}(y,\tau)\in C^2(\prod)$ $(i,j=1,2,\dots,n)$ such that

$$\sum_{i,j=1}^{n} a_{ij}(y,\tau)\xi_i\xi_j\geqslant\alpha|\xi|^2 \qquad \alpha>0 \qquad \forall\xi\in\boldsymbol{R}^n,\ \text{a.e. in } \prod .$$

The a_{ij}'s can be extended periodically to $\boldsymbol{R}^n\times\boldsymbol{R}$. Let us associate the family of operators P^ε to the functions a_{ij} defined by

$$(1.1)_\varepsilon \qquad P^\varepsilon=\frac{\partial}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a_{ij}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\big)\frac{\partial}{\partial x_i}\Big) \qquad (\varepsilon>0)\,,$$

where $x=\varepsilon y$, $t=\varepsilon\tau$. And we set

$$(1.1)_0 \qquad P^0=\frac{\partial}{\partial t}-\sum_{i,j=1}^{n}a_{ij}^0\,\frac{\partial^2}{\partial x_j\,\partial x_i}\,,$$

where a^0_{ij} are suitable constants such that P^0 is the homogenization operator of the P^ε [1].

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Let Ω be a bounded open set of $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$ and $T>0$ fixed; we set $Q=\Omega\times(0,T)$.

For any given function $\psi\in L^2(Q)$, we set

$$K^\psi=\{v\,|\,v\in L^2(0,T;H_0^1(\Omega)),\, v\leqslant\psi \quad \text{a.e. in } Q\}\;. \tag{1.2}$$

Let $u^\varepsilon=S^\varepsilon(\psi,u_0)$ be the weak solution, for any $\varepsilon\geqslant 0$, (in the sense of [**3**]$_3$, [**12**]) of the variational inequality

$$\langle P^\varepsilon u^\varepsilon, v-u^\varepsilon\rangle\geqslant 0 \qquad \forall v\in K^\psi\,, \qquad u^\varepsilon\in K^\psi\,, \qquad u^\varepsilon(0)=u_0\,. \tag{1.3}$$

In a previous paper [**10**], he have given estimates on the rapidity of convergence of $\{u^\varepsilon\}_{\varepsilon>0}$ to u^0, in dependence on the smoothness of ψ and u_0.

In this paper we give homogenization estimates in the case of quasi-variational inequalities connected to problems of stocastic impulse.

Then, for any (ess) lower bounded function φ, we define

$$(M\varphi)(x,t)=\inf_{\xi\geqslant 0,\; x+\xi\in\bar\Omega,\; t\in(0,T)}\varphi(x+\xi,t)\,, \tag{1.4}$$

and we consider, for any $\varepsilon\geqslant 0$

$$\langle P^\varepsilon u^\varepsilon, v-u^\varepsilon\rangle\geqslant\langle f, v-u^\varepsilon\rangle \quad \forall v\in K^{1+Mu^\varepsilon}\,, \quad u^\varepsilon\in K^{1+Mu^\varepsilon}\,, \quad u^\varepsilon(0)=u_0\,. \tag{1.5$_\varepsilon$}$$

We assume

$$0\neq f\in L^\infty(Q)\,; \tag{1.6}$$

$$\text{(a)}\quad u_0\in W^{2,r}(\Omega)\cap H_0^1(\Omega) \quad\text{or}\quad \text{(b)}\quad u_0\in W_0^{1,r}(\Omega)\quad (r>n+1)\,; \tag{1.7}$$

$$u_0<1+Mu_0 \text{ in } \bar\Omega\,; \tag{1.8}$$

$$\|u^\varepsilon\|_{C^{\beta,\beta/2}(\bar Q)}\leqslant C \qquad \forall\varepsilon>0\,. \tag{1.9}$$

From (1.6), (1.7), (1.8) we deduce [**9**] that (1.5)$_\varepsilon$ admits, for any $\varepsilon\geqslant 0$, an unique continuous solution u^ε and, furthermore, by (1.9), we can obtain, via standard methods, that $\{u^\varepsilon\}_{\varepsilon>0}$ converges to u^0 strongly in $C(\bar Q)$.

