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Homogenization of quasilinear parabolic systems in diagonal form, having quadratic growth in the spatial gradient (**)

1 - Introduction

We study the homogenization of quasilinear parabolic systems having their main part in diagonal form and a nonlinear first order term $f(x, t, u, \text{grad}_x u)$ growing quadratically with respect to the spatial gradient $\text{grad}_x u$ of the solution u.

A number of homogenization results have been recently obtained in the nonlinear elliptic case [**2**]$_1$, [**3**], [**4**], [**5**]. In particular Biroli and Mosco in [**3**], Boccardo and Murat in [**4**] consider elliptic variational inequalities with non smooth obstacles and a nonlinear first order term $f(x, u, \text{grad}\, u)$ growing quadratically in $\text{grad}\, u$.

On the other hand a relatively small number of works deal with the parabolic nonlinear case: among these let us mention the paper by M. Biroli [**2**]$_2$, who gives homogenization results for parabolic variational and quasi-variational inequalities, where the first order term $f(x, t, u, \text{grad}_x u)$ is sublinear in u and $\text{grad}_x u$ and the obstacle is Hölder continuous.

The quasilinear parabolic system we consider herein has already been dealt with by M. Struwe in [**9**], who proved the Hölder continuity of bounded weak solutions via a « parabolic hole filling » technique.

In the present paper: first we give a suitable variational formulation of our initial boundary value problems (IBVPbs) and then, by taking into ac-

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count the above mentioned Hölder regularity, we prove that both the homogenization and the energy integral converge. The proofs are carried out by local energy methods. The homogenization result also requires to show the existence of suitable « correctors ».

1.1 - Notations

Whenever possible we use the same notations as [**1**].

We let Ω be a bounded, connected, open subset of R^n with « smooth » boundary $\partial\Omega$.

More precisely we assume that Ω is « of type A », i.e. there exists a constant $A > 0$ s.t. $\forall R > 0$ and $\forall x \in \Omega$

$$|\Omega \cap B_R(x)| \geqslant AR^n, \tag{1.1}$$

where $B_R(x) = \{\tilde{x} \in R^n : |\tilde{x} - x| < R\}$ and $|\cdot|$ stands for the n-dimensional Lebesgue measure of a subset of R^n.

Moreover we assume that Ω satisfies the « Wiener type » condition: there are constants $\gamma > 0$, $R_0 > 0$ s.t., $\forall x_0 \in \partial\Omega$ and $\forall R \in (0, R_0]$

$$B_{2R}(x_0) - \operatorname{cap}\,(c\Omega \cap B_R(x_0)) \geqslant \gamma R^{n-2}. \tag{1.2}$$

(We recall that

$$B_{2R}(x_0) - \operatorname{cap}\,(c\Omega \cap B_R(x_0))$$

$$= \inf\{\int_{B_{2R}} |\operatorname{grad}\varphi|^2\,dx : \varphi \in \mathscr{D}(B_{2R}(x_0)),\ \varphi(x) \equiv 1\ \ \forall x \in c\Omega \cap B_R(x_0)\}).$$

Let $T > 0$ be a fixed positive number; we denote by $Q = \Omega \times (0, T)$ the space-time cylinder in R^{n+1} and by $\Sigma = \partial\Omega \times (0, T)$ the time-like boundary, i.e. the lateral surface of Q.

In order to define the family of IBVPbs to which we shall apply the homogenization procedure we also need to introduce the periodicity cells $Y = \prod_{i=1}^{n} (0, y_i^0)$ in R^n and $Y \times (0, \tau_0)$ in $R^{n+1}(y_i^0, \tau_0 > 0)$ and consider $n \times n$ functions $a_{ij}(y, \tau) \in C^1(\overline{Y} \times [0, \tau_0])$ $(i, j = 1, \dots, n)$ satisfying

$$\begin{aligned} &a_{ij}(y, \tau) \quad \text{is } Y - \tau_0 \text{ periodic},\\ &\exists \lambda > 0 \ \text{s.t.}\ a_{ij}(y, \tau)\,\xi_i \xi_j \geqslant \lambda |\xi|^2 \qquad \forall \xi \in R^n,\ \forall (y, \tau)\ \ (^1). \end{aligned} \tag{1.3}$$

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(1) We adopt here and in the following the usual summation convention over repeated indexes.

We extend $a_{ij}(y, \tau)$ to R^{n+1} by periodicity and set

$$a^{\varepsilon}_{ij}(x, t) = a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \qquad (x, t) \in Q\,, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter.

It follows from (1.3) and (1.4) that $a^{\varepsilon}_{ij} \in C^0(\bar{Q}) \cap W^{1,\infty}(Q)$ and

$$(1.5)_\varepsilon \qquad \begin{aligned} &a^{\varepsilon}_{ij}(x, t) \quad \text{is } \varepsilon Y - \varepsilon\tau_0 \text{ periodic}\,,\\ &a^{\varepsilon}_{ij}(x, t)\, \xi_i \xi_j \geqslant \lambda |\xi|^2 \qquad \forall \xi \in R^n,\ \forall (x, t)\,. \end{aligned}$$

We can now define the family of parabolic operators

$$p^{\varepsilon} = \partial_t + A^{\varepsilon}\,,$$

where

$$A^{\varepsilon} = A^{\varepsilon}\left(\frac{t}{\varepsilon}\right) = -\,\partial_i\big(a^{\varepsilon}_{ij}(x, t)\,\partial_j\big) = -\,\operatorname{div}_x\big(a^{\varepsilon}(x, t)\,\operatorname{grad}_x\big) \ (^2)$$

are the second order elliptic operators associated to the $n \times n$ matrices $a^{\varepsilon}(x, t) = [a^{\varepsilon}_{ij}(x, t)] \in \big(C^0(\bar{Q}) \cap W^{1,\infty}(Q)\big)^{n\times n}$.

We point out that the family $\{A^{\varepsilon}\}$ is ε-uniformly bounded in $\mathscr{L}(L^2(0, T;\ H^1_0(\Omega)), L^2(0, T; H^{-1}(\Omega)))$. We denote by

$$p^0 = \partial_t + A = \partial_t - q_{ij}\,\partial_i\,\partial_j$$

the homogenized operator of p^{ε} [**1**]. We recall that the matrix $[q_{ij}]$ associated to A is constant and fulfils the ellipticity condition

$$q_{ij}\,\xi_i\,\xi_j \geqslant \lambda |\xi|^2 \qquad \forall \xi \in R^n\,, \tag{1.5}$$

where $\lambda > 0$ is the same as in $(1.5)_\varepsilon$.