About the rapidity of convergence, we obtain the following

Theorem 1. *Under the assumption* (1.6), (1.8), (1.9) $(0<\alpha<1)$

$$\|u^\varepsilon-u^0\|_{L^\infty(Q)}\leqslant C\varepsilon^{\alpha/2(n+3\alpha)} \qquad \textit{if (1.7) (a) holds}\,, \tag{1.10}$$

$$\|u^\varepsilon-u^0\|_{L^\infty(Q)}\leqslant C\varepsilon^{\alpha/4(n+3\alpha)} \qquad \textit{if (1.7) (b) holds}\,. \tag{1.11}$$

Theorem 2. *Under the assumption* (1.9) *and*

$$0 \neq f \in L^\infty(0, T; W^{-1,r}(\Omega)) \qquad (r > n + 1)\,, \tag{1.12}$$

$$u_0 \in C_0^\alpha(\bar{\Omega}) \qquad (0 < \alpha < 1)\,, \tag{1.13}$$

there exists a suitable positive number β, $\beta < 1$, *such that*

$$\|u^\varepsilon - u^0\|_{L^\infty(Q)} \leqslant C\varepsilon^{\alpha/4(n+3\alpha)}\,. \tag{1.14}$$

In **4** we proove (1.10), (1.11) and (1.14).

In **3** we show some preliminary results.

In **2** we examine hypothesis (1.9). This property is well known in the variational case [**3**]$_3$.

We proove that it holds if we assume that there exist subsolutions $\underline{u}^\varepsilon$ of $(1.5)_\varepsilon$, $\varepsilon > 0$, such that

$$\exists \delta > 0: \quad \underline{u}^\varepsilon \geqslant -1 + \delta \text{ in } \bar{Q} \qquad \forall \varepsilon > 0\,, \tag{1.15}$$

$$\underline{u}^\varepsilon \in C(\bar{Q}) \qquad \forall \varepsilon > 0\,. \tag{1.16}$$

2 - About the assumption (1.9).

For any $\varepsilon \geqslant 0$ we consider the variational equation

$$(2.1)_\varepsilon \qquad \begin{gathered} \langle P^\varepsilon u^\varepsilon, v - u^\varepsilon\rangle = \langle f, v - u^\varepsilon\rangle \\ \forall v \in L^2(0, T; H_0^1(\Omega))\,, \quad u^\varepsilon \in L^2(0, T; H_0^1(\Omega))\,, \quad u^\varepsilon(0) = u_0\,. \end{gathered}$$

Let $\bar{u}^\varepsilon$ be the unique bounded solution of $(2.1)_\varepsilon$.

We define, for any $\varepsilon \geqslant 0$, $\tilde{u}^\varepsilon$ as the unique bounded solution of the variational inequality

$$\langle P^\varepsilon \tilde{u}^\varepsilon, v - \tilde{u}^\varepsilon\rangle \geqslant \langle f, v - \tilde{u}^\varepsilon\rangle \qquad \forall v \in K^0\,, \quad \tilde{u}^\varepsilon \in K^0\,, \quad \tilde{u}^\varepsilon(0) = u_0\,. \tag{2.2}$$

We can assume that there exists a positive number δ, independent on ε, such that

$$0 < \delta < 1\,, \quad \tilde{u}^\varepsilon \geqslant -1 + \delta \qquad \forall \varepsilon > 0\,. \tag{2.3}$$

Now we set $\underline{u}^\varepsilon = \tilde{u}^\varepsilon - \delta/2$. Then we have

$$-1 + \frac{\delta}{2} \leqslant \underline{u}^\varepsilon \leqslant -\frac{\delta}{2}\,. \tag{2.4}$$

We consider the variational selection

$$z = S^\varepsilon(\varphi)\,, \tag{2.5}$$

wich maps any function $\varphi \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$ into the solution $z \in L^2(0, T; H_0^1(\Omega))$ of the variational inequality

$$\langle P^\varepsilon z, v - z\rangle \geqslant \langle f, v - z\rangle \quad \forall v \in K^{1+M\varphi}\,, \quad z \in K^{1+M\varphi}\,, \quad z(0) = \underline{u}_0\,. \tag{2.6}$$

The variational selection S^ε is non decreasing, then, by (2.4), we have

$$S^\varepsilon\left(-1 + \frac{\delta}{2}\right) \leqslant S^\varepsilon(\underline{u}^\varepsilon) \leqslant S^\varepsilon\left(-\frac{\delta}{2}\right)\,. \tag{2.7}$$