Now let $f^l\colon Y \times (0, \tau_0) \times R^N \times R^{nN} \ni (y, \tau, u, p) \mapsto f^l(y, \tau, u, p) \in R$, $l = 1, \ldots, N$ be a Carathéodory function, i.e. measurable in (y, τ) and continuous in (u, p), which satisfies the growth condition

$$|f^l(y, \tau, u, p)| \leqslant K(1 + |p|^2) \qquad \text{a.e. in } (y, \tau)\ \forall (u, p) \qquad (K > 0)\,. \tag{1.6}$$

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(2) The spatial and time derivatives of a smooth function $v(x, t)$, $(x, t) \in Q$, are respectively denoted by $\partial_j v(x, t)$, $j = 1, \ldots, n$ and $\partial_t v(x, t)$.

We extend f^l to $R^{n+1}\times R^N\times R^{nN}$ by periodicity and assume

$$|f^l(y,\tau,u,p)-f^l(y,\tau,v,q)| \tag{1.7}$$

$$\leqslant c_1(M)\left(|p-q|+|p-q|\,|q|+|p-q|^2\right)+c_2(\eta)\,,$$

if $|u|$, $|v|\leqslant M$ and $|v-u|\leqslant\eta$, where $c_1(M)$, $c_2(\eta)>0$ and $\lim_{\eta\to 0} c_2(\eta)=0$. We set

$$f^l_\varepsilon(x,t,u,p)=f^l\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u,p\Big)\qquad (x,t,u,p)\in Q\times R^N\times R^{nN} \tag{1.8}$$

and note that f^l_ε is a Carathéodory function, $\varepsilon Y-\varepsilon\tau_0$ periodic in (x,t), for which (1.6) and (1.7) hold uniformly in ε.

In order to construct the «homogenized» function f^l_0 of f^l_ε let us consider the operator

$$A_1=-(\partial/\partial y_i)\big(a_{ij}(y,\tau)(\partial/\partial y_j)\big)=-\operatorname{div}_y\big(a(y,\tau)\operatorname{grad}_y\big)\,,$$

acting on the (quotient) Hilbert space (3) $\tilde W(Y)=W(Y)/R$ where $W(Y)=\{\psi\in H^1(Y):\ \psi \text{ is } Y \text{ periodic}\}$; it turns out that $A_1\in\mathscr{L}(\tilde W(Y),\tilde W(Y)^*)$. Let us associate to A_1 the bilinear continuous and coercive form $a_1\colon \tilde W(Y)\times\tilde W(Y)\to R$

$$(\varphi,\psi)\mapsto a_1(\varphi,\psi)={}_{\tilde W(Y)^*}\langle A_1\varphi,\psi\rangle_{\tilde W(Y)}=\int_Y a_{ij}(y,\tau)\,\frac{\partial\varphi}{\partial y_j}\,\frac{\partial\psi}{\partial y_i}\,dy\,.$$

As an application of the Lax-Milgram Lemma, we state the existence of a unique solution $\chi_j(y,\tau)\in\tilde W(Y)$ $\forall\tau$ of the following variational equation

$$a_1(\chi_j(y,\tau),\psi)={}_{\tilde W(Y)^*}\langle A_1 y_j,\psi\rangle_{\tilde W(Y)}\quad \forall\psi\in\tilde W(Y)\quad (j=1,\dots,n),\ \text{ since } A_1y_j\in\tilde W(Y)^*.$$

We choose the additive τ-dependent constant in $\chi_j(y,\tau)$ in such a way that $\chi_j(y,\tau)$ is $Y-\tau_0$ periodic. (We can take e.g. $\mathscr{M}_y(\chi_j(y,\tau))=0$ $\forall\tau$, where $\mathscr{M}_y(\chi_j(y,\tau)):=(1/|Y|)\int_Y\chi_j(y,\tau)\,dy$.)

Due to the regularity assumption on the coefficients $a_{ij}(y,\tau)\in C^1(\overline Y\times[0,\tau_0])$, it turns out that $\chi_j(y,\tau)\in L^\infty(0,\tau_0;\,W^{1,\infty}(Y))$ ([6]).

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(3) On the quotient space $\tilde W(Y)$ the inner product $(\varphi|\psi)_{\tilde W(Y)}=(\partial\varphi/\partial y_i|\partial\psi/\partial y_i)_{L^2(Y)}$ $\forall\varphi,\psi\in\tilde W(Y)$ is well defined.

We can now construct the «homogenized» function

$$f_0^l\colon Q\times R^N\times R^{nN}\ni(x,t,u,p)\mapsto f_0^l(u,p)\in R\qquad l=1,\dots,N\,.$$

We consider p as an $n\times N$ matrix and set

$$f_0^l(u,p)=\mathscr{M}_{y,\tau}\big(f^l(y,\tau,u,(I-\operatorname{grad}_y\chi)(y,\tau)p)\big)$$
$$=\frac{1}{|Y|\tau_0}\int_0^{\tau_0}\int_Y f^l\big(y,\tau,u,(I-\operatorname{grad}_y\chi)(y,\tau)\,p\big)\,\mathrm{d}y\,\mathrm{d}\tau\,,$$

where I is the $n\times n$ identity matrix and $(\operatorname{grad}_y\chi)(y,\tau)\in\big(L^\infty(Y\times(0,\tau_0))\big)^{n\times n}$ is the $n\times n$ matrix $[\operatorname{grad}_y\chi_j(y,\tau)]=[(\partial/\partial y_k)\chi_j(y,\tau)]$, $j,k=1,\dots,n$.

It can be easily shown that also f_0^l satisfies conditions (1.6) and (1.7).

We notice that, if we extend $\chi_j(y,\tau)$ to R^{n+1} by periodicity and define the matrix $(I-\operatorname{grad}_y\chi)^\varepsilon\in(L^\infty(Q))^{n\times n}$

$$(I-\operatorname{grad}_y\chi)^\varepsilon(x,t)=(I-\operatorname{grad}_y\chi)\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big)\qquad(x,t)\in Q\,,$$

then the following relationship holds $\forall(u,p)\in R^N\times R^{nN}$

(1.9) $$f_\varepsilon^l(\cdot,\cdot,u,(I-\operatorname{grad}_y\chi)^\varepsilon p)\rightharpoonup f_0^l(u,p)\quad\text{in } L^2(Q)\quad\text{weakly},\quad l=1,\dots,N\,.$$

1.2 - Problem setting and results

We consider the family of IBVPbs

(1.10$_\varepsilon$) $$p^\varepsilon u_\varepsilon^l=f_\varepsilon^l(\cdot,\cdot,u_\varepsilon,\operatorname{grad}_x u_\varepsilon)\quad\text{in } Q\qquad l=1,\dots,N\,,$$
$$u_\varepsilon(x,t)_{|\Sigma}=0\,,\qquad u_\varepsilon(x,0)=g(x,0)\,,$$

where $u_\varepsilon(x,t)=[u_\varepsilon^l(x,t)]$ is the unknown vector and $\operatorname{grad}_x u_\varepsilon$ is the $n\times N$ matrix $[\operatorname{grad}_x u_\varepsilon^l]=[\partial_j u_\varepsilon^l]$, $j=1,\dots,n$; $l=1,\dots,N$. The parabolic system $(1.10)_\varepsilon$ has its main part in diagonal form, whereas the coupling among equations is due to the first order nonlinear terms, which, by (1.6) and (1.8), grow quadratically with respect to $\operatorname{grad}_x u_\varepsilon$.

To the family $(1.10)_\varepsilon$ we associate the following IBVPb

(1.10) $$p_0u^l=f_0^l(u,\operatorname{grad}_x u)\quad\text{in } Q\qquad l=1,\dots,N\,,$$
$$u(x,t)_{|\Sigma}=0\,,\qquad u(x,0)=g(x,0)\quad\big(u(x,t)=[u^l(x,t)]\big).$$

It has the same properties as $(1.10)_\varepsilon$. We shall show (1.10) is the homogenized problem of $(1.10)_\varepsilon$.

According to [9], by a weak solution of $(1.10)_\varepsilon$, respectively (1.10), we mean a vector $u_\varepsilon \in \mathscr{H}^N = (L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q))^N$, resp. $u \in \mathscr{H}^N$, which satisfies the system $(1.10)_\varepsilon$, resp. (1.10), in the distribution sense and the initial condition $u_\varepsilon(\cdot, 0) = u(\cdot, 0) = g(\cdot, 0)$, where the datum $g(\cdot, 0) = [g^l(\cdot, 0)] \in (L^2(\Omega))^N$.

More explicitly, by choosing test functions in the space $\tilde{\mathscr{H}} = L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)) \cap L^\infty(Q)$, we set problems $(1.10)_\varepsilon$ and respectively (1.10) in the following variational forms

$$(1.11)_\varepsilon \qquad \int_0^T \Big(- {}_{H^{-1}(\Omega)}\!\!\prec \partial_t v(t), u_\varepsilon^l(t) \succ_{H_0^1(\Omega)} + {}_{H^{-1}(\Omega)}\!\!\prec A^\varepsilon\Big(\frac{t}{\varepsilon}\Big) u_\varepsilon^l(t), v(t) \succ_{H_0^1(\Omega)}\Big)\, dt$$

$$= {}_{L^1(Q)}\!\!\prec f_\varepsilon^l(\cdot, \cdot, u_\varepsilon, \mathrm{grad}_x u_\varepsilon), v \succ_{L^\infty(Q)} + \big(g^l(\cdot, 0) \,|\, v(\cdot, 0)\big)_{L^2(\Omega)} \qquad l = 1, \dots, N$$

$$\forall v \in \tilde{\mathscr{H}} \text{ s.t. } v(\cdot, T) = 0\,; \qquad u_\varepsilon \in \mathscr{H}^N, \qquad u_\varepsilon(\cdot, 0) = g(\cdot, 0) \in \big(L^2(\Omega)\big)^N;$$

$$(1.11) \qquad \int_0^T \Big(- {}_{H^{1}(\Omega)}\!\!\prec \partial_t v(t), u^l(t) \succ_{H_0^1(\Omega)} + {}_{H^{-1}(\Omega)}\!\!\prec A u^l(t), v(t) \succ_{H_0^1(\Omega)}\Big)\, dt$$

$$= {}_{L^1(Q)}\!\!\prec f_0^l(u, \mathrm{grad}_x u), v \succ_{L^\infty(Q)} + \big(g^l(\cdot, 0) \,|\, v(\cdot, 0)\big)_{L^2(\Omega)} \qquad l = 1, \dots, N$$

$$\forall v \in \tilde{\mathscr{H}} \text{ s.t. } v(\cdot, T) = 0\,; \qquad u \in \mathscr{H}^N, \qquad u(\cdot, 0) = g(\cdot, 0) \in \big(L^2(\Omega)\big)^N.$$

Remark 1. We notice that the final condition $v(\cdot, T) = 0$ and the inner products $(g^l(\cdot, 0) \,|\, v(\cdot, 0))_{L^2(\Omega)}$ make sense for test functions $v \in \tilde{\mathscr{H}}$, because in this case the mapping $[0, T] \ni t \mapsto v(t) \in L^2(\Omega)$ is continuous [7].

Also the initial conditions $u_\varepsilon(\cdot, 0) = u(\cdot, 0) = g(\cdot, 0) \in (L^2(\Omega)))^N$ make sense for solutions u_ε, $u \in \mathscr{H}^N$, as we shall see later on.

Let us consider the ε-uniformly bounded solutions of $(1.11)_\varepsilon$, (1.11), i.e. solutions u_ε, $u \in \mathscr{H}^N$ such that $\|u_\varepsilon\|_{L^\infty}$, $\|u\|_{L^\infty} \leqslant M$, where $M > 0$ is a fixed positive number (4). Now, if we assume that the above mentioned constant $M > 0$, the ellipticity constant $\lambda > 0$ in $(1.5)_\varepsilon$, (1.5) and the growth constant $K > 0$ in (1.6) are related by

$$(1.12) \qquad 2KM < \lambda$$

and add a smoothness hypothesis on the initial datum g, then we are entitled to apply Struwe's Hölder regularity result [9].

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(4) The existence of such solutions can be proved by standard « truncation » methods.

About the initial datum $g(\cdot, 0)$ we assume it is Hölder continuous

$$g(\cdot, 0) \in \left(C_0^\beta(\bar{\Omega})\right)^N \text{ (5)} . \tag{1.13}$$

We recall that, under this hypothesis, its components $g^l(\cdot, 0)$, $l = 1, \ldots, N$, can be raised up to Q as weak solutions of the linear IBVP*bs* relative to the parabolic operators p^ε and respectively p^0, having homogeneous boundary conditions and initial data $g^l(\cdot, 0)$. Both solutions g_ε^l and g^l, $l = 1, \ldots, N$, belong to $\mathscr{H}$; moreover they are ε-uniformly bounded in $C^{\gamma,\gamma/2}(\bar{Q})$, for some $\gamma > 0$. If β is small, then $\gamma = \beta$ [**6**].