If we assume δ such that $0 \leqslant S^\varepsilon(-1 + \delta/2) \leqslant \delta/2$, then, by (2.7),

$$S^\varepsilon(\underline{u}^\varepsilon) \geqslant 0\,. \tag{2.8}$$

We set $N = \sup_Q \overline{u}^\varepsilon$ and we fix a positive number λ such that $\lambda \leqslant \delta/(2N+\delta)$. For this choosing of λ, we have

$$\lambda \overline{u}^\varepsilon + (1 - \lambda)\,\underline{u}^\varepsilon \leqslant S^\varepsilon \underline{u}^\varepsilon\,. \tag{2.9}$$

We observe that

$$S^\varepsilon \overline{u}^\varepsilon \leqslant \overline{u}^\varepsilon, \qquad S^\varepsilon \underline{u}^\varepsilon \geqslant \underline{u}^\varepsilon\,. \tag{2.10}$$

From (2.9), (2.10) and [**8**] we have that for any choosing of w, $w \in [\underline{u}^\varepsilon, \overline{u}^\varepsilon]$, the iterated functions (defined in [**7**]) $u_n^\varepsilon = S_n^\varepsilon(w)$ converge weakly in $L^2(0, T; H_0^1(\Omega))$ to the continuous solution u^ε of $(1.5)_\varepsilon$.

If we assume in particular $w = \overline{u}^\varepsilon$, we have that $\{S_n^\varepsilon(\overline{u}^\varepsilon)\}_{n\in N}$ is non increasing and converge uniformly in Q to u^ε. In addition [**8**]

$$\|u_n^\varepsilon - u^\varepsilon\|_{L^\infty(Q)} \leqslant C_1 \theta^n \qquad (0 < \theta < 1)\,, \tag{2.11}$$

where the constant C_1 depends only on $\|\overline{u}^\varepsilon\|_{L^\infty(Q)}$.

We observe that, in large hypothesis about f and u_0, [**1**], we have in the variational equation's case $(2.1)_\varepsilon$ $\overline{u}^\varepsilon \to \overline{u}^0$ strongly in $L^\infty(Q)$. Then we can assume in (2.11) C_1 independent on ε.

It is known, moreover, that u_n^ε are Hölder continuous in Q, uniformly respect to n ([**2**], $[\mathbf{3}]_{1,2,3}$).

We can proove now that (1.9) holds. For any $x, x' \in \Omega$ and $t, t' \in (0, T)$ such that $x' = x + h$; $t' = t + k$; $h \in \mathbf{R}^n$, $k \in \mathbf{R}$, we have

$$(2.12) \qquad |u^\varepsilon(x', t') - u^\varepsilon(x, t)|$$
$$\leqslant |u^\varepsilon(x', t') - u_n^\varepsilon(x', t')| + |u_n^\varepsilon(x', t') - u_n^\varepsilon(x, t)| + |u_n^\varepsilon(x, t) - u^\varepsilon(x, t)|$$
$$\leqslant 2C_1\theta^n + nC_2\{\|h\|^\gamma + |k|^{\gamma/2}\} .$$

If we set $\mu = \{\|h\|^\gamma + |k|^{\gamma/2}\}$, there exists an integer $\overline{n}$ such that, for any $n \geqslant \overline{n}$, we have $\theta^n \leqslant \mu$ and, moreover, $\overline{n} - 1 < \log_\theta \mu$.

Then, from (2.12),

$$(2.13) \qquad |u^\varepsilon(x', t') - u^\varepsilon(x, t)| \leqslant C_3\theta^{\overline{n}} + \overline{n}C_2\mu \leqslant C_4\mu + C_2\mu \log_\theta \mu .$$

Assume $R = \max\{\|h\|, |k|\}$ and $0 < \sigma < \gamma/4$. If we consider $R < 1$, we obtain

$$(2.14) \qquad \mu \leqslant \{\|h\|^{\gamma-\sigma} + |k|^{(\gamma-\sigma)/2}\}$$
$$\mu \log_\theta \mu \leqslant C_5 R^{\gamma/2} \log_\theta R^\gamma \leqslant C_5 \frac{\gamma}{\sigma} R^{\gamma/2-\sigma} R^\sigma \log_\theta R^\sigma$$
$$\leqslant C_5 \frac{\gamma}{\sigma} R^{\gamma/2-\sigma} \leqslant C_5 \frac{\gamma}{\sigma} \{\|h\|^{\gamma/2-\sigma} + |k|^{(\gamma/2-\sigma)/2}\}.$$

By (2.13), (2.14)

$$(2.15) \qquad |u^\varepsilon(x', t') - u^\varepsilon(x, t)| \leqslant C_6\{\|h\|^{\gamma/2-\sigma} + |k|^{(\gamma/2-\sigma)/2}\} .$$

Then we have, by (2.15), that (1.9) holds with $\beta = \gamma/2 - \sigma$.