We now summarise Struwe's results which hold under our assumptions. The main result deals with the Hölder regularity up to the boundary of the ε-uniformly bounded solutions of $(1.11)_\varepsilon$ (1.11) (Theorem 1). This requires to prove that the just mentioned solutions are continuous with respect to time as trajectories in $(L^2(\Omega))^N$ (Lemma 1).

Theorem 1. *Let u_ε, $u \in \mathscr{H}^N$ be solutions of* $(1.11)_\varepsilon$ *and* (1.11) *respectively. Assume that* (1.1), (1.2), $(1.5)_\varepsilon$, (1.6), (1.12), (1.13) *hold* (6). *Then there exists a number $\alpha > 0$, which does not depend on u_ε, resp. u, but only on the fixed parameters in the problem, such that u_ε, $u \in \left(C^{\alpha,\alpha/2}(\bar{Q})\right)^N$. Moreover the $\left(C^{\alpha,\alpha/2}(\bar{Q})\right)^N$ norm of u_ε, resp. u, can be a priori estimated in terms of these parameters.*

Lemma 1. *Under the same hypotheses as in Theorem* 1 *the mappings $[0, T) \ni t \mapsto u_\varepsilon(t) \ni \left(L^2(\Omega)\right)^N$, $[0, T) \ni t \mapsto u(t) \in \left(L^2(\Omega)\right)^N$ are continuous.*

Remark 2. We point out that Lemma 1 gives sense to the initial conditions $u_\varepsilon(\cdot, 0) = u(\cdot, 0) = g(\cdot, 0) \in \left(L^2(\Omega)\right)^N$ in (1.11_ε) and (1.11).

Our results relate to the convergence of the homogenization (Theorem 2) and to the convergence of the energy integral (Theorem 3).

Theorem 2. *Assume that the same hypotheses as in Theorem* 1 *are satisfied and* (1.7) *holds.*

Let $\{u_\varepsilon\} \subset \mathscr{H}^N$ be solutions of (1.11_ε). *Then there exists a subsequence $\{u_{\varepsilon'}\} \subset \{u_\varepsilon\}$ such that*

$$u_{\varepsilon'} \rightharpoonup u \text{ in } \left(L^2(0, T; H_0^1(\Omega))\right)^N \text{ weakly}, \qquad u_{\varepsilon'} \to u \text{ in } \left(C^0(\bar{Q})\right)^N \text{ strongly},$$

where $u \in \mathscr{H}^N$ is a solution of (1.11).

(5) i.e. $g(\cdot, 0) \in \left(C^\beta(\bar{\Omega})\right)^N$ and $g(x, 0)_{|\partial\Omega} = 0$.

(6) We recall that Struwe's proof was carried out without any smoothness assumption on the coefficients $a_{ij}^\varepsilon(x, t)$, which were only assumed to be bounded and measurable: $a_{ij}^\varepsilon \in L^\infty(Q)$.

Theorem 3. *Under the same hypotheses as in Theorem 2 the energy integral converges*

$$(1.14)\qquad \int_Q a^{\varepsilon'}(x,t)\,\mathrm{grad}_x u^l_{\varepsilon'}\cdot \mathrm{grad}_x u^l_{\varepsilon'}\,dx\,dt \to \int_Q [q_{ij}]\,\mathrm{grad}_x u^l\cdot \mathrm{grad}_x u^l\,dx\,dt \qquad l=1,\dots,N\,.$$

We carry out the proofs via the following lemmas which are stated under the same assumptions as Theorems 2 and 3.

Lemma 2. *Let* $\{u_\varepsilon\}\subset \mathscr{H}^N$ *be solutions of* $(1.11)_\varepsilon$. *Then there exists a subsequence* $\{u_{\varepsilon'}\}\subset\{u_\varepsilon\}$ *such that* $u_{\varepsilon'}\rightharpoonup u$ *in* $(L^2(0,T;H^1_0(\Omega)))^N$ *weakly,* $u_{\varepsilon'}\to u$ *in* $(C^0(\bar Q))^N$ *strongly,* $p^{\varepsilon'}u^l_{\varepsilon'}\rightharpoonup p^0u^l$ *in the vague topology* $\sigma(\mathscr{M}(Q), C^0_0(Q))$ *of the bounded measures* $\mathscr{M}(Q)$, $l=1,\dots,N$.

Furthermore the «local energy» of $u_{\varepsilon'}$ *converges to the «local energy» of* u

$$(1.15)\qquad a^{\varepsilon'}(x,t)\,\mathrm{grad}_x u^l_{\varepsilon'}\cdot \mathrm{grad}_x u^l_{\varepsilon'} \rightharpoonup [q_{ij}]\,\mathrm{grad}_x u^l\cdot\mathrm{grad}_x u^l \quad \text{in } \mathscr{M}(Q) \text{ vaguely},\ l=1,\dots,N\,.$$

Lemma 3. *Let* $\{u_\varepsilon\}\subset \mathscr{H}^N$ *be solutions of* $(1.11)_\varepsilon$. *Then there exists a subsequence* $\{u_{\varepsilon'}\}\subset\{u_\varepsilon\}$ *such that* $u_{\varepsilon'}\rightharpoonup u$ *in* $(L^2(0,T;H^1_0(\Omega)))^N$ *weakly,* $u_{\varepsilon'}\to u$ *in* $(C^0(\bar Q))^N$ *strongly,* $p^\varepsilon u^l_{\varepsilon'}\rightharpoonup p_0u^l$ *in* $\mathscr{M}(Q)$ *vaguely,* $l=1,\dots,N$, $f^l_{\varepsilon'}(\cdot,\cdot,u_{\varepsilon'},\mathrm{grad}_x u_{\varepsilon'})\rightharpoonup \nu^l$ *in* $\mathscr{M}(Q)$ *vaguely,* $l=1,\dots,N$, *where* $u\in\mathscr{H}^N$ *is a solution of the* IBVP*b*

$$(1.16)\qquad \int_0^T \left(-{}_{H^{-1}(\Omega)}\langle \partial_t v(t), u^l(t)\rangle_{H^1_0(\Omega)} + {}_{H^{-1}(\Omega)}\langle A u^l(t), v(t)\rangle_{H^1_0(\Omega)}\right)dt = \int_Q \nu^l v\,dx\,dt + (g^l(\cdot,0)\,|\,v(\cdot,0))_{L^2(\Omega)} \qquad l=1,\dots,N$$

$\forall v\in\tilde{\mathscr{H}}\cap C^0(\bar Q)$ s.t. $v(\cdot,T)=0$; $u\in(\mathscr{H}\cap C^0(\bar Q))^N$, $u(\cdot,0)=g(\cdot,0)\in(C^\beta_0(\bar\Omega))^N$.

Lemma 4. *The* $n\times n$ *matrix* $(\mathrm{grad}_y\chi)(x/\varepsilon', t/\varepsilon')\in(L^\infty(Q))^{n\times n}$ *works as a first order corrector*

$$(1.17)\qquad \mathrm{grad}_x u^l_{\varepsilon'} = (I-\mathrm{grad}_y\chi)^{\varepsilon'}\,\mathrm{grad}_x u^l + z^l_{\varepsilon'}\,,$$

where $z^l_{\varepsilon'}\to 0$ *in* $(L^2(Q))^n$ *strongly,* $l=1,\dots,N$.

Lemma 5. *The following relationships hold*

$$(1.18)\qquad \nu^l = f^l_0(u,\mathrm{grad}_x u) \quad \text{in } L^1(Q) \quad l=1,\dots,N\,,$$

$$(1.19)\qquad f^l_{\varepsilon'}(\cdot,\cdot,u_{\varepsilon'},\mathrm{grad}_x u_{\varepsilon'}) \rightharpoonup f^l_0(u,\mathrm{grad}_x u) \quad \text{in } L^1(Q) \text{ weakly},\quad l=1,\dots,N\,.$$

Remark 3. We point out that our smoothness assumption on the coefficients $a_{ij}(y, \tau) \in C^1(\overline{Y} \times [0, \tau_0])$ is essential to obtain correctors $\chi_j \in L^\infty(0, \tau_0; W^{1,\infty}(Y))$, $j = 1, \ldots, n$ [6]. Actually we need $(\operatorname{grad}_y \chi)(y, \tau) \in (L^\infty(Y \times (0, \tau_0)))^{n\times n}$ in the statements of Lemmas 4 and 5.

2 - Proof of Lemma 2. As the injection $C^{\alpha,\alpha/2}(\overline{Q}) \to C^0(\overline{Q})$ is compact, it follows from the ε-uniform boundedness of $\{u_\varepsilon\}$ in $(C^{\alpha,\alpha/2}(\overline{Q}))^N$ that there exists a subsequence $\{u_{\varepsilon'}\} \subset \{u_\varepsilon\}$ such that

$$u_{\varepsilon'} \to u \quad \text{in } (C^0(\overline{Q}))^N \text{ strongly}. \tag{2.1}$$

Let us now show that $\{u_{\varepsilon'}\}$ are ε'-uniformly bounded in $(L^2(0, T; H_0^1(\Omega)))^N$ too.

To this end let us choose $u^l_{\varepsilon'}$ as the test function in $(1.10)_\varepsilon$

$$\langle(\partial_t + A^{\varepsilon'})u^l_{\varepsilon'}, u^l_{\varepsilon'}\rangle = {}_{L^1(Q)}\langle f^l_{\varepsilon'}(\cdot, \cdot, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'}), u^l_{\varepsilon'}\rangle_{L^\infty(Q)} \qquad l = 1, \ldots, N. \tag{2.2}$$

This makes sense because $u^l_{\varepsilon'} \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$, whereas $A^{\varepsilon'}u^l_{\varepsilon'} \in L^2(0, T; H^{-1}(\Omega))$ and $\partial_t u^l_{\varepsilon'} \in L^2(0, T; H^{-1}(\Omega)) + L^1(Q)$. We obtain

$$\langle\partial_t u^l_{\varepsilon'}, u^l_{\varepsilon'}\rangle = \int_Q \partial_t u^l_{\varepsilon'} \cdot u^l_{\varepsilon'}\, dx\, dt = \tfrac{1}{2}\int_Q \partial_t(u^l_{\varepsilon'}(x, t))^2\, dx\, dt$$

$$= \tfrac{1}{2}\int_\Omega [(u^l_{\varepsilon'}(x, T))^2 - (u^l_{\varepsilon'}(x, 0))^2]\, dx = \tfrac{1}{2}(\|u^l_{\varepsilon'}(\cdot, T)\|^2_{L^2(\Omega)} - \|g^l(\cdot, 0)\|^2_{L^2(\Omega)});$$

$$\langle A^{\varepsilon'}u^l_{\varepsilon'}, u^l_{\varepsilon'}\rangle = \int_Q a^{\varepsilon'}(x, t)\operatorname{grad}_x u^l_{\varepsilon'} \operatorname{grad}_x u^l_{\varepsilon'}\, dx\, dt$$

(by $(1.5)_\varepsilon$) $\geqslant \lambda \int_Q |\operatorname{grad}_x u^l_{\varepsilon'}|^2\, dx\, dt = \lambda \|u^l_{\varepsilon'}\|^2_{L^2(0,T; H_0^1(\Omega))}$;

$$|{}_{L^1(Q)}\langle f^l_{\varepsilon'}(\cdot, \cdot, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'}), u^l_{\varepsilon'}\rangle_{L^\infty(Q)}|$$

$\leqslant \int_Q |f^l_{\varepsilon'}(x, t, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'})|\, |u^l_{\varepsilon'}(x, t)|\, dx\, dt$ (by (1.6))

$\leqslant MK\int_Q (1 + |\operatorname{grad}_x u_{\varepsilon'}|^2)\, dx\, dt = MK(|Q| + \|u_{\varepsilon'}\|^2_{(L^2(0,T; H_0^1(\Omega)))^N})$ (by (1.12))

$$\leqslant \frac{\lambda}{2}(|Q| + \|u_{\varepsilon'}\|^2_{(L^2(0,T; H_0^1(\Omega)))^N}).$$

Finally, by summing (2.2) over l, we get

$$\|u_{\varepsilon'}\|_{(L^2(0,T; H_0^1(\Omega)))^N} \leqslant L, \tag{2.3}$$

where L does not depend on ε'.

Therefore there exists a subsequence, which we shall denote by $\{u_{\varepsilon'}\}$, such that

$$u_{\varepsilon'} \rightharpoonup u \quad \text{in } (L^2(0, T; H_0^1(\Omega)))^N \text{ weakly}, \tag{2.4}$$

where $u \in (C^0(\bar{Q}) \cap L^2(0, T; H_0^1(\Omega)))^N$.