3 - Preliminary results

In this section we give some results on the rapidity of convergence in the variational equation's case $(2.1)_\varepsilon$.

Lemma 1. *If we assume that, for some* $r > n + 1$,

(3.1) $\quad 0 \neq f \in W^{1,r}(Q)$, $\qquad$ (3.2) $\quad u_0 \in W^{4,r}(\Omega) \cap H_0^1(\Omega)$, *we have*

$$\|u^\varepsilon - u^0\|_{L^\infty(Q)} \leq C\varepsilon \qquad C = C(f, u_0)\,. \tag{3.3}$$

Proof. We can obtain easily (3.3) using the multiple scale method, as in [**10**], Th. 1.

Lemma 2. (a) *If* (1.6), (1.7) (a), (1.9) *hold, then*

$$\|u^\varepsilon - u^0\|_{L^\infty(Q)} \leq C\varepsilon^{1/2} \qquad C = C(f, u_0)\,. \tag{3.4}$$

(b) *If* (1.6), (1.7) (b), (1.9) *hold or if we have* (1.9), (1.12), (1.13), then

$$\|u^\varepsilon - u^0\|_{L^\infty(Q)} \leq C\varepsilon^{1/4}\,. \tag{3.5}$$

Proof. We have to regularize the data f and u_0, and, after, we apply Lemma 1. For this purpose we consider: $g = (P^0)^{-1} f \in W^{1,\infty}(0, T;\, W^{2,r}(\Omega))$, and we define, for any $n > 0$, the regularizing function g^n

$$nP^0 g^n + g^n = g\,. \tag{3.6}$$

We have $g^n \in W^{2,3}_r(Q)$ and so $f^n = P^0 g^n \in W^{1,r}(Q)$.

Following the method of [**10**] (§ **4**) we obtain

$$\|g^n - g\|_{L^\infty(0,T;\, W^{1,r}(\Omega))} \leq Cn^{1/2}\,, \qquad \|f^n\|_{L^\infty(0,T;\, W^{1,r}(\Omega))} \leq Cn^{-1/2}\,. \tag{3.7}$$

By (3.7)

$$\|f^n - f\|_{L^\infty(0,T;\, W^{-1,r}(\Omega))} \leq C_n^{1/2}\,. \tag{3.8}$$

We define now, for any $n > 0$, the regularizing function u_0^n

$$nA^0 u_0^n + u_0^n = u_0\,, \tag{3.9}$$

where A^0 is the elliptic part of P^0.

By (1.7), $u_0^n \in W^{4,r}(\Omega) \cap H^1_0(\Omega)$, and

$$\|u_0^n - u_0\|_{L^\infty(\Omega)} \leq Cn^{1/2}\,, \qquad \|A^0 u_0^n\|_{L\ (\Omega)} \leq Cn^{-1/2}\,. \tag{3.10}$$

In the case in which (1.13) holds, $u_0^n \in W^{3,r}(\Omega) \cap H_0^1(\Omega)$ and we have to consider a new regularizing function, which we obtain applying the same method of (3.9) and we name again u_0^n.

We have $u_0^n \in W^{4,r}(\Omega) \cap H_0^1(\Omega)$ and

$$(3.11) \qquad \|u_0^n - u_0\|_{L^\infty(\Omega)} \leqslant Cn^{1/4}, \qquad \|A^0 u_0^n\|_{L^\infty(\Omega)} \leqslant Cn^{-3/4}.$$

Then we obtain

$$(3.12) \qquad \|u^\varepsilon - u^0\|_{L^\infty(Q)}$$

$$\leqslant \|S^\varepsilon(f, u_0) - S^0(f, u_0)\|_{L^\infty(Q)} \leqslant \|S^\varepsilon(f, u_0) - S^\varepsilon(f^n, u_0)\|_{L^\infty(Q)}$$

$$+ \|S^\varepsilon(f^n, u_0) - S^\varepsilon(f^n, u_0^n)\|_{L^\infty(Q)} + \|S^\varepsilon(f^n, u_0^n) - S^0(f^n, u_0^n)\|_{L^\infty(Q)}$$

$$+ \|S^0(f^n, u_0^n) - S^0(f^n, u_0)\|_{L^\infty(Q)} + \|S^0(f^n, u_0) - S^0(f, u_0)\|_{L^\infty(Q)}.$$

In virtue of Lemma 1, (3.8), (3.10), (3.11), we have (3.4) and (3.5) from (3.12), choosing $n = \varepsilon$.

4 - Proof of Theorem 1 and Theorem 2

We prove first, via Caffarelli-Friedman method [**6**], the estimates (1.10), (1.11).

For any $(x_0, t_0) \in Q$, one of the following two cases occurs:

(i) $u^0(x_0, t_0) < \frac{1}{2} + Mu^0(x_0, t_0)$,

(ii) $\frac{1}{2} + Mu^0(x_0, t_0) \leqslant u^0(x_0, t_0) \leqslant 1 + Mu^0(x_0, t_0)$.

By (1.9), we know that

$$(4.1) \qquad \lim_{\varepsilon \to 0} u^\varepsilon = u^0 \text{ in } L^\infty(Q), \qquad (4.2) \qquad \lim_{\varepsilon \to 0} Mu^\varepsilon = Mu^0 \text{ in } L^\infty(Q),$$

then, if (1) holds, we deduce from (1.9), (4.1), (4.2) that there exists a neighbourhood N_0 of (x_0, t_0), all contained in $\bar{Q}$, such that

$$(4.3) \qquad u^\varepsilon(x, t) \leqslant \sigma + Mu^\varepsilon(x, t) \qquad \text{for any } \varepsilon, \quad 0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0),$$

where $\sigma \leqslant 3/4$. By (4.3), we have that $u^\varepsilon(0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0))$ is the continuous solution of the equation corresponding to $(1.5)_\varepsilon$ in a neighbourhood N_1 of (x_0, t_0),

$N_1 \subset\subset N_0$. Then, by Lemma 2, we have in the case (i)

$$\text{(4.4)} \qquad \text{(a)} \quad \|u^\varepsilon - u^0\|_{L^\infty(N_1)} \leqslant C\varepsilon^{1/2}, \qquad \text{(b)} \quad \|u^\varepsilon - u^0\|_{L^\infty(N_1)} \leqslant C\varepsilon^{1/4}.$$

If (ii) holds, we consider the closed subset of $\bar{Q}$

$$\text{(4.5)} \qquad \Sigma^0 = \{(x_0 + \xi, t_0) \,|\, \xi \geqslant 0;\; u^0(x_0 + \xi, t_0) = Mu^0(x_0, t_0)\}.$$

We observe that $Mu^0(x_0, t_0) \leqslant Mu^0(x_0 + \xi, t_0)$ for any $\xi \geqslant 0$, then (4.3) holds for any $\sigma \leqslant 1$ in N_2, where N_2 is a cylinder wich contains Σ^0.

We consider now the subset I_δ of $\bar{Q}$ (δ suitable positive number), defined by

$$\text{(4.6)} \qquad I_\delta = \{(x_0 + \xi, t_0) \,|\, \xi \geqslant 0;\; u^0(x_0 + \xi, t_0) \geqslant Mu^0(x_0, t_0) + \delta\}.$$

There exists a neighbourhood N_3 of Σ^0, $N_3 \subset\subset N_2$, such that we have, in virtue of (1.9), (4.1), (4.2),

$$\text{(4.7)} \qquad \inf u^\varepsilon(x_0 + \xi, t_0) \geqslant Mu^\varepsilon(x_0, t_0) + \delta,$$

$$(x_0 + \xi, t) \in \bar{Q} \cap CN_3, \qquad \xi \geqslant 0 \quad \text{for any } \varepsilon, \quad 0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0).$$

Then, by (1.9), there exists a neighbourhood V_0 of (x_0, t_0), all contained in Q, such that, for any $(x, t) \in V_0$

$$\text{(4.8)} \qquad \inf u^\varepsilon(x + \xi, t) \geqslant Mu^\varepsilon(x, t) + \delta,$$

$$(x + \xi, t) \in \bar{Q} \cap CN_4, \qquad \xi \geqslant 0 \quad \text{for any } \varepsilon, \quad 0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0),$$

where N_4 is a neighbourhood of Σ^0, $N_4 \subset\subset N_3$. We can argue now from (4.8) that there exists a neighbourhood N_5 of Σ^0, $N_5 \subset\subset N_4$, such that

$$\text{(4.9)} \qquad (Mu^\varepsilon)(x, t) = [M(\eta u^\varepsilon)](x, t), \qquad \forall (x, t) \in V_0; \quad \forall \varepsilon, \quad 0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0),$$