Let us now consider the distributions $p^{\varepsilon'} u^l_{\varepsilon'} \in \mathscr{D}'(Q)$, $l = 1, \dots, N$. We notice that they are bounded measures $p^{\varepsilon'} u^l_{\varepsilon'} \in \mathscr{M}(Q)$, because, by $(1.10)_\varepsilon$ and (1.6), $p^{\varepsilon'} u^l_{\varepsilon'} \in L^1(Q)$. We shall now prove that, owing to (2.3), they are also ε'-uniformly bounded. To this end let $Q' \subset\subset Q$ and $\varphi \in \mathscr{D}(Q)$ such that $0 \leqslant \varphi(x, t) \leqslant 1$, $\varphi(x, t)|_{Q'} = 1$; we have

$$\begin{aligned} (2.5) \qquad |p^{\varepsilon'} u^l_{\varepsilon'}|(Q') &\leqslant {}_{\mathscr{D}'(Q)}\langle |p^{\varepsilon'} u^l_{\varepsilon'}|, \varphi \rangle_{\mathscr{D}(Q)} \\ &= {}_{L^1(Q)}\langle |f^l_{\varepsilon'}(\cdot, \cdot, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'})|, \varphi \rangle_{L^\infty(Q)} \qquad \text{(by (1.6))} \\ &\leqslant K \int_Q (1 + |\operatorname{grad}_x u_{\varepsilon'}|^2)\, dx\, dt = K(|Q| + \|u_{\varepsilon'}\|^2_{(L^2(0,T;\, H_0^1(\Omega)))^N}) \\ \text{(by (2.3))} \qquad &\leqslant K(|Q| + L^2) \qquad l = 1, \dots, N, \end{aligned}$$

where the final constant does neither depend on ε' nor on $Q' \subset\subset Q$.

We notice that the ε'-uniform boundedness of the measures $p^{\varepsilon'} u^l_{\varepsilon'}$ can also be obtained by looking at the left hand side of $(1.10)_\varepsilon$ $\forall Q' \subset\subset Q$, let $\varphi \in \mathscr{D}(Q)$ such that $0 \leqslant \varphi(x, t) \leqslant 1$, $\varphi(x, t)_{|Q'} = 1$, then

$$\begin{aligned} (2.6) \qquad |p^{\varepsilon'} u^l_{\varepsilon'}|(Q') &\leqslant {}_{\mathscr{D}'(Q)}\langle |p^{\varepsilon'} u^l_{\varepsilon'}|, \varphi \rangle_{\mathscr{D}(Q)} \\ &\leqslant \int_Q (|-u^l_{\varepsilon'} \partial_t \varphi| + |a^{\varepsilon'}_{ij}(x, t)\, \partial_j u^l_{\varepsilon'} \partial_i \varphi|)\, dx\, dt \\ &\leqslant \|u^l_{\varepsilon'}\|_{L^2(Q)} \|\partial_t \varphi\|_{L^2(Q)} + \|a^{\varepsilon'}_{ij}\|_{L^\infty(Q)} \|\partial_j u^l_{\varepsilon'}\|_{L^2(Q)} \|\partial_i \varphi\|_{L^2(Q)} \quad \text{(by the Poincaré inequality)} \\ &\leqslant \text{const} \|u^l_{\varepsilon'}\|_{L^2(0,T;\, H_0^1(\Omega))} \|\varphi\|_{H_0^1(Q)} \leqslant c(Q') \qquad \text{(by (2.3))} \qquad l = 1, \dots, N, \end{aligned}$$

where the constant $c(Q')$ does not depend on ε'.

It follows from (2.5) or (2.6) that there exist a subsequence, which we shall still denote by $\{p^{\varepsilon'} u^l_{\varepsilon'}\}$, and a bounded measure $\mu^l \in \mathscr{M}(Q)$ such that $p^{\varepsilon'} u^l_{\varepsilon'} \rightharpoonup \mu^l$ in $\mathscr{M}(Q)$ vaguely, $l = 1, \dots, N$.

It could be shown, by a local energy method [1] and by taking (2.1) into account, that

$$p^{\varepsilon'} u^l_{\varepsilon'} \rightharpoonup p^0 u^l \quad \text{in } \mathscr{M}(Q) \text{ vaguely} \qquad l = 1, \ldots, N\,. \tag{2.7}$$

We now turn to the proof of (1.15). Let $\varphi \in \mathscr{D}(Q)$; it follows from (2.1) that $\varphi u^l_{\varepsilon'} \to \varphi u^l$ in $C^0_0(Q)$ strongly. Then (2.7) implies

$$_{\mathscr{M}(Q)}\langle p^{\varepsilon'} u^l_{\varepsilon'}, \varphi u^l_{\varepsilon'} \rangle_{C^0_0(Q)} \to {}_{\mathscr{M}(Q)}\langle p^0 u^l, \varphi u^l \rangle_{C^0_0(Q)} \qquad l = 1, \ldots, N\,.$$

From this relationship, by identification, we deduce (1.15).

Proof of Theorem 3. We point out that the sequence

$$\{a^{\varepsilon'}(x, t) \operatorname{grad}_x u^l_{\varepsilon'} \cdot \operatorname{grad}_x u^l_{\varepsilon'}\} \qquad l = 1, \ldots, N\,,$$

is ε'-uniformly bounded in $L^1(Q)$, because

$$\operatorname{grad}_x u^l_{\varepsilon'} \rightharpoonup \operatorname{grad}_x u^l \text{ in } (L^2(Q))^n \text{ weakly, } l = 1, \ldots, N \text{ by (2.4) and}$$

$$a^{\varepsilon'}(x, t) \operatorname{grad}_x u^l_{\varepsilon'} \rightharpoonup [q_{ij}] \operatorname{grad}_x u^l \text{ in } (L^2(Q))^n \text{ weakly,} \qquad l = 1, \ldots, N\,, \tag{2.8}$$

which is a straight-forward consequence of (2.7). Moreover, as we have just seen, (1.15) holds, where $[q_{ij}] \operatorname{grad}_x u^l \cdot \operatorname{grad}_x u^l \in L^1(Q)$. Hence (1.14) follows.

Proof of Lemma 3. We notice that, owing to (1.6) and (2.3), the sequence $\{f^l_{\varepsilon'}(\cdot, \cdot, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'})\}$ is ε'-uniformly bounded in $L^1(Q)$

$$\|f^l_{\varepsilon'}(\cdot, \cdot, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'})\|_{L^1(Q)} \leqslant K(|Q| + L^2) \qquad l = 1, \ldots, N\,,$$

this constant being independent of ε'.

Therefore there exist a subsequence, which we shall still denote by $\{f^l_{\varepsilon'}\}$, and a bounded measure $\nu^l \in \mathscr{M}(Q)$ such that

$$f^l_{\varepsilon'}(\cdot, \cdot, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'}) \rightharpoonup \nu^l \quad \text{in } \mathscr{M}(Q) \text{ vaguely} \qquad l = 1, \ldots, N\,. \tag{2.9}$$

We now turn to the proof of (1.16).