where η is a C^∞ function that assume the value 1 in N_5, and the value 0 out of N_4. For any $(x, t) \in V_0$, for any ε, $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0)$, we have

$$\text{(4.10)} \qquad |(Mu^0)(x, t) - (Mu^\varepsilon)(x, t)| = |[M(\eta u^0)](x, t) - [M(\eta u^\varepsilon)](x, t)|$$

$$= \Big|\inf_{(x+\xi, t) \in \bar{Q} \cap N_5;\, \xi \geqslant 0} u^0(x + \xi, t) - \inf_{(x+\xi, t) \in \bar{Q} \cap N_5;\, \xi \geqslant 0} u^\varepsilon(x + \xi, t)\Big| \leqslant \|u^0 - u^\varepsilon\|_{L^\infty(\bar{Q} \cap N_4)}.$$

But $N_4 \subset\subset N_2$ and (4.3) holds for N_2. Then we have (4.4), where we read

$N_1 = \bar{Q} \cap N_4$. By (4.10), (4.4), we deduce

$$\text{(4.11)} \quad \begin{array}{ll} \text{(a)} \ \|Mu^0 - Mu^\varepsilon\|_{L^\infty(V_0)} \leqslant C\varepsilon^{1/2}\,, \\ \text{(b)} \ \|Mu^0 - Mu^\varepsilon\|_{L^\infty(V_0)} \leqslant C\varepsilon^{1/4}\,, \end{array} \qquad \big(0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0)\big)$$

and, by (4.9),

$$\text{(4.12)} \qquad \|Mu^\varepsilon\|_{W^{1,\infty}(V_0)} \leqslant C\,.$$

Then we consider $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0)$ and a neighbourhood V_1 of (x_0, t_0), $V_1 \subset\subset V_0$, and we have

$$\text{(4.13)} \qquad \|u^\varepsilon - u^0\|_{L^\infty(V_1)} = \|S^\varepsilon(Mu^\varepsilon) - S^0(Mu^0)\|_{L^\infty(V_1)}$$

$$\leqslant \|S^\varepsilon(Mu^\varepsilon) - S^0(Mu^\varepsilon)\|_{L^\infty(V_1)} + \|S^0(Mu^\varepsilon) - S^0(Mu^0)\|_{L^\infty(V_1)}\,.$$

From (4.11), (4.12) and [**10**], we obtain by (4.13)

$$\text{(4.14)} \quad \begin{array}{ll} \text{(a)} \ \|u^\varepsilon - u^0\|_{L^\infty(V_1)} \leqslant C\varepsilon^{\alpha/2(n+3\alpha)}\,, \\ \text{(b)} \ \|u^\varepsilon - u^0\|_{L^\infty(V_1)} \leqslant C\varepsilon^{\alpha/4(n+3\alpha)}\,. \end{array} \qquad \big(0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0)\big)$$

We observe that the constant C, appearing in (4.14) does not depend on (x_0, t_0). From (4.4) and (4.14) we conclude that, in any case, for any $(x_0, t_0) \in Q$, there exists a neighbourhood V of (x_0, t_0) such that

$$\text{(4.15)} \quad \begin{array}{ll} \text{(a)} \ \|u^\varepsilon - u^0\|_{L^\infty(\bar{Q}\cap V)} \leqslant C\varepsilon^{\alpha/2(n+3\alpha)}\,, \\ \text{(b)} \ \|u^\varepsilon - u^0\|_{L^\infty(\bar{Q}\cap V)} \leqslant C\varepsilon^{\alpha/4(n+3\alpha)}\,, \end{array} \qquad \big(0 \leqslant \varepsilon \leqslant \bar{\varepsilon}(x_0, t_0)\big)$$

Since $\bar{Q}$ is a compact subset of $\boldsymbol{R}^{n+1}$, every system of open sets covering $\bar{Q}$ contains a finite subsystem, also covering $\bar{Q}$. Then, from (4.15), we have (1.10) or (1.11), for ε small.

If (1.9), (1.12), (1.13) holds, we obtain easily (1.14) from (4.13), having in mind Lemma 2 (b) and [**10**].

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Riassunto

In questo lavoro gli autori estendono, al caso di disequazioni quasi-variazionali del tipo connesso a problemi di impulso stocastico, stime di rapidità di convergenza ottenute, per disequazioni variazionali, in un precedente articolo [**10**].

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