This system of variational equations makes sense because it involves correct dualities. It is obtained from $(1.11)_\varepsilon$ by passing to the limit as $\varepsilon' \to 0$ and taking (2.4), (2.8), (2.9) into account.

Proof of Lemma 4 [7]. It follows from the definition that the matrices $(I-\operatorname{grad}_y\chi)(x/\varepsilon, t/\varepsilon)\in(L^\infty(Q))^{n\times n}$ have the following properties

$$(I-\operatorname{grad}_y\chi)^\varepsilon \rightharpoonup I \text{ in } (L^\infty(Q))^{n\times n} \text{ weak }*;$$

$$a^\varepsilon(I-\operatorname{grad}_y\chi)^\varepsilon \rightharpoonup [q_{ij}] = \mathscr{M}_{y,\tau}(a(y,\tau)(I-\operatorname{grad}_y\chi)(y,\tau)) \quad \text{in } (L^\infty(Q))^{n\times n} \text{ weak}*,$$

where

$$\mathscr{M}_{y,\tau}(a(I-\operatorname{grad}_y\chi)) = [\mathscr{M}_{y,\tau}(a_{ij}(y,\tau)-a_{ik}(y,\tau)\,(\partial\chi_j/\partial y_k)\,(y,\tau))] \quad i,j,k=1,\dots,n;$$

they are ε-uniformly bounded in $(L^\infty(Q))^{n\times n}$

$$\|(I-\operatorname{grad}_y\chi)^\varepsilon\|_{L^\infty}\leqslant c \ \text{[8]} . \tag{2.10}$$

By using these properties and (1.15), the following result is arrived at $\forall\psi\in(\mathscr{D}(Q))^n$

$$a^{\varepsilon'}(x,t)(\operatorname{grad}_x u^l_{\varepsilon'}-(I-\operatorname{grad}_y\chi)^{\varepsilon'}\psi)\cdot(\operatorname{grad}_x u^l_{\varepsilon'}-(I-\operatorname{grad}_y\chi)^{\varepsilon'}\psi)$$
$$\rightharpoonup [q_{ij}](\operatorname{grad}_x u^l-\psi)\cdot(\operatorname{grad}_x u^l-\psi) \quad \text{in } \mathscr{M}(Q) \text{ vaguely}, \quad l=1,\dots,N .$$

By taking (1.14) into account, the above result can be improved $\forall\psi\in(\mathscr{D}(Q))^n$

$$\int_Q a\left(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'}\right)\left(\operatorname{grad}_x u^l_{\varepsilon'}-(I-\operatorname{grad}_y\chi)\left(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'}\right)\psi\right)\cdot\left(\operatorname{grad}_x u^l_{\varepsilon'}-(I-\operatorname{grad}_y\chi)\left(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'}\right)\psi\right)dx\,dt$$
$$\to\int_Q [q_{ij}](\operatorname{grad}_x u^l-\psi)\cdot(\operatorname{grad}_x u^l-\psi)\,dx\,dt \qquad l=1,\dots,N . \tag{2.11}$$

Now let $\delta>0$ be arbitrarily small and $\psi^l\in(\mathscr{D}(Q))^n$ such that

$$\|\operatorname{grad}_x u^l-\psi^l\|_{(L^2(Q))^n}\leqslant\delta . \tag{2.12}$$

It follows from (2.10) and (2.12) that

$$\|(I-\operatorname{grad}_y\chi)^{\varepsilon'}(\operatorname{grad}_x u^l-\psi^l)\|_{(L^2(Q))^n}$$
$$\leqslant\|(I-\operatorname{grad}_y\chi)^{\varepsilon'}\|_{L^\infty}\|\operatorname{grad}_x u^l-\psi^l\|_{L^2}\leqslant c\delta . \tag{2.13}$$

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(7) The proof is similar to the one given by F. Murat for the elliptic case [**8**].
(8) We define

$$\|(I-\operatorname{grad}_y\chi)^\varepsilon\|_{L^\infty}=\operatorname{ess\,sup}\left\{|(I-\operatorname{grad}_y\chi)\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)\xi| : \xi\in R^n,\ |\xi|=1;\ (x,t)\in Q\right\}.$$

Moreover, by using (2.11) and $(1.5)_\varepsilon$, we get $\forall \varepsilon' < \varepsilon_0'$

(2.14)
$$\lambda \|\operatorname{grad}_x u^l_{\varepsilon'} - (I - \operatorname{grad}_y \chi)^{\varepsilon'} \psi^l\|^2_{(L^2(Q))^n} \leq m \|\operatorname{grad}_x u^l - \psi^l\|^2_{(L^2(Q))^n} \leq m\delta^2 ,$$

where $m > 0$ satisfies the condition $|[q_{ij}]\xi| \leq m|\xi|$ $\forall \xi \in R^n$ (9). Let us consider

$$z^l_{\varepsilon'} = \operatorname{grad}_x u^l_{\varepsilon'} - (I - \operatorname{grad}_y \chi)^{\varepsilon'} \operatorname{grad}_x u^l$$
$$= (\operatorname{grad}_x u^l_{\varepsilon'} - (I - \operatorname{grad}_y \chi)^{\varepsilon'} \psi^l) - ((I - \operatorname{grad}_y \chi)^{\varepsilon'} (\operatorname{grad}_x u^l - \psi^l)) .$$

It follows from (2.13) and (2.14) that $\forall \varepsilon' < \varepsilon_0'$

$$\|z^l_{\varepsilon'}\|_{(L^2(Q))^n} \leq \delta \left(c + \sqrt{\frac{m}{\lambda}}\right) .$$

This inequality, which holds $\forall \delta > 0$, implies (1.17).

Proof of Lemma 5. Let us write (1.17) in matrix form

(1.17)′
$$\operatorname{grad}_x u_{\varepsilon'} = (I - \operatorname{grad}_y \chi)^{\varepsilon'} \operatorname{grad}_x u + z_{\varepsilon'} ,$$

where $z_{\varepsilon'} \to 0$ in $(L^2(Q))^{n\times N}$ strongly.

We shall prove (1.19) in two steps: first (a) we show that

(2.15)
$$f^l_{\varepsilon'}(\cdot, \cdot, u_{\varepsilon'}, \operatorname{grad}_x u_{\varepsilon'}) = f^l_{\varepsilon'}(\cdot, \cdot, u, (I - \operatorname{grad}_y \chi)^{\varepsilon'} \operatorname{grad}_x u) + r^l_{\varepsilon'} ,$$

where $r^l_{\varepsilon'} \to 0$ in $L^1(Q)$ strongly ($l = 1, \ldots, N$), and next (b) we show that

(2.16)
$$f^l_{\varepsilon'}(\cdot, \cdot, u, (I - \operatorname{grad}_y \chi)^{\varepsilon'} \operatorname{grad}_x u) \rightharpoonup f^l_0(u, \operatorname{grad}_x u) \quad \text{in } L^1(Q) \text{ weakly,}$$

$l = 1, \ldots, N$.

(a) We recall that $\|u_{\varepsilon'}\|_{L^\infty}$, $\|u\|_{L^\infty} \leq M$; moreover (2.1) holds. Therefore, if we choose $\eta > 0$ arbitrarily small, we have $\|u_{\varepsilon'} - u\|_{L^\infty} \leq \eta$ $\forall \varepsilon' < \varepsilon_0'$, which im-

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(9) If we denote by $\mathscr{L}$ the space $\mathscr{L}(L^2(0, T; H^1_0(\Omega)), L^2(0, T; H^{-1}(\Omega)))$, then $\|A\|_{\mathscr{L}} \leq m$.

plies property (1.7)

$$(2.17)\qquad |r^l_{\varepsilon'}(x,t)| \leqslant c_1(M)\big(|z_{\varepsilon'}| + |z_{\varepsilon'}|\,|(I - \operatorname{grad}_y \chi)(\frac{x}{\varepsilon'}, \frac{t}{\varepsilon'})\operatorname{grad}_x u| + |z_{\varepsilon'}|^2\big) + c_2(\eta) \quad \text{a.e. in } (x,t) \quad l = 1, \ldots, N\,,$$

where $c_1(M)$, $c_2(\eta) > 0$ and $\lim_{\eta\to 0} c_2(\eta) = 0$.

By integrating (2.17) over Q and recalling (2.10), we obtain

$$\|r^l_{\varepsilon'}\|_{L^1(Q)} \leqslant c_1(M)\,\big(|Q|^{\frac12}\,\|z_{\varepsilon'}\|_{L^2} + \|z_{\varepsilon'}\|_{L^2}\|(I - \operatorname{grad}_y\chi)^{\varepsilon'}\operatorname{grad}_x u\|_{L^2} + \|z_{\varepsilon'}\|^2_{L^2}\big)$$

$$+\, c_2(\eta)\,|Q| \leqslant c_1(M)\|z_{\varepsilon'}\|_{L^2}\,\big(|Q|^{\frac12} + c\|\operatorname{grad}_x u\|_{L^2} + \|z_{\varepsilon'}\|_{L^2}\big) + c_2(\eta)\,|Q| \qquad l = 1, \ldots, N;$$

therefore (1.17)′ implies (2.15).

(b) Let $\delta > 0$ be arbitrarily small and $\varphi: Q \to R^N$, $\psi: Q \to R^{nN}$ be staircase functions such that $\|u - \varphi\|_{L^\infty} \leqslant \delta$, $\|\operatorname{grad}_x u - \psi\|_{L^2} \leqslant \delta$.

We notice that

$$(2.18)\qquad \|f^l_{\varepsilon'}(\cdot,\cdot,u,(I - \operatorname{grad}_y\chi)^{\varepsilon'}\operatorname{grad}_x u) - f^l_{\varepsilon'}(\cdot,\cdot,\varphi,(I - \operatorname{grad}_y\chi)^{\varepsilon'}\psi)\|_{L^1(Q)} \leqslant c_1(\delta + c_2(\delta)) \qquad l = 1, \ldots, N\,,$$

where $\lim_{\delta\to 0} c_2(\delta) = 0$. This is obtained by integrating over Q the following inequality, which is an application of (1.7)

$$|f^l_{\varepsilon'}(x,t,u,(I - \operatorname{grad}_y\chi)(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'})\operatorname{grad}_x u) - f^l_{\varepsilon'}(x,t,\varphi,(I - \operatorname{grad}_y\chi)(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'})\psi)|$$

$$\leqslant c_1'\big(|(I - \operatorname{grad}_y\chi)(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'})(\operatorname{grad}_x u - \psi)|$$

$$+\,|(I - \operatorname{grad}_y\chi)(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'})(\operatorname{grad}_x u - \psi)|\cdot|(I - \operatorname{grad}_y\chi)(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'})\psi|$$

$$+\,|(I - \operatorname{grad}_y\chi)(\frac{x}{\varepsilon'},\frac{t}{\varepsilon'})(\operatorname{grad}_x u - \psi)|^2\big) + c_2(\delta) \qquad \text{(by (2.10))}$$

$$\leqslant \tilde{c}_1\big(|\operatorname{grad}_x u - \psi| + |\operatorname{grad}_x u - \psi|\,|\psi(x,t)| + |\operatorname{grad}_x u - \psi|^2\big) + c_2(\delta) \quad \text{a.e. in } (x,t),$$

where $l = 1, \ldots, N$, $c_2(\delta) > 0$ and $\lim_{\delta\to 0} c_2(\delta) = 0$. By (1.7) we also get

$$(2.19)\qquad \|f^l_0(u, \operatorname{grad}_x u) - f^l_0(\varphi,\psi)\|_{L^1(Q)} \leqslant c_1(\delta + c_2(\delta)) \qquad l = 1, \ldots, N\,.$$

Now we recall that, according to (1.9)

$$f^l_{\varepsilon'}(\cdot, \cdot, \varphi, (I - \mathrm{grad}_y \chi)^{\varepsilon'} \psi) \rightharpoonup f^l_0(\varphi, \psi) \text{ in } L^2(Q) \text{ weakly} \qquad l = 1, \ldots, N. \tag{2.20}$$

At last from (2.18), (2.19) and (2.20) we deduce (2.16).

Proof of Theorem 2. This follows immediately from Lemmas 3 and 5. We just notice that (1.19) allows us to extend the test functions' space from

$$L^2(0, T; H^1_0(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \cap C^0(\bar{Q})$$

to

$$L^2(0, T; H^1_0(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \cap L^\infty(Q) .$$

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Sunto

Si considerano sistemi quasilineari, parabolici, in forma diagonale, con termine del primo ordine quadratico nel gradiente spaziale.

Utilizzando il risultato di hölderianità ottenuto in [**9**], *si dimostrano, con metodi del tipo energia e con un opportuno metodo dei « correttori », la convergenza dell'omogeneizzazione e la convergenza dell'integrale dell'energia.*

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